J. S. R. Chisholm
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MATHEMATICAL METHODS IN PHYSICS

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Mathematical Methods in Physics

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PREFACE

This book is primarily designed as a pure mathematics text-book which should serve students of physics, chemistry and engineering; it should be of considerable use also to students specialising in mathematics. While the book is probably best used in conjunction with lecture courses, it is designed to be used for private study by students with a knowledge of algebra and of elementary trigonometry and coordinate geometry. To this end, we have illustrated the text with numerous examples and have included in each chapter several sets of exercises. The book may also be useful to students and research workers in the biological and social sciences, where mathematical techniques are playing an increasingly important role; in particular we devote the final chapter to an introductory account of probability and statistics, a topic which is of importance to every scientist. In the text, certain paragraphs have been 'starred' with an asterisk; these can be omitted on a first reading.

Our two main aims are to teach mathematical techniques and to give a clear account of useful concepts in mathematics and mathematical physics. For the great majority of students, mastery of a mathematical technique can only be obtained by practice; we advise serious students to work through the text and examples with a pencil and paper to hand, and to do a considerable number of each set of exercises.

Some of the concepts we present are purely mathematical, but wherever possible we have linked an abstract mathematical concept with allied physical, or possibly geometrical, concepts. For example, in the discussion of vectors we mention force and velocity vectors; we associate differential coefficients with rates of change, integrals with areas; the Dirac δ-function is introduced as the mathematical analogue of a point charge; the vector operators 'div' and 'curl' in a three-dimensional continuum are pictured in terms of the flow and vorticity of a fluid; and so on. This linkage of mathematical and physical concepts is not merely a trick of teaching technique – it is a fundamental correspondence which is basic to all applied mathematics.

Physical concepts arise by observing the universe in which we live and by trying to describe or understand its behaviour in more or less simple terms, constructing mental 'models' of physical systems. For example, we frequently use a model of a dilute gas which is described in terms of volume (V), pressure (p), temperature (T) and a further small number of quantities such as energy and specific heat, though we are aware that the gas is a complicated structure composed of a large number of fast-moving molecules. When we write down formulae such as '$pV/T =$ constant', we are expressing in precise mathematical form a relation which is approximately true for measurable physical quantities; the law is true in some sense for the model only, the 'ideal gas' in this example, and not for any physical gas.

A mathematical theory is a set of equations containing symbols (such as p, V and T) which corresponds to the model; these equations must satisfy two basic conditions in order to be useful:

(i) the *physical* condition that the model must have properties similar to those observed in physical systems;

(ii) the *mathematical* condition that the equations form a consistent mathematical scheme.

In pure mathematics we take no account of the first condition; the equations we study have no particular 'meaning', the concepts we use become non-physical or 'abstract', and the mathematical consistency of the scheme is the only measure of its validity. In applied mathematics however we must seek to satisfy these two conditions simultaneously, an ideal which is often hard to attain; frequently we make progress in theoretical physics for example, by deliberately relaxing one condition in favour of the other. In writing this book we have borne in mind both conditions: the physical one in relating mathematical concepts to physical ones wherever possible; the mathematical one in dealing carefully with difficult mathematical concepts such as limits and in developing subjects such as the vector and matrix algebras in a logical order. Neither condition has been fully satisfied, and we do not feel that this is either possible or desirable in a textbook of this nature; but we hope that, in addition to presenting an intelligible account of useful mathematical techniques, we have succeeded in demonstrating how each condition contributes to the development and understanding of mathematical theories.

One of the most difficult tasks in writing this book has been to make a selection of material. Not only could we have included complete chapters on, say, group theory, approximation methods and integral equations;

we could also have extended our treatment of several topics, in particular, of the special functions, of operational techniques and of probability and statistics. We hope however that our choice of material will provide a broad basic knowledge of mathematics for students of a variety of subjects.

Acknowledgments

A number of exercises have been taken from University of Wales examination papers, and we are indebted to the University for allowing us to reproduce these questions. We are pleased to acknowledge also a number of useful discussions with members of the Department of Pure Mathematics at University College, Cardiff, in particular with Professor J. L. B. Cooper.

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CONTENTS

CHAPTER 1

FUNCTIONS. LIMITS, CONTINUITY, DIFFERENTIATION

§ 1. Functions: examples

The notion of a relation existing between the values of two variable physical quantities presents itself immediately in the study of any branch of physics, chemistry or mathematics. For example, the pressure p of a gas at a given temperature is related to the density ρ of the gas; the period of oscillation T of a simple pendulum depends on the length l of the pendulum; the time t that a body takes to fall from rest under gravity depends on the height h from which it is dropped. In all these simple examples it is possible to find a definite mathematical formula relating the two quantities; the three formulae being the well-known relations

$$\text{(i)} \quad p = k\rho, \qquad \text{(ii)} \quad T = 2\pi\sqrt{(l/g)}, \qquad \text{(iii)} \quad h = \tfrac{1}{2}gt^2.$$

In these formulae the quantities p, ρ; T, l; h, t are assumed to be able to take different real values; such quantities are said to be *real variables* or more simply *variables*. The other quantities k, g do not change their values and these are referred to as *constants*. In most mathematical expressions the earlier letters of the alphabet $a, b, c, \ldots$ are generally used to denote constants, whilst the later letters u, v, x, y, z are used to denote variables.

§ 1.1. Functions: notation and symbolism

A variable quantity y is said to be a function of another variable quantity x if a rule is laid down which determines one or more values of y when the value of x is given. The rule for calculating y from x is known as a *functional relation* between x and y.

A particularly simple mathematical relation or rule connecting two variables is the one relating the sum S_n to the number of terms n in the

simple series

$$1 + 2 + 3 + \dots + n + \dots .$$

This relation is $S_n = \frac{1}{2}n(n+1)$; given n, S_n is determined, so that S_n is a function of n. In this formula for S_n however, the number n only takes the discrete values 1, 2, 3,.... In general a real variable x in an algebraic expression can take any value within a certain range or interval. If the lowest value of x is a and its highest value is b and x can take any value between a and b we say it is a continuous variable in the range or interval (a, b) and write

$$a \leqslant x \leqslant b. \tag{1.1}$$

The interval defined by this inequality (1.1) is called a *closed* interval, since the end points are included among the values of x which form the interval. The interval defined by the inequality

$$a < x < b, \tag{1.2}$$

is called an *open* interval, the end points not being included among the values of x which form the interval. We shall use the notation $)a, b($ to denote an open interval.

When we speak of a closed interval (a, b) it is understood that a and b are finite. If x can take indefinitely large positive values we replace b in the open interval $)a, b($ by ∞; if x can take any real value, we write $-\infty < x < \infty$.

It is often not possible to find a simple mathematical formula relating two variable quantities; for example, suppose that S'_n is the sum of the first n terms of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots,$$

S'_n is dependent on n and is thus a function of n, but there is no simple algebraic expression for S'_n in terms of n.

In experimental sciences, when one quantity is a function of another, the relation between them is often represented graphically; for example, a barograph automatically records the atmospheric pressure p at any time t, but it may not be possible to find a mathematical formula connecting p and t. Thus in physics we often wish to express the fact that one physical variable y is a function of another physical variable x without being able to write down a specific formula between them.

In mathematics we also need to deal in general with functional re-

lations between variables and we express the fact that y is a function of x by writing for example $y=f(x)$, $y=g(x)$ or $y=\Phi(x)$; the symbolism $f(x)$ does not mean f multiplied by x but is simply an abbreviation of the phrase 'function of x' and the different letters $f, g, \Phi, \ldots$ in the above expressions are used to denote different functional relations. When a functional relation is expressed as $y=f(x)$, y is said to be defined *explicitly* in terms of x; x is called the 'independent variable', and y the 'dependent variable'. We are thinking then of a physical situation or experiment in which we are free to choose the quantity x, and the quantity y is thereby fixed. A relation $y=f(x)$ then implies that, given any values of x, then to some or all of them there corresponds some specific value of y. If to each value of x there corresponds one and only one value of y, as for example when $y=\frac{1}{2}x(x+1)$, the function is said to be *single valued* or *uniform*; but if more than one value of y corresponds to one value of x the function is said to be a *multiform* or *multivalued* function. Examples of multivalued functions (as we shall see later in § 1.3) are $y=\sqrt{x}$ (two valued) and $y=\sin^{-1} x$ (infinitely many valued).

A function is not necessarily defined by a single algebraic expression for all values of x. For example

$$\begin{aligned} y &= x && \text{when} \quad -\infty < x \leqslant 1, \\ y &= 2x - 1 && \text{when} \quad 1 < x < \infty, \end{aligned}$$

defines y as a function of x for all values of x.

It is often not possible to express a relation in the form $y=f(x)$; if for instance x and y are related by

$$x^2 y - x \cos y = 0,$$

it is not possible to express y as a simple function of x; but given a value of x the equation for y can be solved graphically; such a relationship is said to be *implicit*. The functional symbolism for implicit relationships is $f(x, y)=0$, $g(x, y)=0, \ldots$.

Explicit and implicit functional relationships may be extended to more than two variables but for the present we shall confine ourselves to two variables.

Some physical quantities such as mass, length and density never take negative values, whilst others such as height, rotation and speed may be either positive or negative. Similarly if $y=f(x)$ then the value of y may be always positive for all real values of x within a certain range, for example, $y=x^2 \geqslant 0$ for any x in the range $)-\infty, \infty($; or the value of y

may sometimes be positive and sometimes negative as in the following examples:

$$y = \sin x, \qquad 0 \leqslant x \leqslant 2\pi,$$
$$y = x - 1, \qquad -\infty < x < \infty.$$

In all such cases it is often convenient to think of the magnitude of a quantity y, regardless of its sign. For this purpose we define the *modulus* or *absolute magnitude* $|y|$ to equal y when $y \geqslant 0$, and to equal $-y$ when $y < 0$; thus $|y| \geqslant 0$ always. The absolute magnitudes of $\sin x$, $x-1$ and $f(x)$, for instance, are written $|\sin x|$, $|x-1|$ and $|f(x)|$.

§ 1.2. Function of a function

Suppose that the variable x in the relation $y=f(x)$ is itself determined in terms of a third variable t by the relation $x=\Phi(t)$; x is the dependent variable in this latter relation and if a particular value t_1 of t is chosen, then the corresponding value x_1 of x is determined as $x_1=\Phi(t_1)$; and hence the corresponding value y_1 of y is determined as $y_1=f(x_1)$. Thus the values of y are determined ultimately by the chosen values of t. The functional symbolism used in this case is $y=f\{\Phi(t)\}$, and it means that y is a function of a function of t, being really equivalent to the two relations

$$y = f(x), \qquad x = \Phi(t). \tag{1.3}$$

It is important to note here the distinction between the variable y and the function f in the equation $y=f(x)$. What is really meant by this relation is that y is the value of the function when the independent variable has the value x. There is, of course, a difference between the function and the value of the function, and this sometimes leads to ambiguity in the use of functional notation. In most current literature the symbols f, g, Φ alone are used to denote functions and $f(x)$, $g(x)$, $\Phi(x)$ are used to denote the values of the functions when the variable associated with them is x. The distinction is more obvious in the above case of function of a function. If the relations (1.3) hold, it is clear that although y is a function of t, $y \neq f(t)$ in general, since the function relating the variables y and t is not f in general. In fact $f(x)$, $f(t)$ do have very different values in general. For example if

$$y = \sin x, \qquad x = t^2,$$

then $y \neq \sin t$, but in fact $y = \sin x \equiv \sin t^2$, and $\sin x (\equiv \sin t^2)$ and $\sin t$ have very different values. It is clear then that in any work involving

functional relations, especially when changes of variable occur, it is important to specify clearly which variables are related and what is the functional relation between them.

§ 1.3. Inverse functions

If y is a function of x defined by $y=f(x)$ for a given range of values of x, then the graph of y against x can be used to determine the values of x corresponding to given values of y (see figs. 1.1 and 1.2). The values of y may all be within a limited range, for example, $y \geqslant 0$ in fig. 1.1, and $-1 \leqslant y \leqslant 1$ in fig. 1.2; but within such a range we can say that x is a function of y. This function is said to be the *inverse function* of f. The value of the inverse function where it exists may be single valued or multivalued.

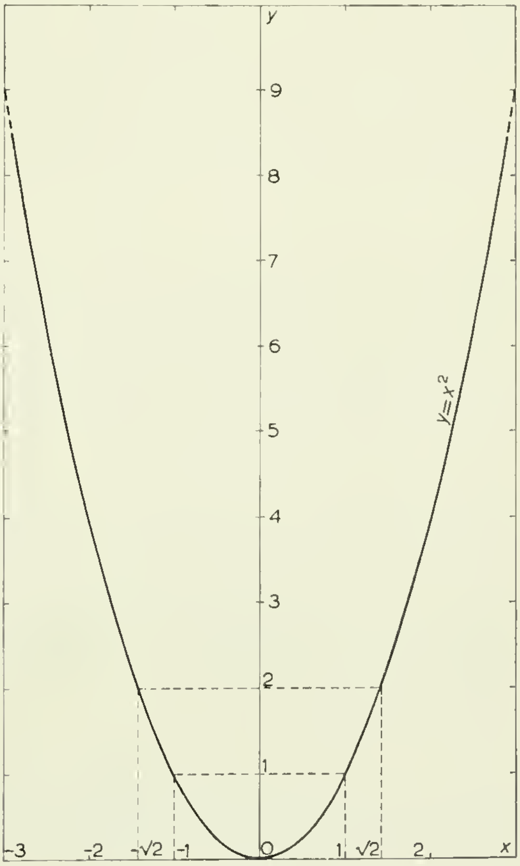

Fig. 1.1

Example 1

If $y=x^2$ (see fig. 1.1) then provided y is positive there are two values of x for every value of y, these are denoted by $\pm y^{\frac{1}{2}}$. If y is negative the inverse function $y^{\frac{1}{2}}$ is not a real function.

Example 2

If $y=\sin x$ (see fig. 1.2), we note first that $|y| \leqslant 1$ and that y assumes all its possible values between -1 and $+1$, each once only, as x increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$. Thus if $-1 \leqslant y \leqslant 1$ there are infinitely many values of x which satisfy the equation $\sin x=y$, but there is only one which also satisfies the relation $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$. This particular value of x is denoted by $\sin^{-1} y$ or arc $\sin y$ and is called the *principal value* of the inverse sine function. The complete solution of the equation $\sin x=y$ is then

$$x = \sin^{-1} y + 2n\pi \quad \text{or} \quad x = (2n-1)\pi - \sin^{-1} y,$$

where n is any positive or negative integer. Thus the inverse sine function is a many valued function.

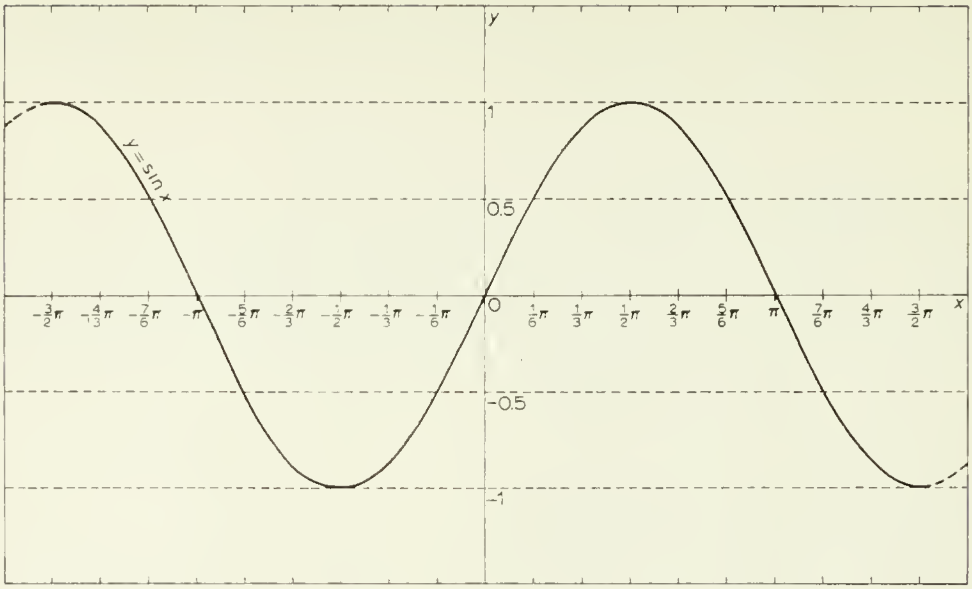

Fig. 1.2

All the inverse trigonometric functions are many valued.

When $y = \tan x$, the principal value of x is such that $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and is denoted by $\tan^{-1} y$. The complete solution is

$$x = \tan^{-1} y \pm n\pi \qquad (n = 0, 1, 2, \ldots).$$

When $y = \cos x$, since $\cos x$ is an even function it has the same values in the range $(0, \pi)$ as it has in the range $(-\pi, 0)$ for the corresponding positive and negative values of x. To include the complete range of values of y each once, we restrict x to the closed interval $(0, \pi)$. The principal value is denoted by $\cos^{-1} y$ and satisfies $y = \cos x$ and $0 \leqslant x \leqslant \pi$. The complete solution is

$$x = \pm(\cos^{-1} y + 2n\pi) \qquad (n = 0, 1, 2, \ldots).$$

Similar results may be written down for the remaining inverse trigonometric functions.

§ 2. Limits of functions

Consider the sequence of numbers

$$\frac{9}{10}, \quad \frac{99}{100}, \quad \frac{999}{1\,000}, \quad \frac{9999}{10\,000}, \quad \ldots. \tag{1.4}$$

We notice immediately that these numbers approach more and more closely to the number 1 in the sense that the difference between the numbers and 1 becomes smaller and smaller as we proceed along the sequence. The nth term of the sequence can be written as

$$\frac{10^n - 1}{10^n} = 1 - \frac{1}{10^n}. \tag{1.5}$$

This term differs from 1 by less than 0.01 if $1/10^n < 0.01$, that is if $n > 2$; so that all terms after the second differ from 1 by less than 0.01. Similarly the nth term differs from 1 by less that 10^{-6} if $n > 6$ and so on. We must notice however that even when n is very large the value of $1/10^n$ is never actually zero, so that no term of the sequence is ever actually equal to 1. Thus we say that the numbers of this sequence approach or *tend to the limiting value* 1 or more briefly *tend to* 1 as the value of n gets larger and larger or as n *tends to infinity*. We write this symbolically in the form

$$\lim_{n \to \infty} \left(1 - \frac{1}{10^n}\right) = 1, \tag{1.6}$$

where 'lim' stands for 'limit' and the arrow is used as a symbol for the words 'approaches' or 'tends to'. Sometimes eq. (1.6) is written in the form

$$1 - \frac{1}{10^n} \to 1 \quad \text{as} \quad n \to \infty.$$

In the above sequence the number n takes the discrete values 1, 2, 3,.... If x is a continuous variable and a function $f(x)$ is defined in a certain range of x, we can likewise define the limit of $f(x)$ as x tends to any given value a say, in the range.

For most ordinary functions $f(x)$, the value $f(a)$ at $x=a$ is given simply by substituting $x=a$ in $f(x)$ and then $f(x)$ approaches the value $f(a)$ as x approaches a. Thus $f(x) \to f(a)$ as $x \to a$. Simple examples are:

(i) $x^2 \to 16$ as $x \to 4$;

(ii) $\sin x \to 1$ as $x \to \frac{1}{2}\pi$;

(iii) $\log_{10} x \to 1$ as $x \to 10$.

It may happen however, that the value of $f(x)$ for some particular value of a cannot be determined simply by putting $x=a$ in the function. For example, if $f(x) = (\sin x)/x$ where x is measured in radians, and we put $x=a=0$, the value of $f(x)$ becomes 0/0 and we cannot say what this

value is. When this difficulty arises, we can define the limiting value of $f(x)$ as x tends to a.

Suppose that $f(x)$ is undefined at $x=a$, but is well-defined for values of x near to a. In fig. 1.3 the dotted lines mark a value of x near to a and the corresponding value of y. If by taking $|x-a|$ small enough we can obtain values of y as close to a number l as we please, we say that

$$\lim_{x\to a} f(x) = l, \tag{1.7}$$

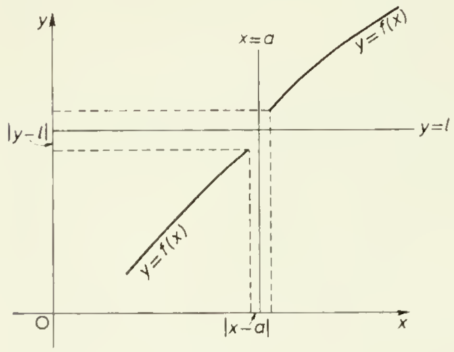

Fig. 1.3

The condition that y is as close to l as we please is best expressed as

$$|y - l| < \varepsilon, \tag{1.8}$$

where ε is any chosen positive number. The condition that the limit in eq. (1.7) exists is that eq. (1.8) holds for all x satisfying the condition

$$|x - a| < \delta, \tag{1.9}$$

where δ is a positive quantity depending on ε; this eq. (1.9) expresses the condition that 'x is sufficiently near to a'.

Example 3

As a simple example of a limit, consider the function

$$f(x) = \frac{1 + \dfrac{1}{x}}{1 + \dfrac{2}{x}},$$

as $x \to 0$. In this function both denominator and numerator become infinite as $x \to 0$, so that $f(x)$ is undefined at $x=0$. Nevertheless, for $x \neq 0$ the function can be written as $(x+1)/(x+2)$, which takes the value $\frac{1}{2}$ when $x=0$. Further

$$|f(x) - \tfrac{1}{2}| = \left| \frac{x}{2(x+2)} \right| < \tfrac{1}{2}|x| \quad \text{for} \quad |x| < 1.$$

So we can ensure that eq. (1.8) is satisfied, or

$$|f(x) - \tfrac{1}{2}| < \varepsilon,$$

for any small ε, by choosing $|x| < 2\varepsilon$, equivalent to choosing $\delta = 2\varepsilon$ in eq. (1.9). Hence

$$\lim_{x\to 0} f(x) = \tfrac{1}{2}.$$

Example 4

The limit mentioned above, namely, the limit of the function

$$f(x) = \frac{\sin x}{x}$$

as $x \to 0$, is very important. It must be clearly understood here that x is measured in radians in the expression $\sin x$ in accordance with the usual convention that angles are measured in radians if no units are mentioned. We determine the limit of this function as $x \to 0$ by considering the areas of the right angled triangle OAC, the circular sector OAB and the isosceles triangle OAB in fig. 1.4. Since we are letting $x \to 0$ we have made $x < \frac{1}{2}\pi$ in the diagram. We have

area $\triangle$OAB < area sector OAB < area $\triangle$OAC,

or

$$\tfrac{1}{2}a^2 \sin x < \tfrac{1}{2}a^2 x < \tfrac{1}{2}a^2 \tan x,$$

so that

$$\sin x < x < \tan x.$$

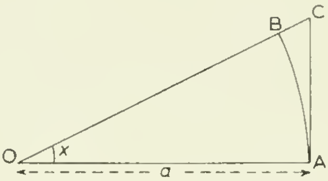

Fig. 1.4

But for $x < \frac{1}{2}\pi$, $\sin x$ is positive, and therefore dividing through by $\sin x$, we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

or

$$1 > \frac{\sin x}{x} > \cos x,$$

Hence the quotient $(\sin x)/x$ lies between the value 1 and the value of $\cos x$. But as $x \to 0$, $\cos x \to 1$, so the difference between $\cos x$ and 1 can be made as small as we please. That is

$$\left| \frac{\sin x}{x} - 1 \right| < |1 - \cos x|,$$

and we know that $1 - \cos x < \varepsilon$ provided $x < \cos^{-1}(1-\varepsilon)$. Thus $\delta = \cos^{-1}(1-\varepsilon)$ in eq. (1.9) and then we deduce that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1.10}$$

In general we note that if

$$\lim_{x \to a} f(x) = l,$$

and we write $f(x) = l + \alpha(x)$, then $f(x) - l = \alpha(x)$. Thus since from eqs. (1.8) and (1.9)

$$|f(x) - l| < \varepsilon,$$

whenever

$$|x - a| < \delta,$$

we can write this as $|\alpha(x)| < \varepsilon$ whenever $|x - a| < \delta$. This is equivalent to

$$\lim_{x \to a} \alpha(x) = 0.$$

If both $P(x)$ and $Q(x)$ are polynomials which vanish at $x = a$, the function $f(x) = P(x)/Q(x)$ becomes 0/0 when $x = a$. The remainder theorem tells us that $(x - a)$ is a factor of both $P(x)$ and $Q(x)$ and $\lim_{x \to a} f(x)$ can usually be evaluated by cancelling this factor before putting $x = a$ in the function. An alternative method of finding the limiting value as $x \to a$ is illustrated in the following simple example.

Example 5

Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{2x - 2}.$$

Put $x = 1 + h$, then as $x \to 1$, $h \to 0$ and

$$\frac{x^2 + x - 2}{2x - 2} = \frac{(1 + h)^2 + (1 + h) - 2}{2(1 + h) - 2} = \frac{h(h + 3)}{2h} = \frac{h + 3}{2}.$$

Thus

$$\lim_{x \to 1} \frac{x^2 + x - 2}{2x - 2} = \lim_{h \to 0} \frac{h + 3}{2} = \frac{3}{2}.$$

As in the above example it is sometimes convenient to change the variable in the function, especially when the value of the variable x is required to increase indefinitely or, as we say, tend to infinity. Thus if we require to find the values of $\lim_{x \to \infty} f(x)$, it is often convenient to write $x = 1/h$; then as $x \to \infty$, $h \to 0$ so that we can write the limit in the form $\lim_{h \to 0} f(1/h)$.

Example 6

$$\lim_{x \to \infty} \frac{2x^3 + 4x^2 - 5x + 1}{7x^3 - 10x - 6}.$$

Putting $x = 1/h$, the limit becomes

$$\lim_{h \to 0} \frac{2h^{-3} + 4h^{-2} - 5h^{-1} + 1}{7h^{-3} - 10h^{-1} - 6} = \lim_{h \to 0} \frac{2 + 4h - 5h^2 + h^3}{7 - 10h^2 + h^3} = \frac{2}{7}.$$

If as $x \to a$, the value of $f(x)$ increases indefinitely so that by making x approach sufficiently close to a, $f(x)$ can be made as large as we please, then we say that $\lim_{x\to a} f(x) = +\infty$. The meaning of $\lim_{x\to a} f(x) = -\infty$ is then obvious.

§ 2.1. Properties of limits

If $f(x)$ and $g(x)$ are any two functions which have finite limits l and m as x tends to a, then

$$\text{(i)} \quad \lim_{x\to a} \{f(x)+g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x), \tag{1.11}$$

$$\text{(ii)} \quad \lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x), \tag{1.12}$$

$$\text{(iii)} \quad \lim_{x\to a} f(x)/g(x) = \{\lim_{x\to a} f(x)\}/\{\lim_{x\to a} g(x)\}. \tag{1.13}$$

These results are intuitively obvious as may be seen in the following way. Given $\lim_{x\to a} f(x) = l$, and $\lim_{x\to a} g(x) = m$, we can write

$$f(x) = l + \alpha(x), \qquad g(x) = m + \beta(x),$$

where $\alpha(x)$, $\beta(x)$ are functions of x which tend to zero as x tends to a. Then

$$\lim_{x\to a} \{f(x) + g(x)\} = \lim_{x\to a} \{(l + m) + \alpha(x) + \beta(x)\} = l + m,$$

since $\alpha(x)+\beta(x)$ may be made as small as we please as $x \to a$.

Also

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{l+\alpha}{m+\beta} = \lim_{x\to a} \left(\frac{l}{m} + \frac{l+\alpha}{m+\beta} - \frac{l}{m}\right)$$

$$= \lim_{x\to a} \left\{\frac{l}{m} + \frac{\alpha m - \beta l}{m(m+\beta)}\right\} = \frac{l}{m},$$

provided $m \neq 0$, since $(\alpha m - \beta l)/m(m+\beta)$ may then be made as small as we please. The result (1.12) may be demonstrated in the same way.

More rigorous proofs of these results, using the definition of a limit given by eqs. (1.7)–(1.9) require the following theorem.

For any two numbers A, B it is always true that

$$|A + B| \leqslant |A| + |B|,$$

and

$$|AB| = |A|\cdot|B|.$$

It can be seen that these results are true in all the following cases, (i) A, B have same sign, (ii) A, B have different signs, (iii) either $A=0$ or $B=0$.

★ The proof of eq. (1.12) is then

$$\begin{aligned}|f(x)\,g(x) - lm| &= |(f(x) - l)(g(x) - m) + l(g(x) - m) + m(f(x) - l)| \\ &\leqslant |f(x) - l|\,|g(x) - m| + |l|\,|g(x) - m| + |m|\,|f(x) - l|.\end{aligned}$$

Now suppose that $\varepsilon_1 > 0$ and that δ_1 and δ_2 be positive numbers such that

$$|f(x) - l| < \varepsilon_1 \quad \text{whenever} \quad |x - a| < \delta_1,$$
$$|g(x) - m| < \varepsilon_1 \quad \text{whenever} \quad |x - a| < \delta_2.$$

Let δ_1 be the smaller of δ_1 and δ_2; then provided $|x-a|<\delta_1$, we have

$$|f(x) - l|\,|g(x) - m| + |l|\,|g(x) - m| + |m|\,|f(x) - l| \leqslant \varepsilon_1^2 + |l|\,\varepsilon_1 + |m|\,\varepsilon_1.$$

Now given any small quantity ε, if ε_1 is chosen such that it is the positive root of

$$\varepsilon_1^2 + |l|\,\varepsilon_1 + |m|\,\varepsilon_1 = \varepsilon,$$

we have from the above results

$$|f(x)\,g(x) - lm| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta_1.$$

This concludes the proof. ★

Example 7

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \sec x = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \sec x = 1.$$

Example 8

$$\lim_{x\to 0} (\cot x - \operatorname{cosec} x).$$

This is of the form $\infty - \infty$ and is not necessarily zero. However we prove that it is zero in this case as follows

$$\lim_{x\to 0} (\cot x - \operatorname{cosec} x) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x},$$

and we can write this as

$$\lim_{x\to 0} \frac{\cos^2 x - 1}{\sin x\,(\cos x + 1)} = \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} = 0.$$

Limits may often be evaluated by algebraic or trigonometric manipulation and the use of properties (1.11)–(1.13) as in Examples 5–8, or may be verified by direct application of the definition as in Examples 9 and 10 following. Other methods of evaluating limits will be given in Ch. 6.

Example 9

Show that

$$\lim_{n\to\infty} \frac{n^2 - 5n + 2}{3n^2 + n + 1} = \frac{1}{3}.$$

By definition this means that

$$\left| \frac{n^2 - 5n + 2}{3n^2 + n + 1} - \frac{1}{3} \right| < \varepsilon, \tag{1.14}$$

provided n is sufficiently large; this may be written, provided

$$n > N, \tag{1.15}$$

where N is a large positive number whose value depends on ε. We have

$$\left| \frac{n^2 - 5n + 2}{3n^2 + n + 1} - \frac{1}{3} \right| = \left| \frac{-16n + 5}{3(3n^2 + n + 1)} \right| = \left| \frac{16n - 5}{3(3n^2 + n + 1)} \right|.$$

For $n > 0$, we then have

$$\left| \frac{16n - 5}{3(3n^2 + n + 1)} \right| < \left| \frac{16n}{9n^2} \right| = \frac{16}{9n},$$

and thus eq. (1.14) is satisfied provided $16/9n < \varepsilon$, that is, whenever $n > N$, where $N = 16/9\varepsilon$.

Example 10

Show that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

By definition we must show that

$$\left| \frac{n!}{n^n} \right| < \varepsilon \quad \text{for} \quad n > N.$$

We have

$$\frac{n!}{n^n} = \frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n}$$

$$= 1\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-3}{n}\right)\left(1 - \frac{n-2}{n}\right)\frac{1}{n}.$$

Every term in a bracket in this last expression is <1. Thus

$$\left|\frac{n!}{n^n}\right| < \frac{1}{n}$$

and if we choose $N = 1/\varepsilon$, the result follows.

§ 2.2. An important special limit

A limit which is of importance in later work is

$$\lim_{b \to a} \frac{b^n - a^n}{b - a}.$$

We shall evaluate this limit when n is (i) a positive integer, (ii) a negative integer, (iii) a rational fraction.

(i) When n is a positive integer

$$\frac{b^n - a^n}{b - a} = b^{n-1} + ab^{n-2} + \dots + a^{n-2}b + a^{n-1}.$$

Thus

$$\lim_{b \to a} \frac{b^n - a^n}{b - a} = na^{n-1}.$$

(ii) When n is a negative integer, let $n = -m$ so that m is a positive integer; then

$$\lim_{b \to a} \frac{b^n - a^n}{b - a} = \lim_{b \to a} \frac{b^{-m} - a^{-m}}{b - a} = \lim_{b \to a} \left(-\frac{1}{a^m b^m}\right)\left(\frac{b^m - a^m}{b - a}\right),$$

and using the property of limits in eq. (1.12) this is

$$\lim_{b \to a} \left(-\frac{1}{a^m b^m}\right) \lim_{b \to a} \frac{b^m - a^m}{b - a} = -\frac{1}{a^{2m}}\, ma^{m-1},$$

using (i). Thus the value is

$$-ma^{-m-1} = na^{n-1}.$$

(iii) When n is a positive or negative rational fraction, let $n = p/q$ where p and q may be positive or negative integers, then

$$\frac{b^n - a^n}{b - a} = \frac{b^{p/q} - a^{p/q}}{b - a}.$$

Put $b=y^q$, $a=x^q$ then the limit becomes

$$\lim_{y\to x}\frac{y^p-x^p}{y^q-x^q}=\lim_{y\to x}\left\{\left(\frac{y^p-x^p}{y-x}\right)\Big/\left(\frac{y^q-x^q}{y-x}\right)\right\}$$

$$=\left\{\lim_{y\to x}\frac{y^p-x^p}{y-x}\right\}\Big/\left\{\lim_{y\to x}\frac{y^q-x^q}{y-x}\right\},$$

and using (i) and (ii) this becomes

$$\frac{px^{p-1}}{qx^{q-1}}=\frac{p}{q}x^{p-q}=\frac{p}{q}a^{(p-q)/q}=na^{n-1}.$$

Thus

$$\lim_{b\to a}\frac{b^n-a^n}{b-a}=na^{n-1}, \tag{1.16}$$

whether n is a positive or negative integer or a rational fraction.

EXERCISE 1.1

Evaluate the limits in Nos. 1–12.

1. $\lim_{x\to\alpha}\dfrac{\sin(x-\alpha)}{x^2-\alpha^2}$.

2. $\lim_{x\to\frac{1}{2}\pi}\{\tan x-(1+x)\sec x\}$.

3. $\lim_{x\to 0}\dfrac{1-\cos 2x}{x}$.

4. $\lim_{x\to 0}\dfrac{\operatorname{cosec} x-\cot x}{\sin x}$.

5. $\lim_{x\to 0}\dfrac{x}{\sqrt{(1+x)}-\sqrt{(1-x)}}$.

6. $\lim_{x\to\infty}\{\sqrt{(1+3x+x^2)}-\sqrt{(1+x^2)}\}$.

7. $\lim_{x\to 0}\dfrac{x^2}{b-\sqrt{(b^2+x^2)}}$.

8. $\lim_{x\to a}\dfrac{x^3+ax^2-2a^3}{x^2-a^2}$.

9. $\lim_{x\to\infty}\dfrac{x^3+2x^2+3}{x^3+4x+1}$.

10. $\lim_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{a+x}-\dfrac{1}{a}\right)$.

11. $\lim_{x\to\infty}x^2\sin\left(\dfrac{1}{x}\right)$.

12. $\lim_{x\to 1}\dfrac{\sqrt{(5x-4)}-\sqrt{x}}{x-1}$.

In Nos. 13–15, find all those positive integers n for which $|A_n-a|<\varepsilon$. If possible, find a positive integer N, such that for all $n>N$, $|A_n-a|<\varepsilon$.

13. $A_n=(-1)^n,\ a=1$ (i) $\varepsilon=2\frac{1}{2}$ (ii) $\varepsilon=\frac{1}{2}$.

14. $A_n=\dfrac{n+2}{n},\ a=1$ (i) $\varepsilon=2$ (ii) $\varepsilon=\frac{1}{4}$.

15. $A_n=\sqrt{(n+1)}-\sqrt{n},\ a=0$ (i) $\varepsilon=1$ (ii) $\varepsilon=\frac{4}{13}$.

In Nos. 16–20 use the ε, N definition of a limit as $n \to \infty$, given in Example 9, to verify the given values of the limits.

16. $\lim_{n\to\infty} (n^2 + n - 1)/(3n^2 + 1) = \frac{1}{3}$.

17. $\lim_{n\to\infty} (6n^3 + 2n + 1)/(n^3 + n^2) = 6$.

18. $\lim_{n\to\infty} (5n - 1)/(7n + 3) = \frac{5}{7}$.

19. $\lim_{n\to\infty} \sqrt{n}\{\sqrt{(n + 1)} - \sqrt{n}\} = \frac{1}{2}$.

20. $\lim_{n\to\infty} \dfrac{5n^2 + 1}{n^3 + 2n} = 0$.

§ 3. Continuous functions

Consider the following two functions of x and their graphs in figs. 1.5 and 1.6,

(i) $f(x) = x + 1$ when $x \geqslant 0$, $f(x) = -x$ when $x < 0$

and

(ii) $g(x) = \tan x$.

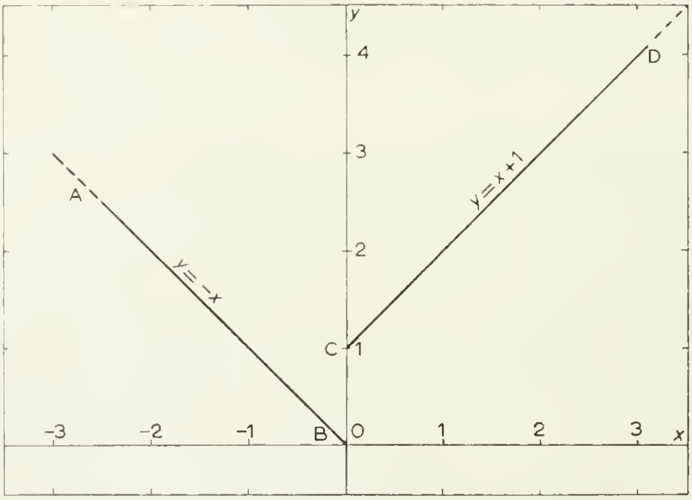

Fig. 1.5

Let us try to find the limit of $f(x)$ as $x \to 0$. We note immediately in fig. 1.5 that the limit obtained by approaching $x=0$ along the line CD will be different from that obtained by approaching along AB; that is, the limits as we approach $x=0$ through positive and negative values of x

will be different. We distinguish these two approaches by calling them $x \to 0+$ and $x \to 0-$ respectively; so for the function $f(x)$ above

$$\lim_{x \to 0+} f(x) = 1; \qquad \lim_{x \to 0-} f(x) = 0.$$

Similarly the function $g(x) = \tan x$ has two limits as $x \to \frac{1}{2}\pi$:

$$\lim_{x \to \frac{1}{2}\pi +} g(x) = -\infty; \qquad \lim_{x \to \frac{1}{2}\pi -} g(x) = +\infty.$$

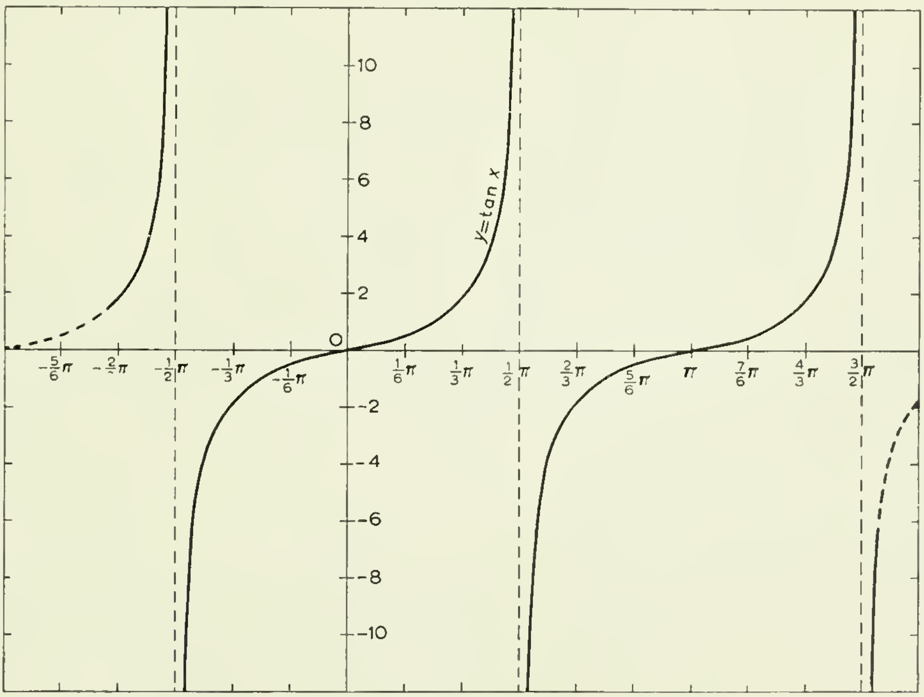

Fig. 1.6

Both these functions have breaks in their graphs at certain points. We say that a function is continuous at a point if it has no break in its graph at that point, and that it is discontinuous at any point where it has a break in its graph. We note that where there are breaks in the graphs of the functions $f(x)$ and $g(x)$ defined above, the functions do not possess unique finite limits. More precisely then we say that a function $f(x)$ is continuous at a point $x=a$ if it possesses a *unique* finite limit at $x=a$ whose value is the value of the function at $x=a$, that is, $f(a)$. In the above notation the condition for continuity of $f(x)$ at $x=a$ is

$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a).$$

A function is then defined to be continuous in a given range $)b, c($ of values of x, if it is continuous for every value of x in the open range.

§ 3.1. Properties of continuous functions

From the properties of limits given in § 2.1 we can derive the following properties of continuous functions.

(i) The sum of two continuous functions is a continuous function.

(ii) The product of two continuous functions is a continuous function.

(iii) If the functions $f(x)$ and $g(x)$ are both continuous at $x=a$, then provided $g(a)\neq 0$, the quotient $f(x)/g(x)$ is continuous at $x=a$.

The reader can easily verify the following results:

All polynomials denoted by $P(x)$, $Q(x)$ are continuous functions.

All rational functions $P(x)/Q(x)$ are continuous in any range which does not include values of x at which $Q(x)=0$.

The circular functions $\sin x$, $\cos x$ are continuous for all values of x.

The functions $\tan x$, $\sec x$ are discontinuous for values of x given by $x=\pm\frac{1}{2}(2n+1)\pi$ $(n=0, 1, 2,\ldots)$, as already seen for $\tan x$ at $x=\frac{1}{2}\pi$ in § 3. Similarly $\cot x$, $\operatorname{cosec} x$ are discontinuous for values of x given by $x=\pm n\pi$ $(n=0, 1, 2,\ldots)$.

§ 4. Derivative of a function

When a variable x or y varies from one value to another the amount by which the new value differs from the former value is called the *increment* of the variable. The symbol used to denote an increment of the variable x is δx.

Suppose now that the variables x and y are related by $y=f(x)$, where $f(x)$ is a finite single valued function of x in a given interval $)a, b($. Let x be given a small positive or negative increment δx; to the new value $x+\delta x$ will correspond a new value $y+\delta y$ of y, given by

$$y + \delta y = f(x + \delta x).$$

So the increment δy corresponding to the increment δx is

$$\delta y = f(x + \delta x) - f(x),$$

and the ratio of the increments is

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}. \tag{1.17}$$

For ordinary functions $f(x)$, the ratio on the right of eq. (1.17) usually tends to a unique finite limit as $\delta x \to 0$. The value of this limit is then called the *derivative* or *differential coefficient* of the function at the point x in the interval $)a, b($ and is denoted by $f'(x)$. This symbolism $f'(x)$ indicates that its value does in general depend on the value of x. As $\delta x \to 0$, the limit of $\delta y/\delta x$ is written as $\mathrm{d}y/\mathrm{d}x$ and is known as the *derivative of y with respect to x*; thus we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x). \tag{1.18}$$

When there is no unique limit $f'(x)$ for a particular value of x in (a, b) we say that the function is not differentiable for that particular value. If $f(x)$ possesses a unique derivative at all points in a range, it is said to be differentiable in the range. Before finding the values of the derivatives of certain simple functions and developing general rules for determining derivatives, we discuss the relation between continuity and differentiability of functions.

If $y = f(x)$ and $f(x)$ is differentiable, then $\delta y/\delta x$ has a unique finite limit as $\delta x \to 0$; therefore $\delta y = f(x+\delta x) - f(x) \to 0$ as $\delta x \to 0+$ or $\delta x \to 0-$; $f(x)$ is therefore a continuous function. The converse is not necessarily true. The value of $\delta y = f(x+\delta x) - f(x)$ may tend to zero as $\delta x \to 0$, but the value of $\delta y/\delta x$ which takes the form $0/0$ in the limit, may not itself have a unique limiting value. An example of this latter case is given by $f(x) = |x|$ at $x = 0$. At $x = 0$ and $\delta x \to 0+$, we have

$$\lim_{\delta x \to 0+} \delta y/\delta x = \lim_{\delta x \to 0+} \delta x/\delta x = 1,$$

whilst at $x = 0$ and $\delta x \to 0-$, we have

$$\lim_{\delta x \to 0-} \delta y/\delta x = \lim_{\delta x \to 0-} (-\delta x)/\delta x = -1.$$

Thus the limit is not unique and the function is not differentiable at $x = 0$ although it is continuous there.

§ 4.1. Some elementary derivatives

Example 11

$y = x^n$, where n is a positive or negative integer, or a rational fraction. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x + \delta x)^n - x^n}{(x + \delta x) - x},$$

and using eq. (1.16) this becomes nx^{n-1}. We note that when $n \leqslant 0$ the derivative is infinite at $x = 0$, so that in this case the function is not differentiable at $x = 0$.

Example 12

$y=a$, where a is a constant, gives $y+\delta y=a$, so that $\delta y=0$ and $\delta y/\delta x=0$. Thus the derivative of a constant is zero.

Example 13

$y=\sin x$. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\lim_{\delta x\to 0}\frac{\sin(x+\delta x)-\sin x}{\delta x}=\lim_{\delta x\to 0}\frac{2\cos(x+\frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\delta x}$$

$$=\lim_{\delta x\to 0}\cos(x+\tfrac{1}{2}\delta x)\lim_{\delta x\to 0}\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x},$$

and using eq. (1.10) with x replaced by $\frac{1}{2}\delta x$, this becomes $\cos x$.

Example 14

$y=\cos x$. As in Example 13 we may prove that in this case

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin x.$$

Example 15

$y=Cf(x)$, where C is a constant. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\lim_{\delta x\to 0}\frac{C\{f(x+\delta x)-f(x)\}}{\delta x}=\lim_{\delta x\to 0}C\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=Cf'(x).$$

In all the limits in the above examples we could have used h instead of δx throughout. The definition of $f'(x)$ is then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \qquad (1.19)$$

provided the limit exists. We shall find it convenient to use h instead of δx in some later chapters.

Also we shall frequently find it convenient to use the notation $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ for the derivative of $f(x)$ instead of $f'(x)$.

Other simple derivatives are found by using properties of derivatives which follow immediately from the properties of limits given in § 2.1.

§ 4.2. Properties of derivatives. Rules for differentiation

The following rules for differentiation are concerned with the sum, product and quotient of two functions of x. Suppose that u, v are two

variables defined as differentiable functions of x in a given interval $)a, b($ by the following relations

$$u = f(x), \qquad v = g(x).$$

Then suppose y is a variable defined as a function of x in the same range. In the following three equations the value of y is defined, and the corresponding value of $\mathrm{d}y/\mathrm{d}x$ is given:

$$y = u + v = f(x) + g(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x}; \tag{1.20}$$

$$y = uv = f(x)\, g(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}; \tag{1.21}$$

$$y = \frac{u}{v} = \frac{f(x)}{g(x)}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v \dfrac{\mathrm{d}u}{\mathrm{d}x} - u \dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}; \tag{1.22}$$

in eq. (1.22) the result is true at all points in the range where $v = g(x) \neq 0$.

We shall prove the result in eq. (1.21) and leave the proofs of the results in eqs. (1.20), (1.22), which are similar, as an exercise for the reader.

Suppose that δu, δv are the increments of u, v respectively corresponding to an increment δx of x. Since $f(x)$, $g(x)$ are differentiable then $\delta u \to 0$, $\delta v \to 0$ as $\delta x \to 0$. Then if

$$y = uv,$$

we have for the corresponding increment of y

$$\delta y = (u + \delta u)(v + \delta v) - uv = \delta u\, v + u\, \delta v + \delta u\, \delta v,$$

and therefore

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} v + u \frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \delta v.$$

Now as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} \to \frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \frac{\delta v}{\delta x} \to \frac{\mathrm{d}v}{\mathrm{d}x},$$

which are both finite, and $\delta v \to 0$. Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}.$$

By means of these rules we are now able to write down the derivatives of the remaining four circular functions

$$y = \tan x = \frac{\sin x}{\cos x}, \qquad \frac{dy}{dx} = \sec^2 x; \tag{1.23}$$

$$y = \cot x = \frac{\cos x}{\sin x}, \qquad \frac{dy}{dx} = -\operatorname{cosec}^2 x; \tag{1.24}$$

$$y = \sec x = \frac{1}{\cos x}, \qquad \frac{dy}{dx} = \sec x \tan x; \tag{1.25}$$

$$y = \operatorname{cosec} x = \frac{1}{\sin x}, \qquad \frac{dy}{dx} = -\operatorname{cosec} x \cot x; \tag{1.26}$$

remembering in results (1.25) and (1.26) that the derivative of the constant unity is zero.

Other examples are:

Example 16: sum of two functions, using (1.20):

$$y = 5x^4 + \cos x, \qquad \frac{dy}{dx} = 20x^3 - \sin x.$$

Example 17: product of two functions, using (1.21):

$$y = x^2 \tan x, \qquad \frac{dy}{dx} = 2x \tan x + x^2 \sec^2 x.$$

Example 18: quotient of two functions, using (1.22):

$$y = \frac{\sec x}{x^3 + 4x}, \qquad \frac{dy}{dx} = \frac{(x^3 + 4x) \sec x \tan x - (\sec x)(3x^2 + 4)}{(x^3 + 4x)^2}.$$

Functions which depend on more than two simple functions can be differentiated by the repeated use of the above rules.

Example 19

$$y = \frac{x^4 \tan x}{x^2 - 1}.$$

First note that

$$\frac{d}{dx}(x^4 \tan x) = 4x^3 \tan x + x^4 \sec^2 x.$$

Then

$$\frac{dy}{dx} = \frac{(x^2-1)(4x^3 \tan x + x^4 \sec^2 x) - (x^4 \tan x) 2x}{(x^2-1)^2} = \frac{2x^3(x^2-2) \tan x + x^4(x^2-1) \sec^2 x}{(x^2-1)^2}.$$

EXERCISE 1.2

Differentiate the following functions with respect to x, simplifying the answer where possible.

1. $x \tan x$. **2.** $\sin x \cos x$. **3.** $(1 + \sin x)/(1 - \sin x)$.

4. $(x^2 - 2x \cos \alpha + 1)/(x^2 + 2x \cos \alpha + 1)$.

5. $(x^3 + 3x^2 + 1) \operatorname{cosec} x$.

6. $(\cot x)/(x^2 - 1)$.

7. $(4x^3 + 2x + 5)/(2x^4 + 3)$.

8. $(ax + b)/(ax^2 + 2hx + b)$.

§ 4.3. Geometrical interpretation of derivative

The notion of a derivative has been given without any reference to the graph representing the function, but in the same way as continuity of a function can be interpreted in terms of the geometry of the curve of the function, so the derivative of a function may be interpreted geometrically. In fact the continuity of a function tells us very little about the direction of its curve at a particular point, but the derivative does just that. In fig. 1.7 let P be a fixed point on a continuous curve $y=f(x)$ and let Q be a variable point on the curve in the neighbourhood of P. Let PM, QN be the ordinates at P, Q respectively, and let PR be drawn parallel to Ox to meet QN in R. If OM$=x$, then PM$=f(x)$. Since Q is in the neighbourhood of P on the curve, we shall suppose that MN$=\delta x$ where δx is a small increment of x. We note that this small increment may be positive or negative, so that Q may be on either side of P, as shown in the diagram. Then QN$=f(x+\delta x)$ and QR$=f(x+\delta x)-f(x)$ which again may be positive or negative.

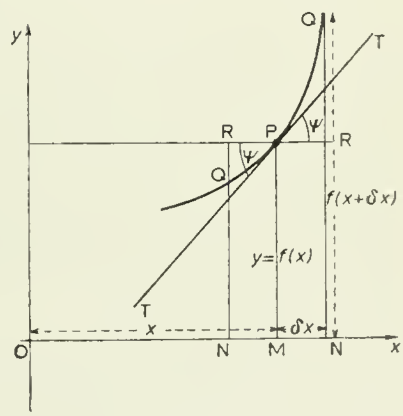

Fig. 1.7

Now we define the tangent to the curve at P as the limit of the chord PQ as Q moves up to P. In fig. 1.7 let the tangent at P be PT. We have

$$\tan \mathrm{RPQ} = \frac{\mathrm{RQ}}{\mathrm{PR}} = \frac{\mathrm{QN} - \mathrm{RN}}{\mathrm{PR}} = \frac{f(x + \delta x) - f(x)}{\delta x},$$

and as Q moves up to P, then $\delta x \to 0$, so that if Ψ is the angle that PT makes with the x-axis, this result gives

$$\tan \Psi = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}. \tag{1.27}$$

If this limit has the same value whether $\delta x \to 0$ through positive or negative values then obviously the direction of the tangent on either side of P has the same value, and the curve does not possess a 'corner', that is, it does not turn through a non-zero angle at P. At some points a curve may possess a tangent which is parallel to the y-axis. Then $\tan \Psi$ is infinite and the curve does not possess a unique *finite* derivative; so points where $\Psi = \pm \frac{1}{2}\pi$ are exceptional. Thus if a function $f(x)$ possesses a unique finite derivative at any point, this derivative measures the gradient of the tangent to the curve $y = f(x)$ at that point.

Example 20

Find the slope of the tangents to the curve

$$4y = x^2 - 20x + 28,$$

at the points P, Q whose abscissae are respectively 4, -6.

We have

$$\frac{dy}{dx} = \tfrac{1}{2}x - 5.$$

When $x = 4$, $dy/dx = -3$; when $x = -6$, $dy/dx = -8$.

§ 4.4. The derivative of a function of a function

Suppose we wish to find the derivative dy/dx when, for example

(i) $y = (2x + 3)^2$.

We can, of course, write this as

$$y = 4x^2 + 12x + 9,$$

and thence find

$$\frac{dy}{dx} = 8x + 12 = 4(2x + 3).$$

Similarly by means of the product formula, we can find the following

derivatives:

(ii) $y = \sin 2x = 2 \sin x \cos x, \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 \cos 2x;$

(iii) $y = \sin^2 x = \sin x \sin x, \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 \sin x \cos x.$

An alternative method involves the use of an intermediate variable t. In examples (i), (ii) and (iii) we write

(i) $y = t^2$ where $t = 2x + 3$,

(ii) $y = \sin t$ where $t = 2x$,

(iii) $y = t^2$ where $t = \sin x$.

In these examples, y is given as 'a function of a function' of x, in the form

$$y = f(t), \qquad t = \Phi(x),$$

so that

$$y = f\{\Phi(x)\}. \tag{1.28}$$

In general we want to find the derivative of y, defined by eq. (1.28), with respect to x. We shall assume that the derivatives $\mathrm{d}y/\mathrm{d}t = f'(t)$ and $\mathrm{d}t/\mathrm{d}x = \Phi'(x)$ exist. Proceeding from the definition of the derivative, we suppose x to be given a small increment δx. Since t and x are related by $t = \Phi(x)$, δx corresponds to an increment δt given by

$$\delta t = \Phi(x + \delta x) - \Phi(x).$$

The increment δt in turn produces an incremental change δy in y:

$$\delta y = f(t + \delta t) - f(t).$$

The ratio of increments in y and x is

$$\frac{\delta y}{\delta x} = \frac{f(t + \delta t) - f(t)}{\delta x}.$$

Define a new function $g(t)$ by

$$g(t) = \frac{f(t + \delta t) - f(t)}{\delta t} - f'(t)$$

where $\delta t \neq 0$. Multiply by δt and rearrange the terms, so that

$$f(t + \delta t) - f(t) = \{g(t) + f'(t)\}\delta t.$$

This result has been derived for $\delta t \neq 0$, but it is also true when $\delta t = 0$, provided we assign some value to $g(0)$. It is natural to assign $g(0)$ to be 0,

since $g(t) \to 0$ as $\delta t \to 0$. Then

$$\frac{\delta y}{\delta x} = \frac{\{g(t) + f'(t)\}\delta t}{\delta x},$$

which is valid even if $\delta t = 0$. When $\delta x \to 0$, then $\delta t/\delta x \to \mathrm{d}t/\mathrm{d}x$ and $g(t) \to 0$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(t)\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x}. \tag{1.29}$$

Using formula (1.29) in the three examples above, we have immediately

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2t \cdot 2 = 2(2x + 3)2 = 4(2x + 3),$

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (\cos t)2 = 2\cos 2x,$

(iii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2t(\cos x) = 2\sin x \cos x.$

The derivatives of the circular functions sec x and cosec x can be determined very easily using the result (1.29)

$$y = \sec x = \frac{1}{\cos x} = \frac{1}{t} \quad \text{where} \quad t = \cos x.$$

So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{t^2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x,$$

and similarly for cosec x.

The above rule for the differentiation of a function of a function may be extended to any number of intermediate variables. Suppose

$$y = f(t), \qquad t = \Phi(u), \qquad u = g(w), \qquad w = F(x).$$

Then since

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}u}, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}w}\,\frac{\mathrm{d}w}{\mathrm{d}x},$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}x},$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}w}\,\frac{\mathrm{d}w}{\mathrm{d}x}; \tag{1.30}$$

this rule for differentiation is aptly called the *chain rule*.

Example 21

$$y = \sin^2(4x - 9).$$

Write

$$y = t^2, \qquad t = \sin u, \qquad u = 4x - 9,$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{du}\frac{du}{dx} = 2t(\cos u)4 = 8\sin(4x-9)\cos(4x-9).$$

With practice the intermediate variables can often be omitted and the working may be written, for example, as

$$\frac{d}{dx}\sin^2(4x-9) = 2\sin(4x-9)\cos(4x-9)\frac{d}{dx}(4x-9)$$
$$= 8\sin(4x-9)\cos(4x-9).$$

EXERCISE 1.3

Differentiate the following functions with respect to x, simplifying the answer where possible.

1. $(1-3x)^{\frac{1}{3}}$. **2.** $(1+x^2)^7$. **3.** $\sin^3 x$.

4. $\tan^3(3x-5)$. **5.** $(\sin 3x)^{\frac{1}{3}}$. **6.** $\sin^2 2x \cos 2x$.

7. $(x^2-5x+8)^{-\frac{1}{2}}$. **8.** $(x+1)(x^2+x+1)^{-\frac{1}{2}}$.

9. $\{\sqrt{(x^2+1)}-x\}/\{\sqrt{(x^2+1)}+x\}$. **10.** $\cos^3(4-3x)$.

11. $x\sqrt{\{(a-x)/(a+x)\}}$. **12.** $\left(1-\dfrac{\sin x}{x}\right)^{\frac{1}{2}}$.

§ 4.5. The derivative of an inverse function

Suppose that

$$y = f(x), \tag{1.31}$$

where $f(x)$ is a single valued continuous function which increases (or decreases) steadily in a range $)x_1, x_2($ of x, as in fig. 1.8. Then it is intuitively obvious † that a unique value of x in this range is determined if y is

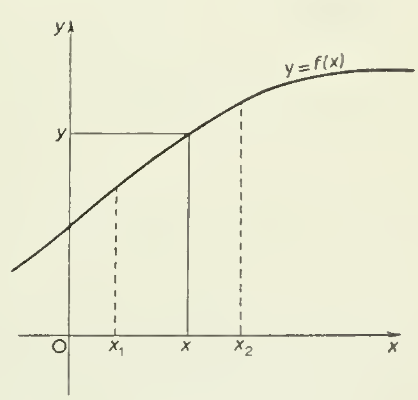

Fig. 1.8

† The analytical proof of this result is not given, but may be found in many mathematical text-books, for example Hardy [1946].

given. So that in this range, x is determined as a single valued function of y, say

$$x = g(y). \tag{1.32}$$

The function $g(y)$ is called the *inverse function* of $f(x)$, and is found by solving eq. (1.31) for x in the range $)x_1, x_2($.

If now δx, δy are corresponding increments of x, y respectively, at any point x in the range, we have

$$\frac{\delta y}{\delta x}\,\frac{\delta x}{\delta y} = 1.$$

Hence proceeding to the limit as $\delta x \to 0$, $\delta y \to 0$ and using eq. (1.12) we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}y} = 1. \tag{1.33}$$

In terms of the functions f and g, this relation is

$$f'(x)\,g'(y) = 1.$$

The values of x and y here are related by (1.32) so the derivative of the inverse function is given by

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'\{g(y)\}}. \tag{1.34}$$

Example 22

$$y = x^2, \qquad x \text{ in the range })0, \infty($$

$f'(x) = 2x$, so that, using eq. (1.34)

$$g'(y) = \frac{1}{2x} = \frac{1}{2y^{\frac{1}{2}}},$$

which is the result obtained if we write $x = y^{\frac{1}{2}}$.

Using the result (1.34) for the derivative of an inverse function, we may note here that we can now find the gradient of the tangent to a curve when its equation is expressed parametrically in the form

$$x = u(t), \qquad y = v(t),$$

t being the parameter. This involves finding the value of $\mathrm{d}y/\mathrm{d}x$ when x and y are expressed in this form. Using the chain rule and the eq. (1.34)

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\Big/\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right).$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v'(t)}{u'(t)},$$

and therefore the value of $\mathrm{d}y/\mathrm{d}x$ is also expressed as a function of the parameter t.

Example 23

The coordinates of a point on a curve are given by

$$x = a(t + \sin t), \qquad y = a(1 - \cos t).$$

Prove that the equation of the tangent at the point where $t = \frac{1}{4}\pi$ is

$$x - y(1 + \sqrt{2}) = \tfrac{1}{4}\pi a.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x} = \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\Big/\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right),$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(1 + \cos t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = a \sin t,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin t}{1 + \cos t}.$$

When $t = \frac{1}{4}\pi$, then

$$x = a\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right), \qquad y = \frac{a}{\sqrt{2}}(\sqrt{2} - 1),$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + \sqrt{2}}.$$

The equation of the tangent is therefore

$$y - \frac{a}{\sqrt{2}}(\sqrt{2} - 1) = \frac{1}{1 + \sqrt{2}}\left(x - a\frac{\pi}{4} - \frac{a}{\sqrt{2}}\right),$$

which gives the required result.

The derivatives of the inverse circular functions are of importance. In order to define x uniquely in terms of y we consider the principal values of these functions as defined in § 1.3, but exclude the end points of the ranges where the functions are neither strictly increasing nor decreasing.

Example 24

If $y = \sin^{-1} x$ where $-1 < x < 1$, then by definition

$$x = \sin y,$$

and is steadily increasing in the range $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$. Since $dx/dy = \cos y$, we deduce from eq. (1.33) that

$$\frac{dy}{dx} = \frac{1}{\cos y}. \tag{1.35}$$

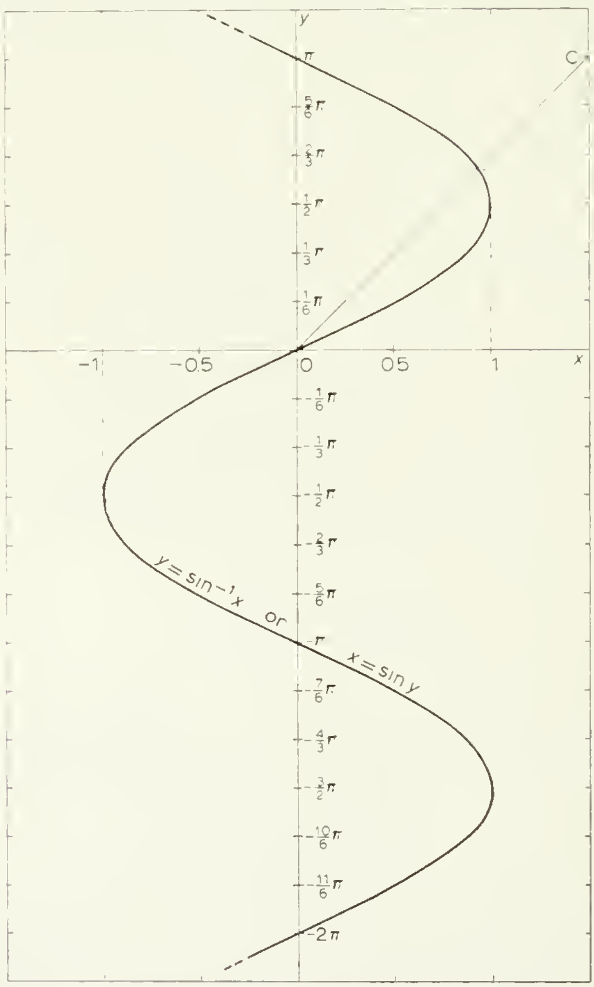

Fig. 1.9

But in the range $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, $\cos y$ is positive and is given by

$$\cos y = (1 - \sin^2 y)^{\frac{1}{2}} = (1 - x^2)^{\frac{1}{2}}.$$

Hence from eq. (1.35)

$$\frac{dy}{dx} = \frac{1}{(1 - x^2)^{\frac{1}{2}}}.$$

This sign is clearly correct for the principal value

$$y = \sin^{-1} x, \qquad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi,$$

as shown in fig. 1.9 †. If the more general solution of $x = \sin y$ were taken then

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{(1 - x^2)}},$$

the ambiguity in sign being obvious from the fact that y is a many valued function of x in the range $-1 < x < 1$. The value of x at any point on the curve $x = \sin y$

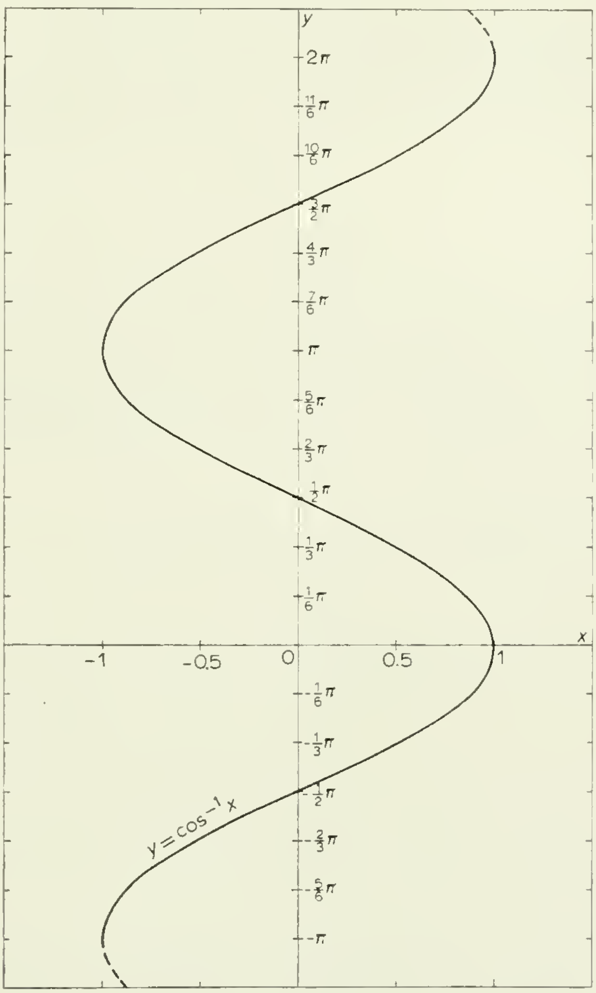

Fig. 1.10

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† If $y = \sin^{-1} x$, then $x = \sin y$ and the graph of this relationship is shown in fig. 1.9. It can be obtained from the graph of $y = \sin x$ by interchanging x and y and this is equivalent to a reflection of the curve $y = \sin x$ in the line OC bisecting the first quadrant as in a mirror. The same will be true of other inverse functions dealt with in Examples 25 and 26.

being the gradient of the tangent there, we see for example in fig. 1.9 that for $-\frac{3}{2}\pi < y < -\frac{1}{2}\pi$, $\mathrm{d}y/\mathrm{d}x$ is negative.

Example 25

Similarly when $y = \cos^{-1} x$, $-1 < x < 1$, with principal value lying in the range $0 < y < \pi$, we have, as shown in fig. 1.10

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(1 - x^2)^{\frac{1}{2}}}.$$

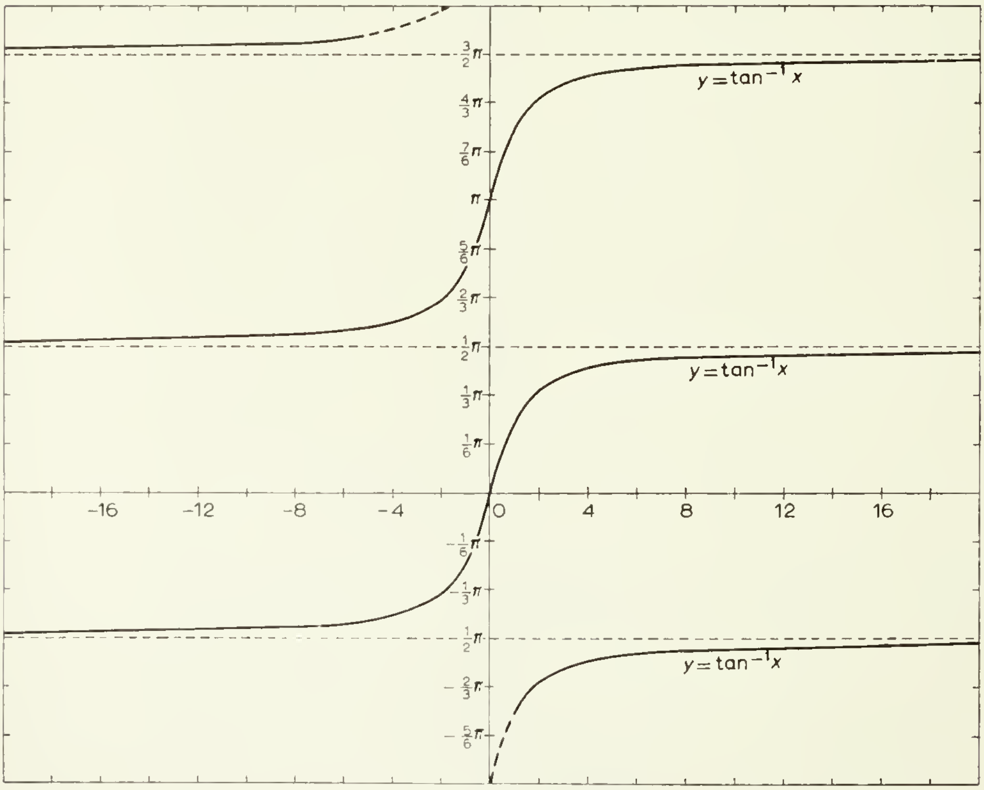

Fig. 1.11

Example 26

When $y = \tan^{-1} x$, there is no restriction on the range of x, and no ambiguity in the sign of the general value of $\mathrm{d}y/\mathrm{d}x$, as shown in fig. 1.11.

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + x^2},$$

for all x.

§ 4.6. Differentiation of implicit functions of two variables

We remarked in § 1.1 that it is not always possible to express a relationship between two variables x and y explicitly in the form

$$y = f(x),$$

but that there may exist an implicit relation

$$f(x, y) = 0 \tag{1.36}$$

between the two variables.

We shall now find the value of $\mathrm{d}y/\mathrm{d}x$ when x and y are implicitly related by (1.36). In this paragraph we shall confine ourselves to simple examples, and give the formal definition of $\mathrm{d}y/\mathrm{d}x$ in terms of $f(x, y)$ later in Ch. 8, eq. (8.34).

Example 27

$$x^2y - x \cos y = 0.$$

This relation defines y as a function of x, so that each term may be regarded as a product of two functions of x and may be differentiated by the rule for products. Differentiating both sides of the relation with respect to x, we get

$$2xy + x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - \cos y - x(-\sin y) \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

or

$$(x^2 + x \sin y) \frac{\mathrm{d}y}{\mathrm{d}x} = \cos y - 2xy.$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos y - 2xy}{x(x + \sin y)}.$$

Example 28

$$x^3 + 2xy^2 + 3y^3 = 1. \tag{1.37}$$

Differentiating both sides of the equation with respect to x

$$3x^2 + 2y^2 + 4xy \frac{\mathrm{d}y}{\mathrm{d}x} + 9y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = - \frac{3x^2 + 2y^2}{y(4x + 9y)}. \tag{1.38}$$

We note that in both these examples the value of $\mathrm{d}y/\mathrm{d}x$ is expressed as a function of both x and y. We must remember that the values of x and y in the expression (1.38) are not independent, but satisfy eq. (1.37). In general we cannot express $\mathrm{d}y/\mathrm{d}x$ in terms of x or y only.

EXERCISE 1.4

Differentiate the functions in Nos. 1–8 with respect to x, simplifying the answer where possible:

1. $\dfrac{a}{x}\tan^{-1}\dfrac{x}{a}$. **2.** $(x\sin^{-1}x)/\sqrt{(1-x^2)}$.

3. $\tan^{-1}\{2x/(1-x^2)\}$. **4.** $\tan^{-1}\{(x^2+1)/x\}$.

5. $\sin^{-1}\{x/(1+x^2)^{\frac{1}{2}}\}$. **6.** $\operatorname{cosec}^{-1}x$.

7. $\cos^{-1}\{\sin\sqrt{(1-x^2)}\}$. **8.** $\tan(\sin^{-1}x)$.

Find the values of $\mathrm{d}y/\mathrm{d}x$ when x and y are defined as in Nos. 9–11:

9. $x = at^2, \quad y = 2at.$

10. $x = a\cos\theta, \quad y = a\sin\theta.$

11. $y = \sec 4t, \quad x = \tan t,$

prove that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{16x(1-x^4)}{(1-6x^2+x^4)^2}$.

12. A curve is given by

$$x = a(2+\cos\theta)(1-\cos\theta), \qquad y = a\sin\theta(1-\cos\theta).$$

Prove that the tangent at the point P whose parameter is θ, is

$$y = \tan\tfrac{1}{2}\theta(x - 2a\sin^2\tfrac{1}{2}\theta).$$

Deduce that the tangents at $\theta = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$ are concurrent.

13. If $y - x = \sin(y+x)$, prove that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cot^2\tfrac{1}{2}(y+x).$$

14. If $x^3 + y^3 - 3axy = 0$, find the value of $\mathrm{d}y/\mathrm{d}x$.

15. If $y^4 + x^4 = 4ax^2y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x^2-2ay)}{x(x^2-3ay)}.$$

§ 5. Derivatives of higher order

If $y=f(x)$ is a differentiable function of x, the derivative $\mathrm{d}y/\mathrm{d}x=f'(x)$ itself will often be a differentiable function of x. The derivative of $\mathrm{d}y/\mathrm{d}x$ may be written as $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$; this is more usually denoted by $\mathrm{d}^2y/\mathrm{d}x^2$,

and is called the *second derivative* of y with respect to x. By repeated differentiation we obtain third, fourth, ..., nth derivatives denoted by $\mathrm{d}^3y/\mathrm{d}x^3$, $\mathrm{d}^4y/\mathrm{d}x^4$, ..., $\mathrm{d}^ny/\mathrm{d}x^n$. We denote the successive derivatives of the function $f(x)$ by $f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x)$ where the n is enclosed in brackets to show that it denotes n primes not the power n. The values of the functions $f'(x), f''(x), \ldots, f^{(n)}(x)$ when x is given a particular value a after differentiation are denoted by $f'(a), f''(a), \ldots, f^{(n)}(a)$.

Example 29

$$y = x^2(1-x)^2,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6x^2 + 4x^3,$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - 12x + 12x^2,$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -12 + 24x,$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 24,$$

$$\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = 0, \qquad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 0, \quad (n \geqslant 5).$$

Example 30

$$y = x^2 \sin x,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x,$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -x^2 \sin x + 4x \cos x + 2 \sin x,$$

and it is obvious that differentiation can be repeated indefinitely.

Example 31

When $f(x) = x^4 - 5x^2 + 2$, find the values of $f'(2)$ and $f'''(1)$.

$$f(x) = x^4 - 5x^2 + 2;$$

$$f'(x) = 4x^3 - 10x, \qquad f'(2) = 32 - 20 = 12;$$

$$f''(x) = 12x^2 - 10;$$

$$f'''(x) = 24x, \qquad f'''(1) = 24.$$

For some special functions a general expression for the nth derivative of the function can be found. Some important examples are:

Example 32

$$f(x) = x^{-1}; \qquad f'(x) = -x^{-2}; \qquad f''(x) = 2x^{-3};$$
$$f'''(x) = -2\cdot 3x^{-4}; \qquad f''''(x) = 2\cdot 3\cdot 4x^{-5}. \tag{1.39}$$

Thus it would appear that

$$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}. \tag{1.40}$$

We can *prove* this last result by the *method of induction*, which is as follows. Assume the result (1.40) to be true for one value of n, say $n=m$, so that

$$f^{(m)}(x) = (-1)^m\, m!\, x^{-(m+1)}. \tag{1.41}$$

Differentiating eq. (1.41) with respect to x, we get

$$f^{(m+1)}(x) = (-1)^{(m+1)}(m+1)!\, x^{-(m+2)}. \tag{1.42}$$

Eq. (1.42) is precisely the same as eq. (1.41) with $m+1$ in place of m, so that we have shown that if (1.40) is true when $n=m$, it is also true when $n=m+1$. But results (1.39) show that it is true when $n=3$, 4 and hence it is true when $n=5$, and so on. Thus it is true for all integral values of n.

This method of induction is used very frequently, as we shall see in the succeeding examples and also the next paragraph.

Example 33

$$f(x) = \sin x,$$
$$f'(x) = \cos x = \sin(x + \tfrac{1}{2}\pi),$$
$$f''(x) = \cos(x + \tfrac{1}{2}\pi) = \sin(x + \pi).$$

Thus generally, by the method of induction,

$$f^{(n)}(x) = \sin(x + \tfrac{1}{2}n\pi). \tag{1.43}$$

Example 34

$$f(x) = \cos x.$$

As in Example 33, $f^{(n)}(x) = \cos(x + \frac{1}{2}n\pi)$.

Example 35

$$f(x) = a_0 + a_1x + \ldots + a_mx^m,$$
$$f'(x) = a_1 + a_2x + \ldots ma_mx^{m-1},$$
$$f''(x) = a_2 + a_3x + \ldots m(m-1)a_mx^{m-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f^{(m)}(x) = m!\, a_m,$$

and therefore

$$f^{(n)}(x) = 0, \quad \text{for} \quad n > m.$$

§ 5.1. Successive derivatives of a product. Leibniz theorem

In § 4.2 we have shown that if

$$u = f(x), \qquad v = g(x), \quad \text{and} \quad y = uv = f(x)g(x),$$

then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx},$$

which in terms of the functions $f(x)$ and $g(x)$ may be written as

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x).$$

For convenience of writing in this result, and what follows, we will write this as

$$\frac{dy}{dx} = f'g + fg', \tag{1.44}$$

where primes are understood to mean differentiation with respect to x and all the functions on the right hand side are functions of x. This means that in general we are using $f^{(n)}$ instead of $f^{(n)}(x)$ and $g^{(n)}$ instead of $g^{(n)}(x)$.

By successive differentiation of eq. (1.44) we find

$$\frac{d^2y}{dx^2} = f''g + 2f'g' + fg'', \tag{1.45}$$

$$\frac{d^3y}{dx^3} = f'''g + 3f''g' + 3f'g'' + fg''', \tag{1.46}$$

and in these two results we note that the coefficients in the expressions on the right are the binomial coefficients in $(f+g)^2$ for d^2y/dx^2 and the binomial coefficients in $(f+g)^3$ for d^3y/dx^3. This suggests a general formula for d^ny/dx^n, the nth derivative of the product uv:

$$\frac{d^ny}{dx^n} = f^{(n)}g + \binom{n}{1}f^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \dots$$
$$+ \binom{n}{r}f^{(n-r)}g^{(r)} + \dots + \binom{n}{1}f'\,g^{(n-1)} + f\,g^{(n)}. \tag{1.47}$$

This formula was established by Leibniz. To prove it we again use the method of induction and this means that we assume eq. (1.47) is true for

$n=m$, so that

$$\frac{d^m y}{dx^m}=f^{(m)}g+\binom{m}{1}f^{m-1}g'+\ldots+\binom{m}{r}f^{(m-r)}g^{(r)}+\ldots+f\,g^{(m)}. \quad (1.48)$$

On differentiating this result with respect to x, we get

$$\frac{d^{m+1}y}{dx^{m+1}}=\{f^{(m+1)}g+f^{(m)}g'\}+\binom{m}{1}\{f^{(m)}g'+f^{(m-1)}g''\}+\ldots$$

$$+\binom{m}{r}\{f^{(m-r+1)}g^{(r)}+f^{(m-r)}g^{(r+1)}\}+\ldots+\{f'\,g^{(m)}+f\,g^{(m+1)}\}$$

$$=f^{m+1}g+\left\{1+\binom{m}{1}\right\}f^{(m)}g'+\ldots$$

$$+\left\{\binom{m}{r-1}+\binom{m}{r}\right\}f^{(m-r+1)}g^{(r)}+\ldots f\,g^{(m+1)}.$$

But it is easily verified that

$$\binom{m}{r-1}+\binom{m}{r}=\binom{m+1}{r},$$

so that we get

$$\frac{d^{m+1}y}{dx^{m+1}}=f^{(m+1)}g+\binom{m+1}{1}f^{(m)}g'+\ldots+\binom{m+1}{r}f^{(m+1-r)}g^{(r)}+\ldots f\,g^{(m+1)}. \quad (1.49)$$

Eq. (1.49) is precisely the same as eq. (1.48) with $(m+1)$ in place of m. But by eq. (1.46) the result is true when $n=3$ and so by the method of induction it is true for all integral values of n.

Example 36

When $y=x^3 \sin x$, find d^3y/dx^3.

Let $f(x)=x^3$, $g(x)=\sin x$, so that

$$f'(x)=3x^2, \qquad f''(x)=6x, \qquad f'''(x)=6,$$

and $f^{(n)}(x)=0$, $n>3$. Also using eq. (1.43)

$$g^{(n)}(x)=\sin(x+\tfrac{1}{2}n\pi). \quad (1.50)$$

Thus from eq. (1.46)

$$\frac{d^3y}{dx^3}=6\sin x+3(6x)\sin(x+\tfrac{1}{2}\pi)+3(3x^2)\sin(x+\pi)+x^3\sin(x+\tfrac{3}{2}\pi)$$

$$=(6-9x^2)\sin x+(18x-x^3)\cos x.$$

Using eq. (1.43) and eq. (1.47) in the reverse order we can evaluate the nth derivative of y:

$$\frac{d^n y}{dx^n} = x^3 \sin(x + \tfrac{1}{2}n\pi) + n(3x^2)\sin\{x + \tfrac{1}{2}(n-1)\pi\}$$

$$+ 3n(n-1)x \sin\{x + \tfrac{1}{2}(n-2)\pi\} + n(n-1)(n-2)\sin\{x + \tfrac{1}{2}(n-3)\pi\}$$

$$= \{x^3 - 3n(n-1)x\} \sin(x + \tfrac{1}{2}n\pi) - \{3nx^2 - n(n-1)(n-2)\} \cos(x + \tfrac{1}{2}n\pi),$$

the remaining terms being zero.

EXERCISE 1.5

Find the nth derivatives of the functions in Nos. 1–7:

1. $\sin 2x$. **2.** $\cos mx$. **3.** $x/(x^2 - a^2)$.

4. $(ax + b)^m$. **5.** x^{2n}. **6.** $x^2 \sin x$.

7. $x^3 \cos 2x$.

8. If $y = \tan^{-1} x$ prove that

$$(1 + x^2)\frac{dy}{dx} = 1.$$

By differentiating this equation with respect to x, n times, show that

$$(1 + x^2)\frac{d^{n+1}y}{dx^{n+1}} + 2nx\frac{d^n y}{dx^n} + n(n-1)\frac{d^{n-1}y}{dx^{n-1}} = 0.$$

9. If $y = (a^2 - x^2)^{\frac{1}{2}} \sin^{-1}(x/a)$ prove that

$$(a^2 - x^2)^{\frac{1}{2}}\frac{dy}{dx} = (a^2 - x^2)^{\frac{1}{2}} - x\sin^{-1}(x/a),$$

and hence that

$$(a^2 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y + 2x = 0.$$

10. If $y = \{x + \sqrt{(x^2 + a^2)}\}^m$ show that

$$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0,$$

and deduce by Leibniz' theorem that

$$(x^2 + a^2)\frac{d^{n+2}y}{dx^{n+2}} + (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2 - m^2)\frac{d^n y}{dx^n} = 0.$$

11. When

$$x = a\cos t(1+\cos t), \quad y = a\sin t(1+\cos t),$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\cot\tfrac{3}{2}t,$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{3}{4a}\sec\tfrac{1}{2}t\,\mathrm{cosec}^3\,\tfrac{3}{2}t.$$

12. If $x=t^2/(1+t)$, $y=t^3/(1+t)$, find $\mathrm{d}y/\mathrm{d}x$, $\mathrm{d}^2y/\mathrm{d}x^2$ as functions of t.

§ 6. Differentials

In § 4 of this chapter we have defined the derivative of the function $y=f(x)$ as the limiting value of the ratio $\delta y/\delta x$ as the increment δx tends to zero. That is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = f'(x). \tag{1.51}$$

In this definition of the derivative $\mathrm{d}y/\mathrm{d}x$, the symbols $\mathrm{d}y$ and $\mathrm{d}x$ had no meaning independent of each other and $\mathrm{d}y/\mathrm{d}x$ was not regarded as a ratio of two quantities $\mathrm{d}y$ and $\mathrm{d}x$. We can however regard $\mathrm{d}y$ and $\mathrm{d}x$ as distinct quantities known as *differentials*, defined in the following way. Since in eq. (1.51) the $\lim_{\delta x\to 0}(\delta y/\delta x)$ has a finite value $f'(x)$ we can write

$$\frac{\delta y}{\delta x} = f'(x) + \varepsilon,$$

where $\varepsilon\to 0$ as $\delta x\to 0$. In this equation δy and δx represent the small increments of y and x respectively; thus we can multiply through by δx to give

$$\delta y = f'(x)\,\delta x + \varepsilon\,\delta x. \tag{1.52}$$

We now *define* the *differential* $\mathrm{d}y$ *of* y to be the first term on the right in this equation. Thus

$$\mathrm{d}y = f'(x)\,\delta x, \tag{1.53}$$

so that the *differential* of y is the product of its derivative $\mathrm{d}y/\mathrm{d}x=f'(x)$ and an arbitrary increment δx (*not necessarily small*) of the independent variable x.

In particular when $f(x)=x$, so that $y=x$ and $f'(x)=1$, we have from eq. (1.53)

$$\mathrm{d}x = f'(x)\,\delta x = \delta x,$$

and hence the *differential* of the *independent* variable is the same as the increment of that variable. Substituting for δx in eq. (1.51) we get

$$\mathrm{d}y = f'(x)\,\mathrm{d}x,$$

where $\mathrm{d}y$, $\mathrm{d}x$ are the differentials of y and x as defined above. Dividing through by $\mathrm{d}x$ we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x), \tag{1.54}$$

and we thus see that the ratio of the differentials is the same as the derivative of the function.

The symbol $\mathrm{d}y/\mathrm{d}x$ thus acquires a double meaning: either it is $\lim_{\delta x\to 0} (\delta y/\delta x)$ or it is the ratio of the differentials $\mathrm{d}y$ and $\mathrm{d}x$. There is no ambiguity in eq. (1.54) since it is true whichever meaning is chosen. The usefulness of the second interpretation is that we can now multiply or divide by quantities like $\mathrm{d}x$ or $\mathrm{d}y$ whenever they occur.

§ 6.1. Geometrical interpretation of a differential

Consider the graph of $y=f(x)$ as in fig. 1.12. If P and Q are two neighbouring points on the curve whose abscissae are x and $x+\delta x$ respectively, then the ordinate at P is $\mathrm{PM}=f(x)$ and the ordinate at Q is $\mathrm{QN}=f(x+\delta x)$. Thus

$$\delta y = f(x+\delta x) - f(x) = \mathrm{QN} - \mathrm{PM}$$
$$= \mathrm{QN} - \mathrm{RN} = \mathrm{QR}. \tag{1.55}$$

If PT is the tangent at P, we know that $f'(x)$ is the gradient of this tangent, that is

$$f'(x) = \mathrm{TR}/\mathrm{PR} = \mathrm{TR}/\mathrm{MN}.$$

Thus by the definition (1.53),

$$\mathrm{d}y = f'(x)\,\delta x = f'(x)\,\mathrm{MN} = \mathrm{TR}, \tag{1.56}$$

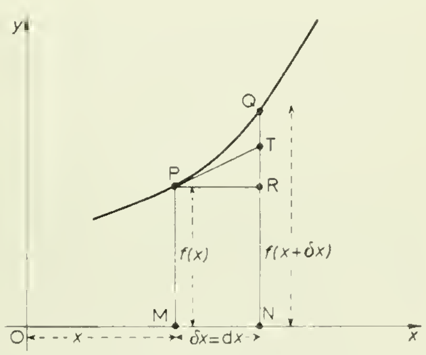

Fig. 1.12

and comparing equations (1.55) and (1.56) we see that δy is the increment of y as we move along the curve, while the differential $\mathrm{d}y$ is the increment of y as we move along the tangent. This geometrical illustration enables us to see why we should expect the result

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x},$$

where the right hand side is the ratio of the differentials dy and dx. As Q approaches P on the curve, the chord PQ and the curve itself approaches its tangent PT. That is, as $\delta x = \mathrm{d}x \to 0$, the value of $\mathrm{QT} \to 0$. But $\mathrm{QT} = \mathrm{QR} - \mathrm{TR} = \delta y - \mathrm{d}y$, and so as $\delta x = \mathrm{d}x \to 0$, $\delta y - \mathrm{d}y \to 0$ and hence the result.

Example 37

$$\mathrm{d}(x^2) = 2x\,\mathrm{d}x.$$

Example 38

$$\mathrm{d}(\sin x) = \cos x\,\mathrm{d}x.$$

Example 39

If $u = f(x)$, $v = g(x)$, then

$$\mathrm{d}(uv) = \frac{\mathrm{d}(uv)}{\mathrm{d}x}\,\mathrm{d}x = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\,v + u\,\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = v\,\mathrm{d}u + u\,\mathrm{d}v,$$

and similarly

$$\mathrm{d}\left(\frac{u}{v}\right) = \frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}.$$

The usefulness of this differential notation will be illustrated in later chapters.

§ 7. Functions of more than one variable. Partial differentiation

So far we have dealt only with derivatives of functions of a single variable. We now wish to discuss the variation of a function which is dependent on more than one variable. Such functions are, of course, very common. For example, (i) the area Δ of a triangle can be expressed in the form $\Delta = \frac{1}{2}bc \sin A$ and is therefore seen to depend on the three quantities b, c and A, Δ is said to be a function of these three quantities; (ii) the pressure p of a dilute gas is related to its volume V and temperature T by $p = RT/V$, where R is constant; the pressure p is therefore a function of the temperature and volume.

In general, if a variable quantity z depends on the value of two other variable quantities x and y, z will be equal to some function of x and y:

$$z = f(x, y). \tag{1.57}$$

If x and y are not connected in any way and can vary independently of each other, we call them the independent variables; z is said to be the dependent variable.

If z depends on x and y through eq. (1.57) and x is changed by the increment δx, while y is kept constant, then z will change by an increment

$$(\delta z)_x = f(x + \delta x, y) - f(x, y).$$

The *partial derivative* of z with respect to x is defined as

$$\lim_{\delta x \to 0} \frac{(\delta z)_x}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}. \tag{1.58}$$

The limit on the right of this equation is the *partial derivative of the function* $f(x, y)$ with respect to x. Clearly the partial derivative is calculated by the same rules as for ordinary derivatives, provided that y is treated as a constant throughout. The partial derivatives of z and $f(x, y)$ with respect to x are denoted by $\partial z/\partial x$ and f_x.

Similarly the partial derivative $\partial z/\partial y$ with respect to y is defined by giving y an increment δy, keeping x fixed. It is equal to f_y formed by ordinary differentiation with respect to y, x being treated as a constant.

Example 40

$$z = x \sin(x - y),$$

$$\frac{\partial z}{\partial x} = \sin(x - y) + x \cos(x - y),$$

$$\frac{\partial z}{\partial v} = -x \cos(x - y).$$

When z depends upon more than two variables we can similarly define partial differentiation of z with respect to any one of the variables, all the other variables being simply treated as constants in the differentiation.

Example 41

$$z = (u^2 + v^2 + w^2)^{\frac{1}{2}},$$

where u, v, w are three independent variables.

$$\frac{\partial z}{\partial u} = u(u^2 + v^2 + w^2)^{-\frac{1}{2}},$$

$$\frac{\partial z}{\partial v} = v(u^2 + v^2 + w^2)^{-\frac{1}{2}},$$

$$\frac{\partial z}{\partial w} = w(u^2 + v^2 + w^2)^{-\frac{1}{2}}.$$

Example 42

If $z=f(y/x)$ where f denotes an arbitrary function, prove that $x(\partial z/\partial x)+y(\partial z/\partial y)=0$.

Here we write

$$z = f(t), \quad t = y/x,$$

and use the ordinary rules for differentiation of a function of a function, but remembering that the derivatives with respect to x and y are partial derivatives, so

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}t}\,\frac{\partial t}{\partial x} = f'(t)\,\frac{\partial t}{\partial x} = -\frac{y}{x^2}\,f'(t),$$

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}t}\,\frac{\partial t}{\partial y} = f'(t)\,\frac{\partial t}{\partial y} = \frac{1}{x}\,f'(t),$$

Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(-\frac{y}{x} + \frac{y}{x}\right)f'(t) = 0.$$

Example 43

Let (x, y) be the rectangular coordinates of a point in a plane, and (r, θ) the polar coordinates, so that

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1.59}$$

These equations define both x and y as functions of r and θ, and since r, θ are independent variables we have

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta, & \frac{\partial y}{\partial r} &= \sin\theta; \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta, & \frac{\partial y}{\partial \theta} &= r\cos\theta. \end{aligned} \tag{1.60}$$

But the equations (1.59) can be solved for (r, θ) in terms of (x, y) to give

$$r = \sqrt{(x^2+y^2)}, \qquad \theta = \tan^{-1} y/x, \tag{1.61}$$

and since x, y can be varied independently in these equations we have

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{(x^2+y^2)}} = \frac{x}{r} = \cos\theta, \\ \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{(x^2+y^2)}} = \frac{y}{r} = \sin\theta, \end{aligned} \tag{1.62}$$

whilst

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r},$$

and similarly

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

We see immediately in Example 43 that

$$\frac{\partial r}{\partial x} \neq \left(\frac{\partial x}{\partial r}\right)^{-1}, \qquad \frac{\partial \theta}{\partial y} \neq \left(\frac{\partial y}{\partial \theta}\right)^{-1},$$

and so on for the other derivatives. The reason for these results can be illustrated geometrically. Consider fig. 1.13 where P is any point in the plane having rectangular coordinates (x, y) and polar coordinates (r, θ). By definition, using eq. (1.58)

$$\frac{\partial r}{\partial x} = \lim_{\delta x \to 0} \frac{(\delta r)_x}{\delta x},$$

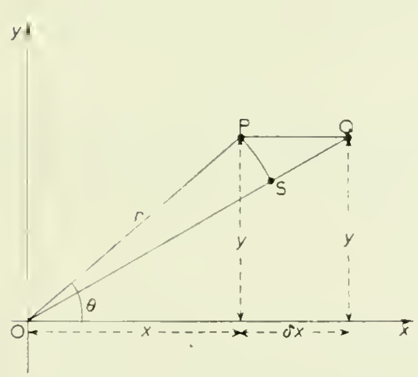

Fig. 1.13

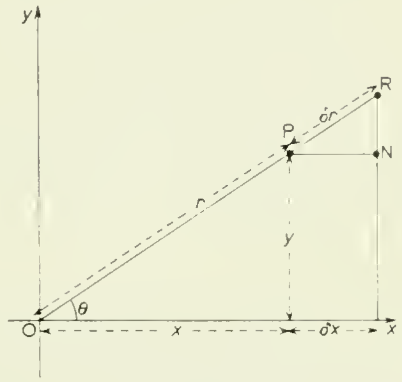

Fig. 1.14

where $(\delta r)_x$ is the increment of r corresponding to an increment δx of x, keeping y constant. Thus in fig. 1.13 if PQ is drawn parallel to the axis of x, thus keeping y constant, and is such that PQ$=\delta x$, the new value of r is OQ. So if OP$=$OS$=r$ we have

$$(\delta r)_x = \text{OQ} - \text{OS} = \text{SQ}.$$

To emphasize the fact that y is kept constant the notation $(\partial r/\partial x)_y$ is often used for this derivative. Thus

$$\left(\frac{\partial r}{\partial x}\right)_y = \lim_{\text{Q}\to\text{P}} \frac{\text{SQ}}{\text{PQ}} = \cos \theta$$

Now consider fig. 1.14 where again P has rectangular coordinates (x, y) and polar coordinates (r, θ). By definition, using eq. (1.58)

$$\frac{\partial x}{\partial r} = \lim_{\delta r \to 0} \frac{(\delta x)_r}{\delta r},$$

where $(\delta x)_r$ is the increment of x corresponding to the increment δr of r,

keeping θ constant. If OP is produced to R such that PR$=\delta r$, this is an increment of r, keeping θ constant. The corresponding increment of x is PN. Again emphasizing the fact that θ is kept constant, we write

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \lim_{\mathrm{R}\to\mathrm{P}} \frac{\mathrm{PN}}{\mathrm{PR}} = \cos\theta.$$

Thus we see why the partial derivatives $(\partial r/\partial x)_y$ and $(\partial x/\partial r)_\theta$ are *not* reciprocals of each other.

In general, if two variables x and y are given as functions of two other variables u and v, and the inverses of these functions can be found, so that u and v are expressed as functions of x and y, then $\partial u/\partial x$, more accurately written as $(\partial u/\partial x)_y$, is not the reciprocal of $\partial x/\partial u$, more accurately written as $(\partial x/\partial u)_v$.

We note therefore, that apart from the above restriction on reciprocals, there are no new techniques involved in finding partial derivatives, and the rules for differentiation of products, quotients etc., will apply to partial derivatives.

§ 7.1. Second partial derivatives

If $z=f(x, y)$, then in general the first partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ are themselves functions of x and y and can again be differentiated partially with respect to either of the variables. We thus obtain four second partial derivatives of z; common notation for these derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

the order of the symbols ∂x, ∂y in the 'denominator' from right to left indicates the order of the differentiation.

If $z=f(x, y)$ we write these in terms of $f(x, y)$ as f_{xx}, f_{xy}, f_{yx}, f_{yy}, the order of the suffixes from right to left indicating the order of the differentiation.

Example 44

$$z = x^2 - xy + y/x^2.$$

The partial derivatives with respect to x and y are

$$\frac{\partial z}{\partial x} = 2x - y - \frac{2y}{x^3}, \qquad \frac{\partial z}{\partial y} = -x + \frac{1}{x^2}.$$

Differentiating again

$$\frac{\partial^2 z}{\partial x^2} = 2 + \frac{6y}{x^4}, \qquad \frac{\partial^2 z}{\partial x \partial y} = -1 - \frac{2}{x^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = -1 - \frac{2}{x^3}, \qquad \frac{\partial^2 z}{\partial y^2} = 0.$$

Example 45

$$r = \sqrt{(x^2 + y^2)}.$$

We have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2)}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{(x^2 + y^2)}},$$

and

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{(x^2 + y^2)} - x^2/\sqrt{(x^2 + y^2)}}{x^2 + y^2} = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}},$$

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x} = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}},$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}.$$

We notice that in these two examples

$$\partial^2 z/\partial x \partial y = \partial^2 z/\partial y \partial x \quad \text{and} \quad \partial^2 r/\partial x \partial y = \partial^2 r/\partial y \partial x.$$

It will be proved later (Ch. 8 § 2) that this result is true for any function which satisfies certain simple conditions.

Example 46

If $z = y^3 f(x/y)$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 6z.$$

Here we regard z as the product of y^3 and $f(t)$ where $t = x/y$. We have

$$\frac{\partial t}{\partial x} = \frac{1}{y}, \qquad \frac{\partial t}{\partial y} = -\frac{x}{y^2}.$$

Treating y as a constant,

$$\frac{\partial z}{\partial x} = y^3 f'(t) \frac{\partial t}{\partial x} = y^3 f'(t) \frac{1}{y} = y^2 f'(t).$$

With x constant,

$$\frac{\partial z}{\partial y} = 3y^2 f(t) + y^3 f'(t) \frac{\partial t}{\partial y} = 3y^2 f(t) - xy f'(t).$$

Also

$$\frac{\partial^2 z}{\partial x^2} = y^2 f''(t)\frac{\partial t}{\partial x} = yf''(t),$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 2yf'(t) + y^2 f''(t)\frac{\partial t}{\partial y} = 2yf'(t) - xf''(t),$$

and

$$\frac{\partial^2 z}{\partial y^2} = 6yf(t) + 3y^2 f'(t)\frac{\partial t}{\partial y} - xf'(t) - xyf''(t)\frac{\partial t}{\partial y} = 6yf(t) - 4xf'(t) + \frac{x^2}{y} f''(t).$$

Therefore

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

$$= x^2 yf''(t) + 4xy^2 f'(t) - 2x^2 yf''(t) + 6y^3 f(t) - 4xy^2 f'(t) + x^2 yf''(t) = 6y^3 f(t) = 6z.$$

EXERCISE 1.6

Differentiate each of the functions in Nos. 1–5 partially with respect to x and partially with respect to y.

1. $x^2 \tan^{-1}(y/x)$. **2.** $xy/(x+y)$. **3.** $(y-x)/(x^2+y^2)$.

4. $xy(ax^2+by^2)/(x+y)$. **5.** $(x^2+y^2)\tan^{-1}(x/y)$.

6. If $z = \sin^{-1}\{(x+y)/(x-y)\}$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

7. If $z = a\tan^{-1}(y/x)$, show that

$$\text{(i)}\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0, \qquad \text{(ii)}\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right).$$

8. If $u = \sin^{-1}\dfrac{\sqrt{(x^2+y^2)}}{x+y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

9. Given that $z = x^3 + 2x^2y - 3xy^2 - \frac{2}{3}y^3$, find $\partial z/\partial x$, $\partial z/\partial y$ and verify that

$$\text{(i)}\quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z, \qquad \text{(ii)}\quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

10. If $u = \sin^{-1}\{(x^2 + y^2)/z\}$ prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \tan u.$$

11. If $r = \sqrt{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}$ and $r \neq 0$, then

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = 0.$$

12. Given $z = \sin(x - y)/\cos(x + y)$, find the first and second partial derivatives with respect to x and y. Prove that these partial derivatives satisfy the following equations:

(i) $$\cos 2x\frac{\partial z}{\partial x} + \cos 2y\frac{\partial z}{\partial y} = 0,$$

(ii) $$\cos 2x\frac{\partial^2 z}{\partial x^2} + \cos 2y\frac{\partial^2 z}{\partial y^2} = 0,$$

(iii) $$\frac{\partial^2 z}{\partial x\partial y} = \frac{2z}{\cos^2(x + y)}.$$

CHAPTER 2

INTEGRATION. SIMPLE METHODS

§ 1. Integration as the inverse of differentiation

In Ch. 1 we defined and studied the process of differentiation of a function of one or more variables. Integration of a function of one variable can be looked upon as the inverse process to differentiation; this means that given the value of the derivative $\mathrm{d}y/\mathrm{d}x$ as a function of x, we wish to find the value of y itself. In other words we want to solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x), \tag{2.1}$$

for y, where $f(x)$ is a given function of x. Suppose that we can find a function $F(x)$ whose derivative $F'(x)$ is equal to $f(x)$. The eq. (2.1) then becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) = F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x).$$

Hence, by simple differentiation, we see that

$$y = F(x) + c, \tag{2.2}$$

where c is any constant, satisfies eq. (2.1).

The constant c in the solution (2.2) is so far not determined. If in addition to eq. (2.1) for y, certain other conditions on the relation between the variables y and x are laid down, for example, $y=0$ when $x=a$, then the constant c in eq. (2.2) must satisfy

$$0 = F(a) + c,$$

so that $c=-F(a)$. Thus the solution (2.2) which satisfies this extra condition is

$$y = F(x) - F(a).$$

However, as far as eq. (2.1) alone is concerned, c is any indeterminate

constant, and is said to be an *arbitrary constant.* When, in addition, y is given for one value of x, c is thereby determined.

The process of finding a function $F(x)$ whose derivative is $f(x)$ is called the process of integration with respect to x. We call $F(x)$ an integral of $f(x)$ and we use the notation

$$F(x) = \int f(x)\,\mathrm{d}x,$$

TABLE 2.1

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$	$\displaystyle\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}; \quad n \neq -1.$ and in particular for $n = 0$ $\displaystyle\int \mathrm{d}x = x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x.$	$\displaystyle\int \cos x\,\mathrm{d}x = \sin x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x.$	$\displaystyle\int \sin x\,\mathrm{d}x = -\cos x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x.$	$\displaystyle\int \sec^2 x\,\mathrm{d}x = \tan x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x.$	$\displaystyle\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x.$	$\displaystyle\int \sec x \tan x\,\mathrm{d}x = \sec x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$	$\displaystyle\int \operatorname{cosec} x \cot x\,\mathrm{d}x = -\operatorname{cosec} x.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{(a^2 - x^2)}}, \quad \lvert x\rvert < a.$	$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}, \quad \lvert x\rvert < a.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\dfrac{x}{a}\right) = -\dfrac{1}{\sqrt{(a^2 - x^2)}}, \quad \lvert x\rvert < a.$	$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)}} = -\cos^{-1}\frac{x}{a}, \quad \lvert x\rvert < a.$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 + x^2}.$	$\displaystyle\int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

the particular symbolism $\int \ldots \mathrm{d}x$ being explained later in § 2.2. The function $f(x)$ is called the *integrand* of the integral.

The above reasoning shows immediately that the function $F(x)$ is not unique, but that any two such functions can only differ by a constant. Thus from any one integral $F(x)$ we can obtain all others in the form $F(x)+c$, where c is an arbitrary constant.

The determination of $F(x)$ for any given function $f(x)$ is in general much more difficult than the determination of derivatives, but there are some simple rules which can be given. In the first place we can write down quite a number of integrals which are known as *standard integrals*, in that they are all obvious from the corresponding results for differentiation.

Table 2.1 gives a preliminary list of standard integrals and in Ch. 5 a further list will be given. In the left hand column we give the derivatives of various functions and in the right hand column the corresponding integrals. It may be noted that the two different results $\sin^{-1}(x/a)$ and $-\cos^{-1}(x/a)$ for the integral $\int (a^2-x^2)^{-\frac{1}{2}}\,\mathrm{d}x$ do differ by a constant only, since

$$\sin^{-1}(x/a) - \{-\cos^{-1}(x/a)\} = \tfrac{1}{2}\pi.$$

§ 1.1. Simple generalisations of standard integrals

It is obvious by differentiation that the above standard forms for integrals may be extended for use in slightly more difficult cases. For example:

(i) If A is any constant, and $f(x)$ any function of x, then

$$\int A\, f(x)\,\mathrm{d}x = A \int f(x)\,\mathrm{d}x.$$

(ii) If $u(x)$, $v(x)$, $w(x)$, ... are any number of functions of x

$$\int \{u(x) + v(x) + w(x) + \ldots\}\mathrm{d}x = \int u(x)\,\mathrm{d}x + \int v(x)\,\mathrm{d}x + \int w(x)\,\mathrm{d}x + \ldots.$$

(iii) The addition of a constant, say b, to x, makes no essential difference to the form of the result, x being replaced by $x+b$ throughout. Therefore

$$\int (x+b)^n\,\mathrm{d}x = \frac{1}{n+1}(x+b)^{n+1}, \qquad n \neq -1,$$

$$\int \cos(x+b)\,\mathrm{d}x = \sin(x+b),$$

and so on for all the standard forms.

(iv) If x is multiplied by a constant k in $f(x)$, then the integral is of the same form as in the table with kx replacing x, except that the value of the integral is divided by this factor k. Thus

$$\int \sin kx \, \mathrm{d}x = -\frac{1}{k} \cos kx,$$

$$\int \frac{\mathrm{d}x}{a^2 + (kx)^2} = \frac{1}{ka} \tan^{-1} \frac{kx}{a}.$$

§ 1.2. Change of variable

One of the most important methods of integration is that known as the method of changing the variable. Suppose we wish to find the integral

$$F(x) = \int f(x) \, \mathrm{d}x. \tag{2.3}$$

Let us write $x = \Phi(t)$ where $\Phi(t)$ is a single valued function of t. The problem is to choose $\Phi(t)$ so that the integral (2.3) reduces to an integral of a simple function of t with respect to t. Using this change of variable, write

$$y = F(x) = F\{\Phi(t)\} \equiv G(t),$$

say. Then from Ch. 1 § 4.4 we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}t} = G'(t).$$

But by definition of the integral (2.3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F'(x) = f(x),$$

so that

$$G'(t) = f(x) \frac{\mathrm{d}x}{\mathrm{d}t} = f\{\Phi(t)\} \, \Phi'(t).$$

This means that

$$G(t) = \int f\{\Phi(t)\} \, \Phi'(t) \, \mathrm{d}t. \tag{2.4}$$

Thus the original integral $F(x)$ is expressed as the function $G(t)$ in the form of an integral with respect to t which may be easier to evaluate. The result $G(t)$ may be transformed into a function of x by using the inverse function of $\Phi(t)$ defining t as a function of x.

By means of such a change of variable, the generalisations (iii) and (iv) of § 1.1 combined may be proved as follows.

Example 1

To evaluate

$$\int (kx + b)^n \, \mathrm{d}x.$$

Write $kx+b=t$, so that $x=(t-b)/k$. Then $\mathrm{d}x/\mathrm{d}t = 1/k$, and in terms of t the integral becomes, using eq. (2.4)

$$\int t^n \frac{1}{k} \, \mathrm{d}t = \frac{1}{k} \int t^n \, \mathrm{d}t = \frac{1}{k} \; \frac{t^{n+1}}{n+1}.$$

Expressing the result in terms of x, we have

$$\int (kx + b)^n \, \mathrm{d}x = \frac{1}{k} \; \frac{(kx + b)^n}{n + 1}.$$

Example 2

By the same change of variable as in Example 1, we can show that

$$\int \cos(kx + b) \, \mathrm{d}x = \frac{1}{k} \sin(kx + b).$$

These results however are such obvious generalisations of the standard integrals that such changes of variable are unnecessary.

Before we consider other changes of variable or substitutions as they are often called, we may note the converse of the given result. Whenever an integral is seen to have the form

$$\int f(t) \frac{\mathrm{d}t}{\mathrm{d}x} \, \mathrm{d}x,$$

we can evaluate it as

$$\int f(t) \, \mathrm{d}t,$$

since we are writing

$$t = \Phi(x), \qquad \frac{\mathrm{d}t}{\mathrm{d}x} = \Phi'(x)$$

in the converse of eqs. (2.3) and (2.4)

Example 3

$$\int f(x^2) x \, \mathrm{d}x = \frac{1}{2} \int f(x^2) \frac{\mathrm{d}(x^2)}{\mathrm{d}x} \, \mathrm{d}x,$$

which may be evaluated as

$$\int f(t) \, \mathrm{d}t,$$

where $t = x^2$.

Example 4

To evaluate

$$\int \frac{x}{1+x^4}\,\mathrm{d}x = \frac{1}{2}\int \frac{(\mathrm{d}x^2/\mathrm{d}x)\,\mathrm{d}x}{1+(x^2)^2},$$

we write $t=x^2$, and evaluate it as

$$\frac{1}{2}\int \frac{\mathrm{d}t}{1+t^2} = \tfrac{1}{2}\tan^{-1} t = \tfrac{1}{2}\tan^{-1} x^2.$$

Example 5

To evaluate

$$\int \frac{x}{\sqrt{(5-x^2)}}\,\mathrm{d}x,$$

we write $t=5-x^2$, so that $\mathrm{d}t=-2x\,\mathrm{d}x$ or $x\,\mathrm{d}x=-\frac{1}{2}\mathrm{d}t$, the integral becomes

$$-\frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = -\sqrt{t} = -\sqrt{(5-x^2)}.$$

EXERCISE 2.1

Using the standard integrals given in § 1 and generalisations of them, integrate with respect to x the functions in Nos. 1–8.

1. $(5x+9)^4$.

2. $(2-7x)^{-6}$.

3. $\sin(2x-5)$.

4. $\sec^2(4-3x)$.

5. $\{36-(2x-3)^2\}^{-\frac{1}{2}}$.

6. $\{16+(2-5x)^2\}^{-1}$.

7. $\sec 2x \tan 2x$.

8. $\operatorname{cosec}^2(3-11x)$.

By a suitable change of variable evaluate the integrals in Nos. 9–16.

9. $\displaystyle\int (2-x)^{\frac{1}{2}}\,\mathrm{d}x$.

10. $\displaystyle\int \frac{x}{9+x^4}\,\mathrm{d}x$.

11. $\displaystyle\int \sin^5 x \cos x\,\mathrm{d}x$.

12. $\displaystyle\int \frac{\sec^2 x\,\mathrm{d}x}{(1+\tan x)^2}$.

13. $\displaystyle\int \cos(7x-5)\,\mathrm{d}x$.

14. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x$.

15. $\displaystyle\int x(4+x^2)^{\frac{1}{2}}\,\mathrm{d}x$.

16. $\displaystyle\int \frac{x^2}{(3+x^3)^{\frac{1}{2}}}\,\mathrm{d}x$.

§ 1.3. Trigonometric substitutions

Consider the standard integral

$$\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)}}.$$

If we write $x=a \sin \theta$, so that $\sqrt{(a^2-x^2)}=a \cos \theta$, $\mathrm{d}x/\mathrm{d}\theta=a \cos \theta$, we get by changing the variable from x to θ,

$$\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)}}=\int \frac{a \cos \theta}{a \cos \theta}\,\mathrm{d}\theta=\int \mathrm{d}\theta=\theta=\sin^{-1}\left(\frac{x}{a}\right).$$

It is, of course, unnecessary to evaluate standard integrals by this method, but such a substitution does enable us to evaluate other integrals involving the same square root.

Example 6

With the same substitution $x=a \sin \theta$, we have

$$\int (a^2-x^2)^{\frac{1}{2}}\,\mathrm{d}x=\int (a \cos \theta)(a \cos \theta)\,\mathrm{d}\theta=\tfrac{1}{2}a^2\int (1+\cos 2\theta)\,\mathrm{d}\theta$$

$$=\tfrac{1}{2}a^2(\theta+\tfrac{1}{2}\sin 2\theta)=\tfrac{1}{2}a^2\left\{\sin^{-1}\left(\frac{x}{a}\right)+\frac{x(a^2-x^2)^{\frac{1}{2}}}{a^2}\right\}. \tag{2.5}$$

Similarly, remembering the standard forms, we could have made the substitution $x=a \cos \theta$ in the integral in Example 6. The result would have differed from the result (2.5) by a constant.

In the same way the square root $(x^2-a^2)^{\frac{1}{2}}$ suggests the substitution $x=a \sec \theta$, whilst $(x^2+a^2)^{\frac{1}{2}}$ suggests the substitution $x=a \tan \theta$, but in Ch. 5 we shall see that other substitutions are usually more convenient for these particular square roots.

Example 7

To evaluate

$$\int (25-9x^2)^{\frac{1}{2}}\,\mathrm{d}x,$$

we write

$$3x=5 \cos \theta,$$

$$3\,\mathrm{d}x/\mathrm{d}\theta=-5 \sin \theta,$$

$$(25-9x^2)^{\frac{1}{2}}=5 \sin \theta.$$

Thus the integral becomes

$$\int 5 \sin \theta(-\tfrac{5}{3} \sin \theta) \, d\theta = -\tfrac{25}{6} \int 2 \sin^2 \theta \, d\theta = -\tfrac{25}{6} \int (1 - \cos 2\theta) \, d\theta$$
$$= -\tfrac{25}{6}(\theta - \tfrac{1}{2} \sin 2\theta) = -\tfrac{1}{6}\left\{25 \cos^{-1} \frac{3x}{5} - 3x(25 - 9x^2)^{\frac{1}{2}}\right\}.$$

§ 1.4. Simple powers and products of some trigonometric functions

In Example 6 we required the value of the integral

$$\int \cos^2 \theta \, d\theta,$$

and this was found by expressing $\cos^2 \theta$ in terms of $\cos 2\theta$. For simple integrals involving squares and cubes of the sine and cosine functions, this method can be applied with the use of the well-known trigonometric formulae

$$2 \cos^2 \theta = 1 + \cos 2\theta; \qquad 4 \cos^3 \theta = 3 \cos \theta + \cos 3\theta;$$
$$2 \sin^2 \theta = 1 - \cos 2\theta; \qquad 4 \sin^3 \theta = 3 \sin \theta - \sin 3\theta.$$

Any simple product of two or more such functions can also be dealt with by similar means.

Example 8

$$\int \sin \theta \cos 2\theta \, d\theta = \tfrac{1}{2} \int (\sin 3\theta - \sin \theta) \, d\theta = \tfrac{1}{2}\{-\tfrac{1}{3} \cos 3\theta + \cos \theta\}.$$

Example 9

$$\int \sin^2 \theta \sin 3\theta \, d\theta = \tfrac{1}{2} \int (1 - \cos 2\theta) \sin 3\theta \, d\theta = \tfrac{1}{4} \int (2 \sin 3\theta - \sin 5\theta - \sin \theta) \, d\theta$$
$$= -\tfrac{1}{6} \cos 3\theta + \tfrac{1}{20} \cos 5\theta + \tfrac{1}{4} \sin \theta.$$

Example 10

$$\int \sin^2 \theta \cos^3 \theta \, d\theta = \tfrac{1}{8} \int (1 - \cos 2\theta)(3 \cos \theta + \cos 3\theta) \, d\theta,$$

and by similar methods as in Examples 8 and 9 this is evaluated as

$$\tfrac{1}{8} \sin \theta - \tfrac{1}{48} \sin 3\theta - \tfrac{1}{60} \sin 5\theta.$$

EXERCISE 2.2

Evaluate the following integrals:

1. $\displaystyle\int \{4 - (3x - 1)^2\}^{\frac{1}{2}} \, dx.$

2. $\displaystyle\int \frac{(x + 2) \, dx}{(5 - 4x - x^2)^{\frac{1}{2}}}.$

3. $\displaystyle\int x(9 - x^2)^{-\frac{1}{2}} \, dx.$

4. $\displaystyle\int x^2(x - 2)^{\frac{1}{2}} \, dx.$

5. $\int \sin^2 \frac{1}{2}x \, dx.$

6. $\int \sin \theta \cos 5\theta \, d\theta.$

7. $\int \tan^4 x \sec^2 x \, dx.$

8. $\int \sec^5 x \tan x \, dx.$

9. $\int (\cos \theta - \sin \theta)^2 \, d\theta.$

10. $\int \sin^2 x \cos 2x \, dx.$

§ 1.5. Integration by parts

This important method of integration is used when we require the integral of the product of two different functions of x. It follows from the rule for the differentiation of a product, namely

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides of this equation with respect to x, we get

$$uv = \int \frac{d(uv)}{dx}\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx,$$

or, rearranging the terms,

$$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx. \tag{2.6}$$

This formula is used when the integrand can be arranged as the product of two factors v and du/dx, the one du/dx being the derivative of some known function given by $u=f(x)$, which means that

$$f(x) = \int \frac{du}{dx}\,dx.$$

Example 11

$$\int x \sin x \, dx.$$

Putting $du/dx = \sin x$ and $v = x$ in eq. (2.6), this integral is evaluated as follows

$$\int x \sin x \, dx = (-\cos x)x - \int (-\cos x)\,dx = -x\cos x + \sin x.$$

We note that in this example, both the functions x and $\sin x$ are immediately integrable, but if we had put $v = \sin x$ and $du/dx = x$ in the eq. (2.6), so that $u = \frac{1}{2}x^2$, then the integral on the right of eq. (2.6) would be

$$\int u\frac{dv}{dx}\,dx = \int \tfrac{1}{2}x^2 \cos x \, dx,$$

and this integral would be harder to evaluate than the original one. In general, polynomials in powers of x are usually treated as v in the rule (2.6) so that integration by parts produces a polynomial dv/dx of lower power; successive applications of (2.6) will eventually reduce the powers of x to zero.

Example 12

$$\int x^3 \sin 2x \, dx = (-\tfrac{1}{2}\cos 2x)x^3 + \tfrac{3}{2}\int x^2 \cos 2x \, dx.$$

Then, repeating the rule (2.6)

$$\int x^2 \cos 2x \, dx = (\tfrac{1}{2}\sin 2x)x^2 - \int x \sin 2x \, dx,$$

and again repeating the rule (2.6)

$$\int x \sin 2x \, dx = (-\tfrac{1}{2}\cos 2x)x + \tfrac{1}{2}\int \cos 2x \, dx = -\tfrac{1}{2}x\cos 2x + \tfrac{1}{4}\sin 2x.$$

Thus altogether

$$x^3 \sin 2x \, dx = -\tfrac{1}{2}x^3 \cos 2x + \tfrac{3}{4}x^2 \sin 2x + \tfrac{3}{4}x \cos 2x - \tfrac{3}{8}\sin 2x.$$

The formula (2.6) enables us to evaluate the integrals of some of the inverse trigonometric functions.

Example 13

$$\int \sin^{-1} x \, dx.$$

We use eq. (2.6) with $v = \sin^{-1} x$ and $du/dx = 1$, so that $u = x$. Thus

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{(1-x^2)^{\frac{1}{2}}} \, dx.$$

The integral on the right hand side in this result is evaluated by writing $t = 1 - x^2$. It becomes

$$\int \frac{x}{(1-x^2)^{\frac{1}{2}}} \, dx = -\tfrac{1}{2}\int t^{-\frac{1}{2}} \, dt = -t^{\frac{1}{2}} = -(1-x^2)^{\frac{1}{2}},$$

and thus

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + (1-x^2)^{\frac{1}{2}}.$$

Example 14

As in Example 13, we can show that

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - (1-x^2)^{\frac{1}{2}}.$$

EXERCISE 2.3

Integrate with respect to x the functions in Nos. 1–6:

1. $x^2 \cos x$. **2.** $x \cos^{-1} x$. **3.** $x \tan^{-1} x$.

4. $x^2 \sin^{-1} x$. **5.** $x^2 \sin 2x$. **6.** $x \cos^2 x$.

§ 2. Definite integrals: areas beneath plane curves

Integration simply as the inverse of differentiation has no apparent application to any physical or mathematical problem; to relate it to any such problem, we require the definition of a *definite integral.* This is most simply introduced by the evaluation of the area beneath a plane curve.

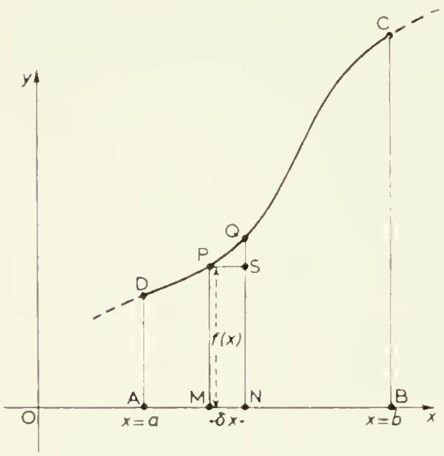

Fig. 2.1

Let $f(x)$ be a continuous function of x which is positive for values of x in the range $a \leqslant x \leqslant b$. The graph of $y=f(x)$ is therefore some curve such as DC in fig. 2.1 where AD is the ordinate at $x=a$ and BC the ordinate at $x=b$. The area beneath the curve in this range is then the area ABCD bounded by the curve $y=f(x)$, the axis of x, and the ordinates at $x=a$, $x=b$. Let M be any point on the x-axis with abscissa x in the range $a<x<b$, and let MP be the ordinate at M cutting the curve DC in P. Suppose z denotes the value of the area AMPD beneath the curve. Then the value of z varies as the position of M varies, so that z is a function of x, say, $z=g(x)$. Consider a point N near to M such that $\mathrm{MN}=\delta x$. Let NQ be the ordinate at N cutting the curve in Q. As the variable x in $z=g(x)$ increases by δx, let z increase by δz, so that $z+\delta z$ represents the area ANQD. Thus by subtraction, δz is the area beneath the arc PQ of the curve, or

$$\delta z = \text{area MNQP}.$$

Now the area MNQP is approximately equal to the area of the rectangle MNSP, which is $\mathrm{MP}\cdot\mathrm{MN}=f(x)\,\delta x$, so that we can write

$$\delta z \approx f(x)\,\delta x.$$

We feel intuitively that this approximate result is more accurate the smaller δx becomes, provided the curve behaves reasonably near P. Thus let us assume tentatively that as $\delta x \to 0$, we can write

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta z}{\delta x} = f(x). \tag{2.7}$$

This equation is of the same form as eq. (2.1), and therefore

$$z = F(x) + c, \tag{2.8}$$

where $F(x)$ is an integral of $f(x)$, and c is an arbitrary constant. Here also, there is a further condition imposed between z and x, namely that $z =$ area AMPD is zero when M is at A, that is when $x=a$. Thus, as in § 1, c must satisfy the equation

$$0 = F(a) + c,$$

or $c=-F(a)$. Hence eq. (2.8) becomes

$$z = F(x) - F(a). \tag{2.9}$$

The total area ABCD will then be the value of z when M is at B, that is when $x=b$, and its value is therefore

$$F(b) - F(a).$$

It is desirable to be able to express this result in terms of the function $f(x)$. We therefore use the notation

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2.10}$$

The expression

$$\int_a^b f(x)\,\mathrm{d}x,$$

is called a *definite integral*; a and b are called the *lower* and *upper* limits; $f(x)$ is called the *integrand* and the interval (a, b) the *range of integration.* A definite integral depends on a, b and the form of the function $f(x)$ only, and is not a function of x. The function

$$F(x) = \quad f(x)\,\mathrm{d}x,$$

an integral of $f(x)$, is sometimes called an *indefinite integral* of $f(x)$ to distinguish it more precisely from a definite integral. In evaluating a definite integral it is convenient to write

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a). \tag{2.11}$$

We remember now that $F(x)$ is not unique, so we should verify that whatever indefinite integral of $f(x)$ is used, the value of the area is always the same.

Suppose $F_1(x)$ and $F_2(x)$ are two integrals of $f(x)$. Then we know that we can write

$$F_2(x) = F_1(x) + c_1, \tag{2.12}$$

where c_1 is some constant. Using $F_1(x)$, we have

$$\int_a^b f(x)\,\mathrm{d}x = [F_1(x)]_a^b = F_1(b) - F_1(a).$$

Using $F_2(x)$, we have

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= [F_2(x)]_a^b = [F_1(x) + c_1]_a^b \\ &= \{F_1(b) + c_1\} - \{F_1(a) + c_1\} = F_1(b) - F_1(a).\end{aligned}$$

Thus the value of a definite integral is independent of which indefinite integral $F(x)$ is used. Further since its value is independent of x, we can replace the variable x in the definite integral $\int_a^b f(x)\,\mathrm{d}x$, by any other letter, say t; the integral $\int_a^b f(t)\,\mathrm{d}t$ has exactly the same value, namely $F(b)-F(a)$. As a consequence of this we note that the area AMPD above, given by eq. (2.9)

$$z = F(x) - F(a), \tag{2.13}$$

could be written in the form

$$z = \int_a^x f(t)\,\mathrm{d}t \tag{2.14}$$

the variable t in the integrand being used to avoid confusion with the upper limit. Here the upper limit x is regarded as variable and z is a function of this upper limit. In fact we have from eq. (2.13)

$$\frac{\mathrm{d}z}{\mathrm{d}x} = F'(x) = f(x). \tag{2.15}$$

Further the solution (2.14) of the equation (2.15) may be regarded as a solution in which the arbitrary constant is written in the particular form $-F(a)$ and is therefore included by specifying the lower limit.

Finally, we have expressed the area under a curve $y=f(x)$ as an integral only when $f(x)$ is positive. If $f(x)$ is negative in the given range $a \leqslant x \leqslant b$ the curve $y=f(x)$ lies below the x-axis, and we still require the value of the area bounded by the curve $y=f(x)$, the axis of x and the ordinates $x=a$, $x=b$. The value of this area will be the same as that of its reflection in the x-axis, namely the area beneath the curve $y=-f(x)$. This area is

$$\int_a^b \{-f(x)\}\,\mathrm{d}x = [-F(x)]_a^b = -F(b) + F(a) = -\int_a^b f(x)\,\mathrm{d}x.$$

Thus when $f(x)$ is negative, the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

gives the correct numerical value for the required area, but with a negative sign. Thus care must be taken in finding the area between a curve and the x-axis when the range of variation of x is such that $f(x)$ has positive and negative values in the range.

The results of this paragraph are based on the intuitive eq. (2.7). A slightly more rigorous approach to this equation and the definition of an area will be given in § 2.2.

For the present we will simply evaluate some definite integrals, after stating some simple properties.

2.1. Properties of definite integrals

From the definition of a definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any indefinite integral of $f(x)$, the following properties are immediately obvious.

(i) $\int_b^a f(x)\,\mathrm{d}x = F(a) - F(b) = -\int_a^b f(x)\,\mathrm{d}x,$

(ii) $\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x,$

(iii) $\int_a^b \{f(x) + g(x)\}\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$

Example 15

$$\int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{\sqrt{(1-4x^2)}} = [\tfrac{1}{2}\sin^{-1} 2x]_0^{\frac{1}{2}} = \{\tfrac{1}{2}\cdot\tfrac{1}{2}\pi - 0\} = \tfrac{1}{4}\pi.$$

Example 16

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \sin 4x \cos 3x\,\mathrm{d}x &= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\sin 7x + \sin x)\,\mathrm{d}x = \tfrac{1}{2}[-\tfrac{1}{7}\cos 7x - \cos x]_0^{\frac{1}{2}\pi}\\ &= \tfrac{1}{2}[\tfrac{1}{7} + 1] = \tfrac{4}{7}.\end{aligned}$$

Example 17

Using integration by parts

$$\int_0^{\frac{1}{2}\pi} x^2 \sin x \, dx = [-x^2 \cos x]_0^{\frac{1}{2}\pi} + \int_0^{\frac{1}{2}\pi} 2x \cos x \, dx$$

$$= 0 + [2x \sin x]_0^{\frac{1}{2}\pi} - 2\int_0^{\frac{1}{2}\pi} \sin x \, dx = \pi - 2[-\cos x]_0^{\frac{1}{2}\pi} = \pi - 2.$$

Example 18

Find the area between the curve $y=\sin x$ and the x-axis, in the range $0 \leqslant x \leqslant 2\pi$,

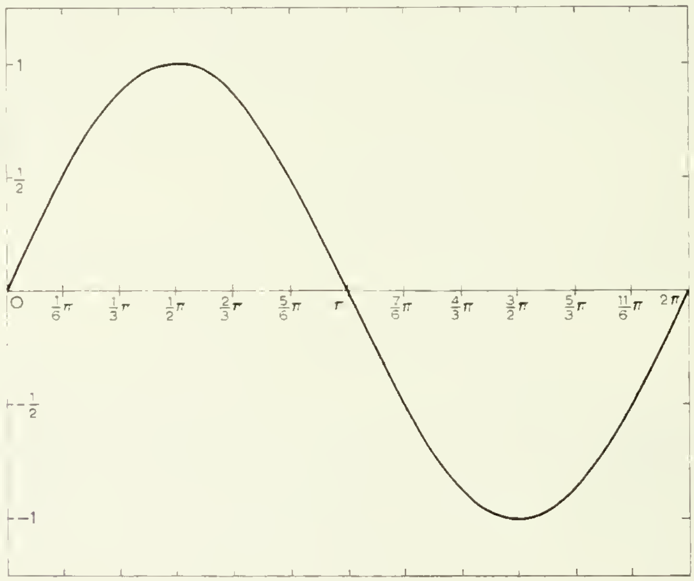

Fig. 2.2

The curve is shown in fig. 2.2. For $0<x<\pi$, $\sin x$ is positive, whilst for $\pi<x<2\pi$. $\sin x$ is negative. The reader can verify that $\int_0^{2\pi} \sin x \, dx$ gives the *algebraic* sum of the areas, which from symmetry is zero. To find the actual magnitude of the two areas, we must take

$$\int_0^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx = [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = 4.$$

The change of variable or substitution rule is used in the following way:

$$\text{(iv)} \quad \int_{x_1}^{x_2} f(x) \, dx = \int_{t_1}^{t_2} f\{\Phi(t)\} \, \Phi'(t) \, dt, \tag{2.16}$$

where $x=\Phi(t)$ is such that when $x=x_1$, $t=t_1$ and when $x=x_2$, $t=t_2$.

To prove this we suppose that $F(x)$ is an integral of $f(x)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x).$$

Making the substitution $x=\Phi(t)$ in $F(x)$, so that it becomes $F\{\Phi(t)\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} F\{\Phi(t)\} = \frac{\mathrm{d}}{\mathrm{d}x}\{F(x)\}\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)\frac{\mathrm{d}x}{\mathrm{d}t} = f\{\Phi(t)\}\,\Phi'(t).$$

Thus

$$\begin{aligned}\int_{t_1}^{t_2} f\{\Phi(t)\}\,\Phi'(t)\,\mathrm{d}t &= [F\{\Phi(t)\}]_{t_1}^{t_2} = F\{\Phi(t_2)\} - F\{\Phi(t_1)\} \\ &= F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x, \qquad (2.17)\end{aligned}$$

which establishes the rule (iv).

Example 19

$$\int_0^3 x(9 - x^2)^{\frac{1}{2}}\,\mathrm{d}x.$$

Let $t=9-x^2$ so that $\mathrm{d}t=-2x\,\mathrm{d}x$ or $x\,\mathrm{d}x=-\frac{1}{2}\mathrm{d}t$. Also when $x=0$, $t=9$ and when $x=3$, $t=0$. The integral is

$$-\tfrac{1}{2}\int_9^0 t^{\frac{1}{2}}\,\mathrm{d}t = \tfrac{1}{2}\int_0^9 t^{\frac{1}{2}}\,\mathrm{d}t = [\tfrac{1}{3}t^{\frac{3}{2}}]_0^9 = 9.$$

Example 20

$$\int_0^1 \frac{x^2\,\mathrm{d}x}{4 + x^6}.$$

Let $t=x^3$, then $\mathrm{d}t=3x^2\,\mathrm{d}x$. Also when $x=0$, $t=0$; $x=1$, $t=1$. The integral is

$$\frac{1}{3}\int_0^1 \frac{\mathrm{d}t}{4 + t^2} = [\tfrac{1}{6}\tan^{-1}\tfrac{1}{2}t]_0^1 = \tfrac{1}{6}\tan^{-1}\tfrac{1}{2}.$$

Example 21

$$\int_0^2 \sqrt{(4 - x^2)}\,\mathrm{d}x.$$

Let $x=2\sin\theta$ so that $\mathrm{d}x=2\cos\theta\,\mathrm{d}\theta$. When $x=0$, $\theta=0$; $x=2$, $\theta=\frac{1}{2}\pi$. The integral is

$$\int_0^{\frac{1}{2}\pi} 4\cos^2\theta\,\mathrm{d}\theta = 2\int_0^{\frac{1}{2}\pi}(1 + \cos 2\theta)\,\mathrm{d}\theta = [2\theta + \sin 2\theta]_0^{\frac{1}{2}\pi} = \pi.$$

Returning to eq. (2.17) it is obvious also that since

$$[F\{\Phi(t)\}]_{t_1}^{t_2} = [F(x)]_{x_1}^{x_2},$$

then the actual change of variable in the limits need not be made. This is convenient when such changes are clumsy. All that is required is that the indefinite integral in the form $F\{\Phi(t)\}$ shall be transformed back into a function of x, using the inverse of $x = \Phi(t)$.

Example 22

$$\int_1^3 \sqrt{(16 - x^2)}\, dx.$$

We write $x = 4 \sin \theta$; $dx = +4 \cos \theta \, d\theta$. The integral is written as

$$\int_{x=1}^{x=3} 16 \cos^2 \theta \, d\theta = 8 \int_{x=1}^{x=3} (1 + \cos 2\theta)\, d\theta = [4(2\theta + \sin 2\theta)]_{x=1}^{x=3}$$

$$= [8 \sin^{-1} \tfrac{1}{4}x + \tfrac{1}{2}x(16 - x^2)^{\frac{1}{2}}]_1^3 = \tfrac{1}{2}[16 \sin^{-1} \tfrac{3}{4} + 3\sqrt{7} - 16 \sin^{-1} \tfrac{1}{4} - \sqrt{15}].$$

EXERCISE 2.4

Evaluate the integrals in Nos. 1–8.

1. $\int_0^a \sqrt{(2ax - x^2)}\, dx$; use the substitution $a - x = a \cos \theta$.

2. $\int_0^{\frac{1}{2}\pi} x^2 \sin 3x \, dx$.

3. $\int_0^{\frac{1}{2}\pi} (1 + \cos x - \sin x) \sin x \, dx$.

4. $\int_2^4 (x + 2) \sqrt{(2x - 3)}\, dx$.

5. $\int_0^{\frac{1}{2}\pi} (1 + 2 \sin^2 x) \cos^2 x \, dx$.

6. $\int_0^{\pi} \cos \tfrac{1}{2}x \cos x \, dx$.

7. $\int_0^{\frac{1}{2}\pi} x \sin^3 x \cos x \, dx$; use integration by parts on $v = x$, $du/dx = \sin^3 x \cos x$.

8. $$\int_0^2 \frac{x + 1}{\sqrt{(9 - x^2)}}\, dx.$$

9. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

10. Find the area enclosed by the two curves $y = \sin x$, $y = \cos x$ between any two consecutive points at which they intersect.

11. Find the area common to the two parabolas $y^2 = 4ax$, $x^2 = 4ay$.

12. Find the area between the curve $x(y - a)^2 = a^3$ and the two lines $x = a$, $x = 4a$.

§ 2.2. Definite integrals: limit of a sum

The integral calculus may be said to have been begun by the Greek mathematicians who strove to find the area of a circle. They did not see the relationship between this problem and integration as we know it, but it soon becomes obvious that there is a direct connection. If two n-sided polygons are inscribed in and circumscribed about a given circle, their areas can be calculated by Euclidean methods and it is reasonable to suppose that the area of the circle has a value between the areas of these two polygons. An equivalent method of calculating the area of a circle, and one which is capable of extension to the evaluation of other areas is as follows (cf. fig. 2.3). OAB is that part of the circle

$$x^2 + y^2 = a^2,$$

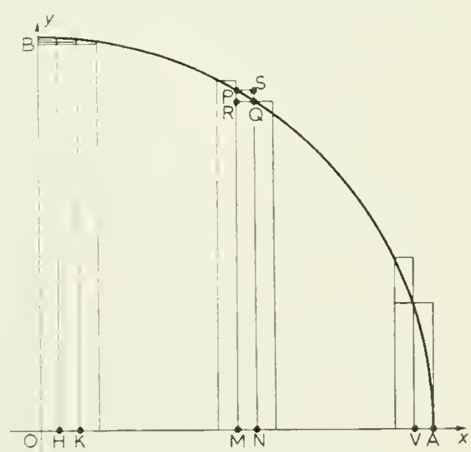

Fig. 2.3

which lies in the first quadrant. Suppose OA to be divided into n equal parts at the points H, K,... M, N... V. Through these points ordinates MP, NQ are drawn to meet the circle in P, Q and a set of rectangles formed by drawing lines PS, QR parallel to the x-axis. Let S denote the sum of the areas of all the rectangles like MNSP going beyond the circle, and s the sum the areas of all the rectangles like MNQR enclosed by the circle. Then the area of the circle lies between the values $4S$ and $4s$. By making n larger, these two values would be nearer each other. Therefore as we make n increase indefinitely or, as we say let $n \to \infty$, it seems plausible to suppose that $4S$ and $4s$ would tend to the same limit. This common limiting value would be the area of the circle.

Let us now use this idea to define the area ABCD beneath a plane curve $y=f(x)$ (fig. 2.4). Let the segment AB be divided into n parts, not necessarily equal, by points whose abscissae are

$$x_0(=a),\ x_1,\ x_2,\dots\ x_{r-1},\ x_r,\dots\ x_n(=b).$$

Let R be the point x_{r-1}, S the point x_r, and RP, SQ the ordinates at R, S respectively. Also let $x_r-x_{r-1}=\delta x_r$. Fig. 2.5 shows a magnified picture of the small area RSQP with $y=f(x)$ taking the possible form shown between P and Q. Suppose that m_r is the least value, and M_r the greatest value of $f(x)$ in the range $x_{r-1}<x<x_r$. Then it is obvious

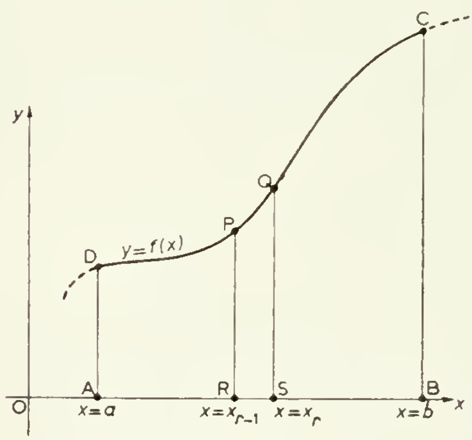

Fig. 2.4

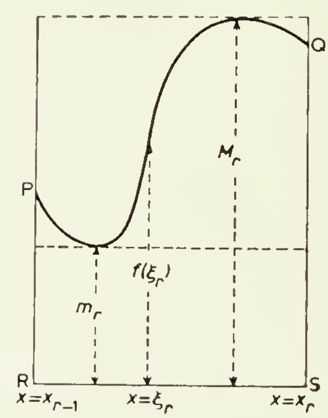

Fig. 2.5

from fig. 2.5 that the area RSQP lies between the values

$$M_r(x_r - x_{r-1}) = M_r\delta x_r \quad \text{and} \quad m_r(x_r - x_{r-1}) = m_r\delta x_r.$$

To conform with the notation of § 2, denote this area RSQP by δz_r. Then

$$m_r\delta x_r \leqslant \delta z_r \leqslant M_r\delta x_r. \tag{2.18}$$

But δx_r being positive, this gives

$$m_r \leqslant \frac{\delta z_r}{\delta x_r} \leqslant M_r, \tag{2.19}$$

and as $\delta x_r \to 0$, both m_r and M_r tend to the value of $f(x)$ at R, that is, $f(x_r)$, provided $f(x)$ is a continuous function. Thus using x instead of x_r for any point in the range, and δx, δz as the corresponding increments, eq. (2.19) gives immediately

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta z}{\delta x} = f(x),$$

the result which was tentatively assumed in § 2. Further by summing equations of the form (2.18) for all values of r, using

$$S = \sum_r M_r \delta x_r = \text{sum of the outer rectangles},$$

$$s = \sum_r m_r \delta x_r = \text{sum of the inner rectangles},$$

and

$$\sum_r \delta z_r = \sum \text{areas RSPQ} = \text{area ABCD},$$

we get

$$s \leqslant \text{area ABCD} \leqslant S, \tag{2.20}$$

the equality sign holding only if the curve $y=f(x)$ is a straight line parallel to the x-axis.

Thus, if these two sums s and S have a common limit as the number of subdivisions increases in such a way that the largest of the subdivisions δx_r tends to zero, then this limit is the area of ABCD.

Moreover, if we take some intermediate point on the curve between P and Q where the abscissa is ξ_r and ordinate $f(\xi_r)$, then

$$m_r \delta x_r \leqslant f(\xi_r)\,\delta x_r \leqslant M_r \delta x_r,$$

and so again by summation for all δx_r

$$s \leqslant \sum_r f(\xi_r)\,\delta x_r \leqslant S. \tag{2.21}$$

Thus if s and S have a common limit as $\delta x_r \to 0$, this limit will be

$$\lim_{\delta x_r \to 0} \sum_r f(\xi_r)\,\delta x_r. \tag{2.22}$$

Comparing eqs. (2.20) and (2.21) we have

$$z = \text{area ABCD} = \lim_{\delta x_r \to 0} \sum_r f(\xi_r)\,\delta x_r. \tag{2.23}$$

This result explains the notation for an integral; the integral sign $\int$ replaces the summation sign $\sum_r$ in the limit when $\delta x_r \to 0$, $\xi_r \to x_r$.

The conditions under which s and S have a common limit depend, of course, on the function $f(x)$. It can be shown that the conditions are fulfilled if $f(x)$ is a continuous function, but they are satisfied also in some other cases.

If s and S have a common limit $f(x)$ is said to be *Riemann-integrable*, and the integral defined in § 2 is a Riemann Integral. The theory goes back to the German mathematician Bernhard Riemann. Other types of integral, such as Stieltjes, Lebesgue, are beyond the scope of this book.

§ 3. Improper integrals

In the definition of the definite integral

$$\int_a^b f(x)\,\mathrm{d}x,$$

it was assumed that the limits of integration a, b were finite and also that the function $f(x)$ was finite and continuous throughout the range $a \leqslant x \leqslant b$. We proceed to explain how, under certain conditions, we may extend the definition of an integral.

§ 3.1. One of the limits infinite

Let $f(x)$ be finite and continuous for all values of $x \geqslant a$ and let us consider the integral

$$\int_a^A f(x)\,\mathrm{d}x, \tag{2.24}$$

where a is fixed, but the upper limit A may become arbitrarily large. If this integral has a finite limit l as $A \to \infty$, then we denote the limit by

$$\int_a^\infty f(x)\,\mathrm{d}x, \tag{2.25}$$

and say that it is an infinite integral convergent to the limit l.

If a finite value l does not exist, but the value of the integral $\to +\infty$ as $A \to +\infty$ then we say that the integral is divergent to $+\infty$, and we can give a similar definition of divergence to $-\infty$.

It may happen that the integral tends to no particular limit as $A \to \infty$, but oscillates over either a finite or an infinite range of values. We then say that the integral oscillates finitely or infinitely as $A \to \infty$.

If $f(x)$ is finite and continuous for all values of $x \leqslant b$, we can give similar definitions for

$$\int_{-\infty}^b f(x)\,\mathrm{d}x.$$

Example 23

$$\int_1^\infty \frac{\mathrm{d}x}{x^4} = \lim_{A\to\infty} \int_1^A \frac{\mathrm{d}x}{x^4} = \lim_{A\to\infty} \left[-\frac{1}{3x^3}\right]_1^A = \tfrac{1}{3}.$$

Example 24

$$\int_{-\infty}^{0} \frac{dx}{a^2 + b^2x^2} = \lim_{A\to\infty} \int_{-A}^{0} \frac{dx}{a^2 + b^2x^2}.$$

We have

$$\int_{-A}^{0} \frac{dx}{a^2 + b^2x^2} = \frac{1}{b^2} \int_{-A}^{0} \frac{dx}{x^2 + a^2/b^2} = \left[\frac{1}{ab} \tan^{-1} \frac{bx}{a}\right]_{-A}^{0}$$

$$= -\frac{1}{ab} \tan^{-1}\left(-\frac{bA}{a}\right) = \frac{1}{ab} \tan^{-1} \frac{bA}{a}.$$

As $A \to \infty$, $\tan^{-1}(bA/a) \to \frac{1}{2}\pi$ and thus we write

$$\int_{-\infty}^{0} \frac{dx}{a^2 + b^2x^2} = \frac{\pi}{2ab}.$$

§ 3.2. Discontinuity in $f(x)$

When $f(x)$ has a finite discontinuity in the range $a \leqslant x \leqslant b$ there is no difficulty in extending the concept of the finite integral between these limits. All we need to do is to define the integral of the function as the sum of the integrals over the separate sub-intervals in which the function is continuous. The meaning in terms of area is then also obvious from fig. 2.6, $f(x)$ having a finite discontinuity at $x = x_1$ where $a < x_1 < b$.

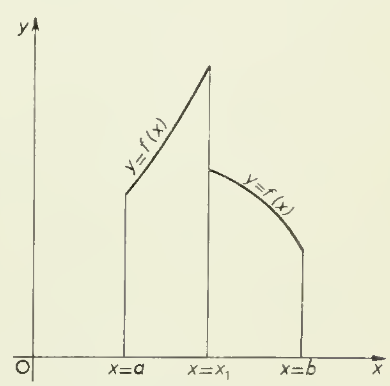

Fig. 2.6

This method of defining the integral can be extended to any function $f(x)$ which has only a finite number of finite discontinuities in the range $a \leqslant x \leqslant b$.

Suppose now that $f(x)$ becomes infinite for some value of x in the range $a \leqslant x \leqslant b$.

It will be sufficient to consider the case where there is only one value of x for which $f(x) \to \infty$. When $f(x)$ becomes infinite at a finite number of isolated points in the range, we can break up the range into a finite number of sub-intervals each containing only one value of x for which $f(x) \to \infty$.

If $f(x)$ becomes infinite at the upper limit b only, we consider in the

first place the integral

$$\int_a^{b-\varepsilon} f(x)\,\mathrm{d}x, \tag{2.26}$$

where ε is positive. If as $\varepsilon \to 0$ this integral tends to a definite limiting value, this value is taken as the definition of

$$\int_a^{b} f(x)\,\mathrm{d}x. \tag{2.27}$$

A similar definition applies when $f(x)$ becomes infinite at the lower limit a.

If $f(x)$ becomes infinite when $x=c$ where $a<c<b$, we consider the sum

$$\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon'}^{b} f(x)\,\mathrm{d}x, \tag{2.28}$$

where ε and ε' are positive.

If as $\varepsilon \to 0$ and $\varepsilon' \to 0$ each of these integrals tends to a finite limiting value the sum of the two limiting values is taken as the definition of the integral $\int_a^b f(x)\,\mathrm{d}x$.

Example 25

$$\int_{\frac{1}{2}\pi}^{\infty} x^2 \cos x\,\mathrm{d}x = \lim_{A\to\infty} \int_{\frac{1}{2}\pi}^{A} x^2 \cos x\,\mathrm{d}x.$$

We have

$$\int_{\frac{1}{2}\pi}^{A} x^2 \cos x\,\mathrm{d}x = [x^2 \sin x]_{\frac{1}{2}\pi}^{A} - 2\int_{\frac{1}{2}\pi}^{A} x \sin x\,\mathrm{d}x = [x^2 \sin x + 2x\cos x]_{\frac{1}{2}\pi}^{A} - 2\int_{\frac{1}{2}\pi}^{A} \cos x\,\mathrm{d}x$$

$$= [x^2 \sin x + 2x\cos x - 2\sin x]_{\frac{1}{2}\pi}^{A} = [A^2 \sin A + 2A\cos A - 2\sin A - \tfrac{1}{4}\pi^2 + 2].$$

As $A\to\infty$, the limit of this function oscillates infinitely.

Example 26

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{(1-x^2)}}.$$

The integral becomes infinite at the upper limit. We therefore consider

$$\int_0^{1-\varepsilon} \frac{x\,\mathrm{d}x}{\sqrt{(1-x^2)}} = -\frac{1}{2}\int_0^{1-\varepsilon} \frac{-2x\,\mathrm{d}x}{\sqrt{(1-x^2)}} = [-\sqrt{(1-x^2)}]_0^{1-\varepsilon} = [-\sqrt{(2\varepsilon-\varepsilon^2)}+1].$$

As $\varepsilon \to 0$ this expression tends to 1 and hence we write

$$\int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{(1-x^2)}} = 1.$$

Example 27

$$\int_0^{2\pi} \frac{dx}{1+\cos x}.$$

Since $\cos x = -1$ when $x=\pi$, the integrand is discontinuous at $x=\pi$. We write the integral as

$$\int_0^{\pi} \frac{dx}{1+\cos x} + \int_{\pi}^{2\pi} \frac{dx}{1+\cos x}.$$

In the second integral put $x=2\pi-u$; it becomes

$$\int_0^{\pi} \frac{du}{1+\cos u} = \int_0^{\pi} \frac{dx}{1+\cos x}.$$

Thus

$$\int_0^{2\pi} \frac{dx}{1+\cos x} = 2\int_0^{\pi} \frac{dx}{1+\cos x} = \lim_{\varepsilon\to 0} \int_0^{\pi-\varepsilon} \frac{2dx}{1+\cos x}.$$

We have

$$\int_0^{\pi-\varepsilon} \frac{2dx}{1+\cos x} = \int_0^{\pi-\varepsilon} \frac{dx}{\cos^2 \frac{1}{2}x} = \int_0^{\pi-\varepsilon} \sec^2 \tfrac{1}{2}x \, dx = [2\tan \tfrac{1}{2}x]_0^{\pi-\varepsilon} = 2\tan(\tfrac{1}{2}\pi - \tfrac{1}{2}\varepsilon).$$

As $\varepsilon\to 0$, $\tan(\frac{1}{2}\pi-\frac{1}{2}\varepsilon)\to\infty$.

§ 4. Related integrals

There are certain other results concerning definite integrals, which are worth mentioning since they reduce the work of evaluation. We list here three such results, assuming in all cases that $f(x)$ is a finite continuous function of x in the given ranges.

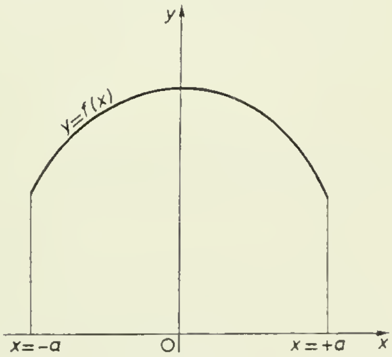

Fig. 2.7

(i) If $f(x)$ is an *even function* of x, meaning that

$$f(-x) = f(x),$$

then

$$\int_{-a}^{+a} f(x)\,dx = 2\int_0^{a} f(x)\,dx. \tag{2.29}$$

This is obvious in terms of areas as shown in fig. 2.7 because of symmetry about the y-axis. Otherwise, we have

$$\int_{-a}^{+a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x.$$

In the first integral write $x=-u$ and it becomes

$$\int_{a}^{0} -f(-u)\,\mathrm{d}u = \int_{0}^{a} f(u)\,\mathrm{d}u = \int_{0}^{a} f(x)\,\mathrm{d}x.$$

Hence the result.

(ii) If $f(x)$ is an *odd function* of x defined by

$$f(-x) = -f(x),$$

then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0. \tag{2.30}$$

The proof follows as in (i).

(iii)

$$\int_{0}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} f(a-x)\,\mathrm{d}x. \tag{2.31}$$

This is proved by writing $a-x=u$ in the second integral. It then becomes

$$-\int_{a}^{0} f(u)\,\mathrm{d}u = \int_{0}^{a} f(u)\,\mathrm{d}u = \int_{0}^{a} f(x)\,\mathrm{d}x.$$

An important application of this rule is

$$\int_{0}^{\frac{1}{2}\pi} f(\sin\theta)\,\mathrm{d}\theta = \int_{0}^{\frac{1}{2}\pi} f(\cos\theta)\,\mathrm{d}\theta, \tag{2.32}$$

since $\sin(\frac{1}{2}\pi-\theta)=\cos\theta$. Thus

$$\int_{0}^{\frac{1}{2}\pi} \sin^2\theta\,\mathrm{d}\theta = \int_{0}^{\frac{1}{2}\pi} \cos^2\theta\,\mathrm{d}\theta,$$

and each of these integrals is therefore equal to their average

$$\tfrac{1}{2}\int_{0}^{\frac{1}{2}\pi} (\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta = \tfrac{1}{2}\int_{0}^{\frac{1}{2}\pi} \mathrm{d}\theta = \tfrac{1}{4}\pi.$$

We note also that because of the result (i), Example 24 § 3.1 implies

$$\int_{-\infty}^{0} \frac{dx}{a^2 + b^2x^2} = \int_{0}^{\infty} \frac{dx}{a^2 + b^2x^2} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{a^2 + b^2x^2} = \frac{\pi}{2ab}$$

(iv) Either by examining the graph of $y=f(x)$ or by simple change of variable in (i) and (ii), we can show that if

$$f(a - x) = f(x),$$

then

$$\int_{0}^{a} f(x)\,dx = 2 \int_{0}^{\frac{1}{2}a} f(x)\,dx; \tag{2.33}$$

whilst if

$$f(a - x) = -f(x),$$

then

$$\int_{0}^{a} f(x)\,dx = 0. \tag{2.34}$$

In particular

$$\int_{0}^{\pi} f(\sin \theta)\,d\theta = 2 \int_{0}^{\frac{1}{2}\pi} f(\sin \theta)\,d\theta, \tag{2.35}$$

$$\int_{0}^{\pi} f(\cos \theta)\,d\theta = 0, \tag{2.36}$$

and

$$\int_{0}^{\pi} f(\tan \theta)\,d\theta = 0, \tag{2.37}$$

since $\sin(\pi-\theta)=\sin \theta$, $\cos(\pi-\theta)=-\cos \theta$ and $\tan(\pi-\theta)=-\tan \theta$ respectively.

EXERCISE 2.5

Find, when they exist, the values of the integrals in Nos. 1–6.

1. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^2}$.

2. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^{\frac{1}{2}}}$.

3. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2 + 9}$.

4. $\displaystyle\int_{0}^{2} \frac{dx}{\sqrt{(4 - x^2)}}$.

5. $\displaystyle\int_{0}^{\frac{1}{2}\pi} \frac{\sin x}{\sqrt{(\cos x)}}\,dx$.

6. $\displaystyle\int_{1}^{\infty} x^n\,dx, \quad n \neq -1$.

7. Show that the integral

$$\int_0^\pi \frac{\sin x}{(1 + 2\cos x)^2}\, dx$$

does not exist.

8. Prove that

$$\int_0^\pi x \sin^2 x\, dx = \int_0^\pi (\pi - x)\sin^2 x\, dx,$$

and hence or otherwise evaluate the integral on the left hand side.

THE CONVERGENCE OF INFINITE SERIES

§ 1. Summation of finite series

In text-books of elementary algebra there may be found many examples of summation of series to a finite number of terms. We shall assume that the reader is familiar with the following elementary results:

(i) For the arithmetical progression

$$a,\ a + d,\ a + 2d,\ \ldots,\ a + (n - 1)d, \tag{3.1}$$

the sum to n terms S_n, is given by

$$S_n = \tfrac{1}{2}n\{2a + (n - 1)d\}. \tag{3.2}$$

(ii) For the geometrical progression

$$a,\ ax,\ ax^2,\ \ldots,\ ax^{n-1}, \tag{3.3}$$

$$S_n = a(1 - x^n)/(1 - x). \tag{3.4}$$

(iii) $$\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n + 1),$$

$$\sum_{r=1}^{n} (r^2) = 1 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$

We shall also assume that the reader has had some experience of summation of other more general series such as (iv) arithmetic-geometrical series, (v) series whose rth terms u_r can be expressed as polynomials in r, so that results such as (iii) can be used, (vi) series whose rth terms can be expressed in the form $u_r = v_{r-1} - v_r$, so that $S_n = v_0 - v_n$.

In this chapter we are not concerned with these methods, but such experience will enable the reader to appreciate more fully the nature of

the problems involved in the consideration of series which have an infinite number of terms.

§ 1.1. Infinite series. Convergence and divergence

We shall use the notation u_r for the rth term of the series and S_n for the sum of the first n terms, subsequently referred to as the *partial sum:*

$$S_n = u_1 + u_2 + \dots + u_r + \dots + u_n = \sum_{r=1}^{n} u_r. \tag{3.5}$$

Consider the arithmetical progression (3.1) for which $S_n = \frac{1}{2}n\{2a+(n-1)d\}$. We see that if $d>0$, then by taking a sufficiently large number n of terms of the series we can make S_n as large as we please. Remembering the definition of a limit as n tends to ∞, this means that for the arithmetical progression with $d>0$

$$\lim_{n\to\infty} S_n = \infty.$$

If however we consider the series in which

$$u_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1},$$

for which $S_n = 1 - 1/(n+1)$, we see that by making n sufficiently large we can make this value of S_n differ from 1 by as small a number as we please, or as we say

$$\lim_{n\to\infty} S_n = 1,$$

for this particular series.

These two series are simple examples of what are called *divergent* and *convergent* series respectively.

When n, the number of terms, is to be made as large as we please, the series is called an *infinite series* and is written as

$$u_1 + u_2 + \dots + u_r + \dots = \sum_{r=1}^{\infty} u_r. \tag{3.6}$$

If the partial sum S_n has a unique finite limit S (say) as n tends to infinity, then the series is said to be convergent to the value S. If a series is not convergent it is divergent. Divergent series may be classified further as:

(i) Properly divergent, when $\lim_{n\to\infty} S_n$ is either $+\infty$ or $-\infty$.

(ii) Finitely oscillating, when $\lim_{n\to\infty} S_n$ oscillates between two finite values.

(iii) Infinitely oscillating, when $\lim_{n\to\infty} S_n$ oscillates between unbounded or infinite values.

Example 1

Convergent. The geometric progression

$$1 + \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^{r-1}} + \dots,$$

has common ratio $\frac{1}{2}$;

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n+1}}.$$

So

$$S = \lim_{n\to\infty} S_n = 2.$$

Example 2

Properly divergent. The arithmetic progression

$$1 + 2 + 3 + 4 + \dots + r + \dots,$$

has partial sum

$$S_n = \tfrac{1}{2}n(n + 1),$$

and

$$S = \lim_{n\to\infty} S_n = +\infty.$$

Example 3

Finitely oscillating. The series

$$1 - 1 + 1 - 1 + \dots + (-1)^{r-1} + \dots,$$

is such that

$$S_{2n+1} = +1, \qquad S_{2n} = 0;$$

so the sum simply oscillates between the finite values $+1$ and 0.

Example 4

Infinitely oscillating. The series

$$1 - 2 + 3 - 4 + 5 \dots + (-1)^{r-1}r + \dots,$$

is such that

$$S_{2n+1} = n + 1, \qquad S_{2n} = -n.$$

Thus according to whether the number of terms is odd or even the sum tends to $+\infty$ or $-\infty$.

This problem of the convergence or divergence of an infinite series is important when we need to use series expansions for certain finite expressions. For example, the binomial series for a positive integer n is given by

$$(1 + x)^n = 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \ldots + x^n, \tag{3.7}$$

and is a valid result for all values of x provided n is a positive integer. If however n is a negative integer or fraction this expansion becomes

$$(1 + x)^n = 1 + nx + \binom{n}{1}x^2 + \ldots + \binom{n}{r}x^r + \ldots, \tag{3.8}$$

and is an infinite series, some terms of which may be positive and some negative. Only if this infinite series is a convergent series will it properly represent or be a valid series for $(1+x)^n$. A particular example of the binomial expansion is the geometric progression

$$1 + x + x^2 + x^3 + \ldots + x^{r-1} + \ldots, \tag{3.9}$$

for which we know that

$$S_n = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{x^n}{1 - x}.$$

The behaviour of this series can now be shown to depend on the value of the common ratio x.

(i) If $-1<x<1$, then $1/(1-x)$ is finite and $x^n \to 0$ as $n \to \infty$, so that

$$\lim_{n\to\infty} S_n = \frac{1}{1 - x},$$

and the series is therefore convergent.

(ii) If $x=1$, the series becomes

$$1 + 1 + 1 + \ldots, \tag{3.10}$$

and $S_n=n$, so that $S_n \to +\infty$ as $n \to +\infty$ and the series is properly divergent.

(iii) If $x>1$, each term of the series is greater than the corresponding term of the series (3.10), so that $S_n>n$ and the series is again properly divergent.

(iv) If $x = -1$ the series is

$$1 - 1 + 1 - 1 \ldots,$$

and S_n is zero or $+1$ according to whether n is odd or even and the series is finitely oscillating.

(v) If $x < -1$, we put $x = -y$, so that $y > 1$ and the series in terms of y is

$$1 - y + y^2 - y^3 + \ldots (-1)^r y^r + \ldots,$$

with

$$S_n = 1 - y + y^2 \ldots (-1)^{n-1} y^{n-1} = \frac{1 + (-1)^{n+1} y^n}{1 + y} = \frac{1}{1+y} + \frac{(-1)^{n+1} y^n}{1+y}.$$

Since $y > 1$ and therefore $y^n \to \infty$ as $n \to \infty$, the sum S_n tends to $+\infty$ when n is odd, and to $-\infty$ when n is even and the series is infinitely oscillating.

Thus summing up, the series

$$1 + x + x^2 + \ldots,$$

is convergent only when $-1 < x < 1$ and since in this case

$$S = \lim_{n \to \infty} S_n = \frac{1}{1 - x} = (1 - x)^{-1},$$

we can properly write

$$(1 - x)^{-1} = 1 + x + x^2 + \ldots;$$

this is the binomial expansion of $(1-x)^n$ with $n = -1$.

The more general binomial expansion $(1+x)^n$ will be dealt with later.

§ 2. General results concerning infinite series

In the preceding paragraph we have seen in a general way that the convergence or divergence of an infinite series depends on the properties of $\lim_{n\to\infty} S_n$. If it were possible to find S_n as an expression in terms of n for any series the problem of convergence would be a comparatively simple one. Unfortunately, as was noted in Ch. 1 § 2 this is not generally possible and other methods have to be devised for deciding whether series are convergent or divergent. Before developing these methods, known as *tests of convergence*, we shall deduce some general properties

of infinite series. As in Ch. 1 § 2 Example 9, we have $\lim_{n\to\infty} S_n = S$, if for any positive number ε

$$|S_n - S| < \varepsilon, \tag{3.11}$$

for all integers n exceeding a certain number N (which depends on ε). A series is divergent if no such number N exists for some particular value of ε. From condition (3.11) it is easy to show that:

(i) If the series $\sum_{r=1}^{\infty} u_r$ is convergent and has the sum S, the series $a+\sum_{r=1}^{\infty} u_r$, where a is any constant, is convergent and has the sum $a+S$.

(ii) If the series $\sum_{r=1}^{\infty} u_r$ is convergent and has the sum S, the series $\sum_{r=m}^{\infty} u_r$ where m is some finite integral number, is convergent and has the sum $S-\sum_{r=1}^{m-1} u_r$.

(iii) If the series $\sum_{r=1}^{\infty} u_r$ is divergent, then so are the series $a+\sum_{r=1}^{\infty} u_r$ and $\sum_{r=m}^{\infty} u_r$ as described in (i) and (ii).

(iv) If the series $\sum_{r=1}^{\infty} u_r$ is convergent to a sum S, and the series $\sum_{r=1}^{\infty} v_r$ is convergent to a sums σ, then the series $\sum_{r=1}^{\infty} (u_r \pm v_r)$ are convergent having sums $S \pm \sigma$.

(v) If either of the series $\sum_{r=1}^{\infty} u_r$, $\sum_{r=1}^{\infty} v_r$ is divergent, and the other convergent, then the series $\sum_{r=1}^{\infty} (u_r+v_r)$ is divergent.

(vi) If k is any finite non-zero constant then the series $\sum_{r=1}^{\infty} ku_r$ has the same convergence properties as the series $\sum_{r=1}^{\infty} u_r$.

The properties (i), (ii), (iii) mean that we can add to or subtract from a series any finite quantity or finite number of terms without altering the convergence properties of the series.

The property (iv) follows from Ch. 1 eq. (1.11). We can use eq. (3.11) to give a formal proof. Let $S_n = \sum_{r=1}^{n} u_r$ and $\sigma_n = \sum_{r=1}^{n} v_r$; then if ε is a small positive number, we can find numbers N_1 and N_2 such that

$$|S_n - S| < \tfrac{1}{2}\varepsilon \quad \text{if} \quad n > N_1,$$

and

$$|\sigma_n - \sigma| < \tfrac{1}{2}\varepsilon \quad \text{if} \quad n > N_2,$$

using $\frac{1}{2}\varepsilon$ instead of ε in eq. (3.11). Let N be the larger of N_1 and N_2; then

$$\begin{aligned} |(S_n \pm \sigma_n) - (S \pm \sigma)| &= |(S_n - S) \pm (\sigma_n - \sigma)| \\ &\leqslant |S_n - S| + |\sigma_n - \sigma| < \varepsilon \quad \text{if} \quad n > N, \end{aligned}$$

and therefore

$$\lim_{n\to\infty} (S_n \pm \sigma_n) = S \pm \sigma.$$

One further property, which requires a little more explanation than the others, is

(vii) If the series $\sum_{r=1}^{\infty} u_r$ is convergent, it follows that

$$\lim_{n\to\infty} u_n = 0. \tag{3.12}$$

Since $u_n = S_n - S_{n-1}$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0,$$

if the series is convergent. Thus the condition (3.12) is *necessary* for convergence. It can be shown however that it is not *sufficient*, since many series for which eq. (3.12) is satisfied can be shown to be divergent. A simple example will suffice. The series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} + \dots, \tag{3.13}$$

has partial sum

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}.$$

In S_n, each term except the last is greater than $1/\sqrt{n}$ and therefore

$$S_n \geqslant \frac{n}{\sqrt{n}} = \sqrt{n},$$

thus

$$\lim_{n\to\infty} S_n = +\infty.$$

So the series (3.13) is a divergent series although

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1/\sqrt{n}) = 0.$$

The necessity of the condition (3.12) means that if $\lim_{n\to\infty} u_n \neq 0$, the series $\sum_{r=1}^{\infty} u_r$ is divergent. This is the simplest test for the divergence of a series. For example, in the series

$$1 - 1 + 1 \dots,$$

$\lim_{n\to\infty} u_n \neq 0$; the series is therefore divergent, as we have already seen in § 1.1 of this chapter.

§ 3. Series of positive terms

The series $\sum_{r=1}^{\infty} u_r$ is said to be a series of positive terms, if u_n is positive for all values of n. For such a series the sum S_n increases with n, since

$$S_{n+1} - S_n \equiv u_{n+1} > 0. \tag{3.14}$$

Remembering properties (i)–(iii) of § 2 we will note here that the criteria for testing whether a series of positive terms is convergent need not be confined to series in which *all* the terms are positive. The series $\sum_{r=m}^{\infty} u_r$ behaves in the same way as the series $\sum_{r=1}^{\infty} u_r$ where m is some definite number. Thus if the terms $u_1, u_2, \ldots u_{m-1}$ of a series contain some negative terms, but all the remaining terms $r=m, m+1, \ldots$ are positive, then the series is one in which its terms are ultimately positive and the same tests may be used on such series as are used for a series of positive terms only. Thus we may broaden the condition (3.14) by discussing series for which

$$S_{n+1} - S_n \equiv u_{n+1} > 0, \tag{3.15}$$

for $n>m$, where m is some finite number. When eq. (3.15) is satisfied the series cannot oscillate and it must either converge or be properly divergent with $\lim_{n\to\infty} S_n = +\infty$.

There is a fundamental test for convergence of these series: if there is a finite number K such that $S_n<K$ for all values of n, then the series is convergent and its sum S obeys $S \leqslant K$. If no such finite number K exists, then the series is divergent.

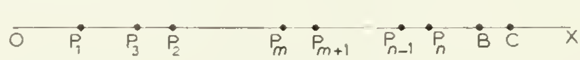

Fig. 3.1

This result is very important, as many other tests of convergence are based on it. It may be illustrated graphically in the following way. On a straight line OX (fig. 3.1) mark off points $P_1, P_2 \ldots, P_n$ such that

$$OP_1 = S_1, \quad OP_2 = S_2, \quad \ldots \quad OP_m = S_m, \quad \ldots \quad OP_n = S_n,$$

with the usual sign convention. Since eq. (3.15) holds for these points when $n>m$, then S_n increases with n. Therefore

$$OP_n > OP_{n-1} > \ldots > OP_{m+1} > OP_m.$$

If $S_n<K$, let C be the point with $OC=K$, and then all the points $P_1 \ldots P_m \ldots P_n$ lie to the left of C however large n may be. Therefore

there must be some point B to the left of C or coincident with it, beyond which P_n never passes, and which is such that $P_nB \to 0$ as $n \to \infty$; if $OB = \alpha$, then $\alpha \leqslant K$ and $\lim_{n\to\infty} S_n = \alpha$.

The analytical proof of this result will be given in § 4.2 Corollary 1; certain theorems on sequences which are required for the proof and which are also required in subsequent paragraphs, are given in §§ 4, 4.1. Before proceeding with these paragraphs, we will make use of the result to prove a very useful lemma.

§ 3.1. Lemma

If the positive term series (i) $\sum_{r=1}^{\infty} u_r$ is such that $u_1 > u_2 > u_3 \ldots$, the series (i) and the series (ii) $\sum_{r=1}^{\infty} mu_m$ where $m = 2^{r-1}$ are either both convergent or both divergent. The nth term of the series (ii)

$$\sum_{r=1}^{\infty} mu_m = u_1 + 2u_2 + 4u_4 + 8u_8 + \ldots,$$

is mu_m where $m = 2^{n-1}$. Let $\sigma_n = \sum_{r=1}^{n} mu_m$. Let S_N be the sum of the first N terms of series (i) where $N = 2^n - 1$, so that

$$S_N = u_1 + u_2 + \ldots + u_N = u_1 + (u_2 + u_3) + (u_4 + \ldots + u_7) + (u_8 + \ldots + u_{15}) + \ldots + (u_m + \ldots + u_N).$$

Then since $u_1 > u_2 > u_3 > \ldots$, we have

$$S_N < u_1 + 2u_2 + 4u_4 + \ldots + mu_m,$$

or $S_N < \sigma_n$. But also

$$S_N = u_1 + u_2 + (u_3 + u_4) + (u_5 + \ldots + u_8) + \ldots + (\ldots + u_m) + (u_{m+1} + \ldots + u_N).$$

and again, since $u_1 > u_2 \ldots$ and all the terms are positive, including those in the last bracket, we have

$$S_N > u_1 + \tfrac{1}{2}(2u_2 + 4u_4 + \ldots + mu_m),$$

or

$$S_N > u_1 + \tfrac{1}{2}(\sigma_n - \sigma_1) = \tfrac{1}{2}(u_1 + \sigma_n).$$

Thus altogether

$$\tfrac{1}{2}(u_1 + \sigma_n) < S_N < \sigma_n,$$

and $N = 2^n - 1 \to \infty$ as $n \to \infty$.

Suppose now that series (ii) is convergent, having a sum σ, so that as $n \to \infty$, $\sigma_n \to \sigma$. Since $S_N < \sigma_n$, we have $S_N < \sigma$ and therefore from the result in § 3, the series (i) is also convergent. On the other hand when the series (ii) is divergent, $\sigma_n \to \infty$ as $n \to \infty$ and since $S_N > \frac{1}{2}(u_1 + \sigma_n)$ we have $S_N \to \infty$ as $n \to \infty$ and therefore the series (i) is also divergent.

★ § 4. Sequences: convergence

In the preceding paragraphs we have seen that the convergence or divergence of any infinite series depends on the value of $\lim_{n\to\infty} S_n$, where $S_n = \sum_{r=1}^{n} u_r$. Any ordered set of numbers

$$S_1, S_2, \ldots, S_n,$$

is called a sequence and is denoted by (S_n). The value of each S_n in the sequence depends, of course, on the value of n and we note that these values are integral. If $\lim_{n\to\infty} S_n$ exists, it is called the *limit of the sequence* (S_n) as $n \to \infty$. As usual, the sequence (S_n) is said to possess a limit S if, for any $\varepsilon > 0$, there is a number N depending on ε such that

$$|S_n - S| < \varepsilon,$$

whenever $n > N$. Such a sequence is said to be a convergent sequence.

In order that the reader may not think that sequences are always presented as partial sums of series, we will now use the notation

$$\alpha_1, \alpha_2, \ldots \alpha_n$$

or (α_n) for any sequence, whose nth term α_n depends on the positive integral number n. The sequence (α_n) converges to the finite limit α as $n \to \infty$, if for any $\varepsilon > 0$, there is a number N, depending on ε such that

$$|\alpha_n - \alpha| < \varepsilon, \tag{3.16}$$

whenever $n > N$.

Divergent and oscillating sequences are defined in the same way as for series. The value of $\lim_{n\to\infty} \alpha_n$ may be found or verified by the methods of Ch. 1 §§ 2, 2.1.

Example 5

$$\alpha_n = \frac{n^2+3}{n^3-1}, \qquad \lim_{n\to\infty} \alpha_n = 0,$$

(α_n) is convergent.

Example 6

$$\alpha_n = \frac{n}{n+1}, \qquad \lim_{n\to\infty} \alpha_n = 1,$$

(α_n) is convergent.

Example 7

$\alpha_n = (-1)^n n$, (α_n) is infinitely oscillating.

Example 8

$\alpha_n = n^2$, (α_n) is properly divergent. ★

★ § 4.1. Bounds of a sequence

The *upper bound* U of a sequence (α_n) is the least number which is greater than or equal to every member of the sequence. This means that $\alpha_n \leqslant U$ for all values of n, and also if ε is any small positive number, then $U-\varepsilon$ which is less that U cannot be greater or equal to every α_n. Thus

$$\alpha_n > U - \varepsilon, \tag{3.17}$$

for at least one value of n.

If $\lim_{n\to\infty} \alpha_n = +\infty$, it is obvious from the above definition that the sequence (α_n) does not possess a finite upper bound. The upper bound is then said to be plus infinity. The *lower bound* L may be similarly defined: $\alpha_n \geqslant L$ for all values of n, while for every number $\varepsilon > 0$

$$\alpha_n < L + \varepsilon,$$

for at least one value of n.

If $\lim_{n\to\infty} \alpha_n = -\infty$ the lower bound is said to be minus infinity. ★

★ § 4.2. Monotonic sequences

If (α_n) is a sequence such that

$$\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \ldots \leqslant \alpha_n \leqslant \cdots,$$

then the sequence is said to be *monotonic increasing*. A sequence is said to be *strictly monotonic increasing* if consecutive terms are not equal, so that

$$\alpha_1 < \alpha_2 < \alpha_3 \ldots < \alpha_n < \cdots.$$

Similarly if (α_n) is a sequence such that

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n \geqslant \cdots,$$

it is said to be *monotonic decreasing*, and *strictly monotonic decreasing* if consecutive terms are not equal.

THEOREM 1. *A monotonic increasing sequence tends to its upper bound.*

Let (α_n) be a monotonic increasing sequence and let its upper bound be U. Then either U is finite or U is plus infinity.

If U is plus infinity then by eq. (3.17) some member α_N of the sequence (α_n) is greater than any given number however large. Hence α_{N+1}, $\alpha_{N+2}, \ldots$ are all greater than the chosen number, and $\lim_{n\to\infty} \alpha_n = +\infty$.

If U is finite we have

$$\alpha_n \leqslant U$$

for all values of n, whilst for $n=N$ say,

$$\alpha_N > U - \varepsilon.$$

But since (α_n) is monotonic increasing

$$\alpha_n \geqslant \alpha_N,$$

if $n>N$, and therefore

$$\alpha_n > U - \varepsilon \quad \text{or} \quad U - \alpha_n < \varepsilon,$$

for all $n>N$. This means that for any $\varepsilon>0$, there is a number N such that $|\alpha_n - U| < \varepsilon$ for all $n>N$. This is the criterion that (α_n) is a convergent sequence having a limit U.

COROLLARY 1. *If (α_n) is a monotonic increasing sequence, and there is a finite number K such that $\alpha_n<K$ for all n, then the sequence (α_n) converges to some number $U \leqslant K$.*

By the definition of the upper bound U of (α_n) it must be less than or equal to K, and by the above theorem the sequence (α_n) tends to its upper bound U.

Similarly it may be proved that a monotonic decreasing sequence tends to its lower bound L; and a corresponding corollary holds.

This Corollary 1 is precisely the result stated in § 3 for the convergence of the positive term series $\sum_{r=1}^{\infty} u_r$.

Since $S_n = \sum_{r=1}^{n} u_r$ is such that $S_n<K$ for all values of n, then by Corollary 1, the sequence (S_n) tends to its upper bound U. This means that

$$\lim_{n\to\infty} S_n = U,$$

where $U \leqslant K$. Thus the series is a convergent series having a sum $\leqslant K$. ★

§ 5. Tests for convergence: standard series

The first test that we shall use for positive term series is the comparison test, by which series are shown to converge or diverge more rapidly than known series. The geometrical series is one that is frequently used as the known series in comparison tests; another useful series is

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots. \tag{3.18}$$

We shall show that this series is convergent if $p>1$, divergent if $p\leqslant 1$. In this series (3.18) we have $u_n=n^{-p}$, and we see at once that if $p\leqslant 0$ the series is divergent since $\lim_{n\to\infty} u_n\neq 0$. If $p>0$, $\lim_{n\to\infty} u_n=0$, so the series might converge. For $p>0$, we write $p=1+\alpha$ and use the lemma of § 3.1. With $u_r=r^{-(\alpha+1)}$ we form the series $\sum_{r=1}^{\infty} mu_m$ with $m=2^{r-1}$ having partial sums σ_{n+1}. We find

$$\begin{aligned}\sigma_{n+1} &= \frac{1}{1^{1+\alpha}}+\frac{2}{2^{1+\alpha}}+\frac{4}{4^{1+\alpha}}+\cdots\frac{2^n}{2^{n(1+\alpha)}}\\ &= \frac{1}{1^{\alpha}}+\frac{1}{2^{\alpha}}+\frac{1}{(2^{\alpha})^2}+\cdots\frac{1}{(2^{\alpha})^n}.\end{aligned}$$

This series is a geometrical progression of common ratio $2^{-\alpha}$ and is therefore convergent provided $2^{-\alpha}<1$ or $\alpha>0$ and divergent if $2^{-\alpha}\geqslant 1$ or $\alpha\leqslant 0$. Since $p=1+\alpha$, the series (3.18) is convergent if $p>1$ and divergent if $p\leqslant 1$.

When $p=1$, the series is

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{r}+\ldots,$$

and is known as the harmonic series. It is a divergent series.

§ 6. Comparison test

In the tests which follow in this paragraph and §§ 6–9 we shall use the abbreviated notation $\sum u_n$ and $\sum v_n$ to denote two infinite series of positive terms and S_n, σ_n will denote their partial sums respectively.

§ 6.1. Comparison test: first form

If $\sum v_n$ is a convergent series and $u_n\leqslant kv_n$ for all values of n, where k is a finite positive constant, then $\sum u_n$ is convergent.

Let the sum of the series v_n be σ, so that $\lim_{n\to\infty} \sigma_n = \sigma$. Then $\sigma_n < \sigma$ since the terms are positive. Also

$$S_n = \sum_{r=1}^{n} u_r \leqslant \sum_{r=1}^{n} kv_r = k\sigma_n. \tag{3.19}$$

Thus $S_n \leqslant k\sigma_n < k\sigma$, so that S_n is less than a finite number $k\sigma$; the series u_n is therefore convergent.

If $\sum v_n$ is divergent and $u_n \geqslant kv_n$, where k is a finite positive constant, then $\sum u_n$ is divergent. Here

$$S_n = \sum_{r=1}^{n} u_r \geqslant \sum_{r=1}^{n} kv_r = k\sigma_n,$$

and since $\sum v_n$ is divergent, $\lim_{n\to\infty} \sigma_n = +\infty$, so that $\lim_{n\to\infty} S_n = +\infty$ and therefore $\sum u_n$ is divergent.

Example 9

The series $\sum (n+n^{\frac{1}{2}})^{-2}$ is convergent.

Here $u_n = (n+n^{\frac{1}{2}})^{-2}$, $v_n = n^{-2}$ are such that

$$\frac{1}{(n+n^{\frac{1}{2}})^2} < \frac{1}{n^2}.$$

From eq. (3.18) with $p > 1$, we know that $\sum v_n$ is a convergent series. Therefore $\sum u_n$ is convergent.

Example 10

The series $\sum n/(4n-1)^2$ is divergent.

Here $u_n = n/(4n-1)^2$, $v_n = n^{-1}$ are such that

$$\frac{n}{(4n-1)^2} > \frac{n}{(4n)^2} = \frac{1}{16n},$$

and again from eq. (3.18) with $p = 1$, $\sum n^{-1}$ is divergent.

§ 6.2. Comparison test: second form

The comparison test can be stated in the following alternative form.

If $\sum u_n$, $\sum v_n$ are two series of positive terms such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l, \tag{3.20}$$

where l is a finite but *non-zero* number, then the two series are either both convergent or both divergent.

The result (3.20) means that given a small positive number ε, we can find a number N such that

$$\left|\frac{u_n}{v_n} - l\right| < \varepsilon, \tag{3.21}$$

for $n>N$. The result (3.21) is now written in the form

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon. \tag{3.22}$$

Since u_n, v_n are positive, l is positive and we can find a small positive number ε such that $l-\varepsilon$ is positive. Using the first form of the test (§ 6.1) we then have: since $u_n<(l+\varepsilon)v_n$, $\sum u_n$ is convergent when $\sum v_n$ is convergent; but also, since $u_n>(l-\varepsilon)v_n$, $\sum u_n$ is divergent when $\sum v_n$ is divergent.

Example 11

$$u_n = (n+3)/(n^3-2).$$

Although the first term of this series is negative it can be treated as a positive term series and we see that when n is large, u_n behaves like $1/n^2$, so we choose $v_n=1/n^2$. Then

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n+3)n^2}{n^3-2} = 1.$$

But $\sum v_n = \sum (1/n^2)$ is convergent, so the series $\sum u_n$ is convergent.

The comparison test may be used for all series $\sum u_n$ when

$$u_n = \{P(n)\}^s / \{Q(n)\}^t,$$

where s and t are positive integers or fractions, and $P(n)$, $Q(n)$ are polynomials in n. The comparison series is $\sum v_n = \sum n^{-p}$ dealt with in § 5.

EXERCISE 3.1

Prove that $\sum u_n$ is convergent when u_n has the values given in Nos. 1–8.

1. $u_n = \dfrac{1}{(2n+1)^3}$.

2. $u_n = \dfrac{n}{(3n+4)^4}$.

3. $u_n = \dfrac{1}{n\sqrt{(n+1)}}$.

4. $u_n = \dfrac{1}{n(n+1)}$.

5. $u_n = \dfrac{n}{n^3-2}$.

6. $u_n = \dfrac{n+2}{n^3+3}$.

7. $u_n = \dfrac{n^{\frac{3}{2}}}{n^3 + n^2 + 1}$. **8.** $u_n = \dfrac{n^4 + 3n^2 - 2}{n^6 + 9}$.

Prove that $\sum u_n$ is divergent when u_n has the values given in Nos. 9–14.

9. $u_n = \dfrac{n^2 + 9}{n^2\sqrt{n} + 3}$. **10.** $u_n = \dfrac{\sqrt{(n + 1)}}{\sqrt{(n^2 + 1)}}$.

11. $u_n = \left(\dfrac{n^2 + 2}{n^3 + 3}\right)^{\frac{2}{3}}$. **12.** $u_n = \dfrac{n + 1}{2n^2 + 1}$.

13. $u_n = \dfrac{n^2 + 2}{n^3 - 5}$. **14.** $u_n = \dfrac{(n^2 + n - 1)^{\frac{1}{2}}}{(n^3 - 1)^{\frac{1}{2}}}$.

15. Using the lemma in § 3.1 show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log_{10} n)^{1+\alpha}},$$

is convergent if $\alpha > 0$, divergent when $-1 \leqslant \alpha < 0$. Show also that it is divergent when $\alpha < -1$.

§ 7. Ratio tests

If we are unable to find a series $\sum v_n$ with which to compare the series $\sum u_n$ for convergence or divergence, then tests using the series $\sum u_n$ only are used.

§ 7.1. D'Alembert's ratio test

This test can be applied to a series $\sum u_n$ of positive terms:
(i) If $u_n/u_{n+1} > k$, where k is a constant greater than 1, then $\sum u_n$ is convergent.
(ii) If $u_n/u_{n+1} < 1$, then $\sum u_n$ is divergent.

If we assume the conditions in (i) to hold, then

$$u_2 < k^{-1}u_1, \qquad u_3 < k^{-2}u_1, \ldots\ u_n < k^{-n+1}u_1, \ldots;$$

since $k > 1$,

$$\sum_{n=1}^{\infty} k^{-n+1}u_1 = ku_1 \sum_{n=1}^{\infty} k^{-n},$$

is a convergent geometrical progression; the series $\sum u_n$ is therefore convergent by the comparison test. The test (ii) can be proved similarly.

Remembering the results of § 2 of this chapter, we must realise that

we can ignore any finite number of terms in any series which do not satisfy the given conditions. Thus, although we have assumed the results (i) and (ii) to be true for all values of n, the test still holds if (i) and (ii) are true for all values of n greater than some finite integral number m.

D'Alembert's ratio test may also be stated in the following limit form, which is more useful in application: if

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l, \tag{3.23}$$

then $\sum u_n$ is convergent if $l>1$ and $\sum u_n$ is divergent if $l<1$.

The formal proof of this result, using the original form of the test follows similar lines to that used for the limit form of the comparison test. The proof is therefore left as an exercise for the reader.

The result of § 2 (vii), namely that $u_n \leqslant u_{n+1}$ gives divergence, enables us to remember that $l<1$ in eq. (3.23) gives divergence.

Example 12

When $u_n = 1/n!$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} = n+1.$$

Therefore $l>1$ in eq. (3.23) and $\sum u_n$ is convergent.

Example 13

Find the values of x for which $\sum_{n=1}^{\infty} \{(n+1)/(n+2)\}x^n$ converges.

We have a positive term series if x is positive, and

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)(n+3)}{(n+2)(n+2)x} = \frac{1}{x}.$$

Therefore $\sum u_n$ is convergent if $0<x<1$, and divergent if $x>1$. When $x=1$, the series is divergent from the comparison test.

When a series $\sum u_n$ gives $l=1$ in eq. (3.23) the test breaks down and the following test can be tried.

§ 7.2. Raabe's test

This test can be applied to a series $\sum u_n$ of positive terms: if

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l, \tag{3.24}$$

then $\sum u_n$ is convergent if $l>1$, and $\sum u_n$ is divergent if $l<1$.

Note that here we have given the test immediately in the limit form, and the limit involves $(u_n/u_{n+1})-1$ which will tend to zero as $n \to \infty$ if D'Alembert's test fails.

This test and D'Alembert's test are special cases of a more general test which is sometimes spoken of as Kummer's test. We will therefore prove Kummer's test.

§ 7.3. Kummer's test

$\sum_{n=0}^{\infty} (a_n)^{-1}$ is a divergent series of positive terms; $\sum u_n$ is a series of positive terms. If for $n>m$, where m is a fixed positive integer

$$\text{(i)} \quad \frac{a_n u_n}{u_{n+1}} - a_{n+1} > k, \tag{3.25}$$

where k is a constant >1, then $\sum u_n$ is convergent,

$$\text{(ii)} \quad \frac{a_n u_n}{u_{n+1}} - a_{n+1} < 0, \tag{3.26}$$

then $\sum u_n$ is divergent.

Consider eq. (3.25); we have for $n=m+1, m+2 \ldots$,

$$\begin{aligned} a_{m+1}\, u_{m+1} - a_{m+2}\, u_{m+2} &> k\, u_{m+2}, \\ a_{m+2}\, u_{m+2} - a_{m+3}\, u_{m+3} &> k\, u_{m+3}, \\ \cdots \qquad \cdots \qquad &\cdots \\ a_{m+N-1}\, u_{m+N-1} - a_{m+N}\, u_{m+N} &> k\, u_{m+N}, \end{aligned}$$

so that by addition

$$a_{m+1}\, u_{m+1} - a_{m+N}\, u_{m+N} > k \sum_{r=2}^{N} u_{m+r}. \tag{3.27}$$

Since $a_{m+N}\, u_{m+N}$ is positive, and using $S_n = \sum_{r=1}^{n} u_r$, eq. (3.27) gives

$$S_{m+N} - S_{m+1} < \frac{a_{m+1}\, u_{m+1}}{k},$$

hence

$$S_{m+N} < S_{m+1} + \frac{a_{m+1}\, u_{m+1}}{k}. \tag{3.28}$$

Since m is a fixed positive integer, the right hand side of eq. (3.28) is a positive constant (K say). So S_{m+N} $(N=1, 2, \ldots)$ is a monotonic increasing

sequence bounded above by K, and therefore satisfies Corollary 1 § 4.2. Therefore u_n is convergent.

The condition (3.26) gives

$$a_{n+1}\,u_{n+1} > a_n\,u_n > a_{n-1}\,u_{n-1} > \dots > a_{m+1}\,u_{m+1},$$

where m is some fixed positive integer. Thus $a_{m+1}\,u_{m+1}$ is a finite constant (C say), and therefore

$$u_n > C/a_n.$$

But $\sum a_n^{-1}$ is a divergent series and so by the comparison test, $\sum u_n$ is divergent.

Again this test can be given in the limit form. If $\sum a_n^{-1}$ is a divergent series of positive terms and

$$\lim_{n\to\infty}\left(\frac{a_n\,u_n}{u_{n+1}} - a_{n+1}\right) = l, \tag{3.29}$$

then $\sum u_n$ is convergent if $l>1$, and $\sum u_n$ is divergent if $l<1$.

The formal proof of this theorem, based on conditions (3.25) and (3.26) is left to the reader.

We see immediately that putting $a_n=n$ gives Raabe's test, whilst $a_n=1$ gives D'Alembert's test.

Example 14

Prove that the series

$$1 + \frac{a\cdot b}{c\cdot d}\,x + \frac{a(a+1)\,b(b+1)\cdot}{c(c+1)\,d(d+1)}\,x^2 + \dots, \qquad (a, b, c, d > 0),$$

converges when $0<x<1$, that it converges when $x=1$ provided that $c+d>a+b+1$, and that it diverges when $x=1$ provided that $c+d<a+b+1$.

Provided x is positive, this is a positive term series. For convenience we denote the first term of the series by u_0 instead of by u_1, this is equivalent to thinking of the first term as a constant added to the series. We then have

$$\frac{u_n}{u_{n+1}} = \frac{(c+n)(d+n)}{(a+n)(b+n)x},$$

or

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Thus by D'Alembert's ratio test (3.23), the series is convergent if $0<x<1$.

When $x=1$, D'Alembert's test breaks down, but then

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left\{\frac{cd - ab + (c+d-a-b)n}{(a+n)(b+n)}\right\},$$

so that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = c + d - a - b.$$

Thus by Raabe's test (3.24), the series is convergent if $c+d>a+b+1$ and divergent if $c+d<a+b+1$.

EXERCISE 3.2

1. Prove that each of the series

$$\Sigma (n+1)x^n, \qquad \Sigma \frac{n+1}{n+2} x^n, \qquad \Sigma \frac{n+1}{(n+2)(n+3)} x^n,$$

converges when $0 < x < 1$, but diverges when $x = 1$.

2. Prove that the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots,$$

converges for all positive values of x.

3. Show that each of the series

$$1 + \frac{a}{2}x + \frac{a+1}{2\cdot 4}x^2 + \frac{a+2}{2\cdot 4\cdot 6}x^3 + \dots, \quad (a > 0),$$

and

$$1 + \frac{a}{2}x + \frac{2(a+1)}{2\cdot 4}x^2 + \frac{3(a+2)}{2\cdot 4\cdot 6}x^3 + \dots, \quad (a, b > 0),$$

converges for any positive value of x.

4. Show that the series

$$\frac{a}{b}x + \frac{2(a+1)}{b^2}x^2 + \frac{3(a+2)}{b^3}x^3 + \dots, \quad (a > 0),$$

converges when $0 < x < b$ and diverges when $x \geqslant b$.

5. Prove that each of the series

$$\Sigma \frac{n!}{3\cdot 5\cdots(2n+1)} x^n, \qquad \Sigma \frac{(n+1)!}{3\cdot 5\cdots(2n-1)} x^n,$$

converges for $0 < x < 2$ and diverges when $x \geqslant 2$.

6. Examine for convergency the series

$$\Sigma\, n!\left(\frac{x}{n}\right)^n, \qquad \Sigma \frac{(2n)!\,x^n}{(n!)^2}, \qquad \text{when } x > 0.$$

§ 8. Cauchy's root test

This test can be applied to a series $\sum u_n$ of positive terms:

If for $n>m$, where m is a fixed positive number

$$\text{(i)} \quad u_n^{1/n} < k, \tag{3.30}$$

where k is a constant <1, then $\sum u_n$ is convergent,

$$\text{(ii)} \quad u_n^{1/n} > 1, \tag{3.31}$$

then $\sum u_n$ is divergent.

Consider eq. (3.30), we have for $n=m+r$, $u_{m+r}<k^{m+r}$, thus

$$\sum_{r=1}^{N} u_{m+r} < \sum_{r=1}^{N} k^{m+r} = k^m \sum_{r=1}^{N} k^r. \tag{3.32}$$

The right hand side of eq. (3.32) is a geometrical progression of common ratio k (<1) and is therefore convergent. Thus by the comparison test the series $\sum_{r=1}^{\infty} u_{m+r}$ is convergent, and therefore the series $\sum_{r=1}^{\infty} u_r$ is convergent.

The condition (3.31) gives $u_n>1$ for $n>m$ and therefore

$$\lim_{n\to\infty} u_n \neq 0,$$

so that the series u_n is divergent (§ 2.vii).

Expressed in the limit form, this test states that if

$$\lim_{n\to\infty} u_n^{1/n} = l,$$

then $\sum u_n$ is convergent when $l<1$, and $\sum u_n$ is divergent when $l>1$.

Example 15

If $a_n>0$ and $\lim_{n\to\infty} a_n^{1/n}=R^{-1}$, then $\sum_{n=1}^{\infty} a_n x^n$ is convergent when $0<x<R$. The result follows immediately from Cauchy's root test.

§ 9. Integral test

The integral test involves the evaluation of the integral

$$I_n = \int_1^n f(x)\,\mathrm{d}x, \tag{3.33}$$

as $n \to \infty$. If the $\lim_{n\to\infty} I_n$ is finite the integral is said to converge. Otherwise it is said to diverge. The integral test can then be stated in the following way.

If $f(x)$ is a positive function of x for $x > 1$, and decreases as x increases, then the positive term series $\sum_{n=1}^{\infty} f(n)$ and the integral I_n are either both convergent or both divergent.

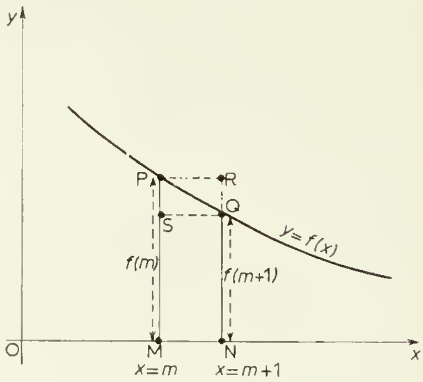

Fig. 3.2

Let $S_n = \sum_{n=1}^{n} f(n)$. Since $f(x) > 0$ for $x \geqslant 1$, the sequence S_n is monotonic increasing. Also from the graph $y = f(x)$, fig. 3.2 the sequence I_n is monotonic increasing.

Let P, Q be points on the curve $y = f(x)$ such that $x = m$, $m+1$ respectively. Then from the diagram

$$\text{area rect. MNQS} < \text{area MNQP} < \text{area rect. MNRP}.$$

Hence, since $\mathrm{MN} = 1$, $\mathrm{MP} = f(m)$, $\mathrm{NQ} = f(m+1)$ we have

$$f(m+1) < \int_m^{m+1} f(x)\,\mathrm{d}x < f(m). \tag{3.34}$$

Writing down (3.34) for all the special values $m = 1, 2, \ldots (n-1)$ and adding, we get

$$S_n - f(1) < I_n < S_n - f(n). \tag{3.35}$$

Now suppose that I_n is convergent so that $\lim_{n\to\infty} I_n = I$ where I is finite. Since (I_n) is monotonic increasing, we have $I_n \leqslant I$ (§ 4.2), and so from (3.35)

$$S_n < I_n + f(1) \leqslant I + f(1).$$

Therefore using Corollary 1 (§ 4.2), (S_n) is a monotonic increasing sequence which has a limit S such that $S \leqslant I + f(1)$. Therefore $\sum_{n=1}^{\infty} f(n)$ is a convergent series.

Similarly if the series is convergent, so that $\lim_{n\to\infty} S_n = S$, then from (3.35)

$$I_n < S_n - f(n) < S_n < S,$$

and hence (I_n) is a convergent sequence and $\lim_{n\to\infty} I_n = I$.

If $I_n \to \infty$ as $n \to \infty$, then from (3.35) $S_n > I_n + f(n)$, so $S_n \to \infty$; and if $S_n \to \infty$, we have from (3.35) $I_n > S_n + f(1)$ and so $I_n \to \infty$.

This completes the proof of the integral test.

Example 16

The integral test furnishes us with a simpler proof that the series (§ 5)

$$\frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots,$$

is convergent for $p > 1$ and divergent if $p < 1$. We have $f(n) = n^{-p}$. Thus

$$I_n = \int_1^n x^{-p}\,\mathrm{d}x = \left[\frac{x^{-p+1}}{1-p}\right]_1^n = \left[\frac{n^{-p+1} - 1}{1-p}\right] \to \frac{-1}{1-p},$$

as $n \to \infty$ if $p > 1$. If $p < 1$, $I_n \to \infty$ as $n \to \infty$ and so the series is divergent.

Example 17

If $f(n) = (n^2 + \alpha^2)^{-1}$ in the integral test, then

$$I_n = \int_1^n \frac{\mathrm{d}x}{x^2 + \alpha^2} = \left[\frac{1}{\alpha}\tan^{-1}\frac{x}{\alpha}\right]_1^n = \frac{1}{\alpha}\tan^{-1}\frac{n}{\alpha} - \frac{1}{\alpha}\tan^{-1}\frac{1}{\alpha}.$$

As $n \to \infty$, $I_n \to \frac{1}{2}\pi\alpha^{-1} - \alpha^{-1}\tan^{-1}\alpha^{-1}$ and is finite. Thus the series $\sum_{n=1}^{\infty}(n^2+\alpha^2)^{-1}$ is convergent.

§ 10. Alternating series

The series

$$u_1 - u_2 + u_3 - u_4 + \dots, \tag{3.36}$$

where each u_r is positive, is called an *alternating series*. There is a simple test of convergence for such series: that the terms decrease steadily in magnitude and $\lim_{n\to\infty} u_n = 0$.

Let us suppose that this condition is satisfied, then we have

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}),$$

and each term in this sum is positive; so that (S_{2n}) is a monotonic increasing sequence. Further

$$S_{2n} = u_1 - (u_2 - u_3) - \dots - (u_{2n-2} - u_{2n-1}) - u_{2n},$$

so that $0 < S_{2n} < u_1$ and therefore sequence (S_{2n}) has a finite limit as $n \to \infty$.

In the same way it can be shown that $0<S_{2n+1}<u_1$; so (S_{2n+1}) has a finite limit as $n\to\infty$. But

$$\lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} u_{2n+1} = 0.$$

Thus $\lim_{n\to\infty} S_{2n+1}=\lim_{n\to\infty} S_{2n}=S$ (say) and the series (3.36) is therefore convergent to the value S.

Example 18

The series

$$\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \dots, \tag{3.37}$$

is convergent if $p>0$, since $(n+1)^{-p}<n^{-p}$ and $\lim_{n\to\infty} n^{-p}=0$ if $p>0$. Compare this with the corresponding series of positive terms (§ 5), in particular when $p=1$, the harmonic series.

Example 19

The series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots,$$

is convergent.

§ 11. Series in general. Absolute convergence

If the series

$$\sum_{r=1}^{\infty} u_r = u_1 + u_2 + u_3 + \dots, \tag{3.38}$$

is a series of mixed positive and negative terms, then the series

$$\sum_{r=1}^{\infty} |u_r| = |u_1| + |u_2| + |u_3| + \dots, \tag{3.39}$$

formed by taking the modulus of each term in (3.38) is a series of positive terms. We cannot apply the tests of convergence given in (§§ 6–9) to the series (3.38), but we can apply these tests to the series (3.39). The series $\sum_{r=1}^{\infty} u_r$ is said to be *absolutely convergent* if the series $\sum_{r=1}^{\infty} |u_r|$ is convergent.

We can now prove that if a series is absolutely convergent then it is also convergent, but the converse is not necessarily true. If a series is convergent it is not necessarily absolutely convergent as we saw in

Example 18, comparing the series (3.37) with $p=1$ and the harmonic series.

Let

$$a_n = u_n \quad (u_n \geqslant 0)$$
$$= 0 \quad (u_n \leqslant 0),$$

and

$$b_n = -u_n \quad (u_n \leqslant 0)$$
$$= 0 \quad (u_n \geqslant 0).$$

Then clearly $a_n \geqslant 0$, $b_n \geqslant 0$ and

$$|u_n| = a_n + b_n, \qquad u_n = a_n - b_n.$$

From the first of these relations it follows that

$$a_n \leqslant |u_n|, \qquad b_n \leqslant |u_n|.$$

Since $\sum_{r=1}^{\infty} |u_r|$ is convergent, it follows from the comparison test that $\sum_{r=1}^{\infty} a_r$ is convergent and $\sum_{r=1}^{\infty} b_r$ is convergent. Hence by (iv) of § 2 $\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} (a_r - b_r)$ is convergent.

For series in general consisting of positive and negative terms, the tests given in §§ 6–9 become tests for absolute convergence.

§ 12. Power series

A series of the type

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \ldots, \tag{3.40}$$

where the coefficients a_r are independent of x is called a power series. As we have already seen in many examples and exercises in this chapter, the series will, in general, converge for certain values of x and diverge for others. Using D'Alembert's ratio test (§ 7.1), the series (3.40) is absolutely convergent when $l > 1$ where

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{a_{n-1}\, x^{n-1}}{a_n\, x^n} \right| = l.$$

It is divergent when $l < 1$.

Suppose $\lim_{n\to\infty} |a_{n-1}/a_n|$ exists and is equal to R, so that

$$\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = R \tag{3.41}$$

and

$$\lim_{n\to\infty} \left| \frac{a_{n-1}\, x^{n-1}}{a_n\, x^n} \right| = \frac{R}{|x|}. \tag{3.42}$$

Then the series (3.40) is absolutely convergent when $|x| < R$ and is divergent when $|x| > R$. Hence a power series is always *absolutely convergent* for all values of x in the open interval $)-R, R($. The value R is called the *radius of convergence* of the series; the reason for the expression 'radius' will be explained in Ch. 17. Whether a power series is convergent when $|x| = R$ will depend on the particular series.

Example 20

To consider the convergenc~ of the binomial expansion of $(1+x)^m$ where m is any positive or negative rational number.

We have

$$(1 + x)^m = 1 + \binom{m}{1} x + \ldots + \binom{m}{n} x^n + \ldots. \tag{3.43}$$

Using eq. (3.41)

$$\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{m - n + 1}{n} \right| = 1,$$

so that $R = 1$ and therefore the binomial expansion (3.43) is valid when $|x| < 1$. If $|x| > 1$ the series on the right hand side of (3.43) is not absolutely convergent. When $|x| = 1$ the convergence depends on the value of m; this is left as an exercise for the reader.

Example 21

Consider the series

$$\sum_{r=1}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \ldots.$$

We have

$$\lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n - 1)!} \right| = +\infty.$$

Thus $R = \infty$ and therefore the series is convergent for all values of x. Notice that this means that $\lim_{u\to\infty} |u_n| = 0$, which in this case becomes

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

EXERCISE 3.3

Test the series in Nos. 1–8 for convergence or divergence.
The series in Nos. 4–10 should be attempted after reading Chs. 4 and 5.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n+1)}}$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4}$.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{(n-1)}}$.

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

6. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n \log n\{\log(\log n)\}^2}$.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{(\log n)^2}{n^3}$.

9. Show that the series $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p}$ is convergent if $p > 1$, divergent if $p \leqslant 1$.

10. If $f(x)$ is an increasing function of x for $x \geqslant 1$, and if

$$S_n = \sum_{r=1}^{n} f(r), \qquad I_n = \int_1^n f(x)\,dx,$$

show that

$$I_n + f(1) \leqslant S_n \leqslant I_n + f(n).$$

By taking $f(x) = \log x$ find the value of

$$\lim_{n\to\infty} \left\{ \frac{\sum_{r=1}^{n} \log r}{n \log n} \right\}.$$

11. Show that the series

$$\sum_{n=0}^{\infty} n^2 x^n, \qquad \sum_{n=0}^{\infty} n^3 x^n,$$

are convergent for $-1 < x < 1$ and for no other real values of x.

CHAPTER 4

THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS

§ 1. Introduction

It will certainly occur to the reader of the preceding chapters who has had previous knowledge of differentiation and integration, that certain very important functions have been entirely ignored in these chapters. For example in Ch. 2 § 1 the standard integral of x^n, $n \neq -1$ has been given as $x^{n+1}/(n+1)$, but no mention has been made of an integral of x^n when $n = -1$. Also when considering the differentiation and integration of the trigonometric or circular functions, no mention was made of the corresponding hyperbolic functions, so that the simplest form of the integrals $\int (a^2+x^2)^{-\frac{1}{2}}\,\mathrm{d}x$ and $\int (x^2-a^2)^{-\frac{1}{2}}\,\mathrm{d}x$ which are usually thought of as standard integrals were not given. All these questions are related to each other and are bound up with the definitions of the logarithmic and exponential functions.

We are all familiar with the logarithm of a number to the base 10, the common or Briggian logarithm, and at first it may be difficult to see the connection between this familiar idea and the definition of the logarithmic function that we will shortly give. In writing a book of this nature, there are many ways open for the introduction of the logarithmic and exponential functions, nearly every one of which presents difficulties, partly because of the logical sequence of proof that is required throughout the book and partly because of the inherent mathematical difficulties, such as continuity and limits. The method we have chosen is that used by HARDY [1947] and also by COURANT [1937]; it has not often been used in books for physicists, but it provides a definition of the logarithmic function which arises naturally out of problems in physics, chemistry, economics and other subjects. In the course of the chapter we shall set down and correlate some of the different definitions of the two functions.

§ 1.1. Definition of the logarithmic function

We shall begin by considering an integral of x^n when $n = -1$. The definite integral is

$$z(x) = \int_a^x \frac{\mathrm{d}t}{t}, \tag{4.1}$$

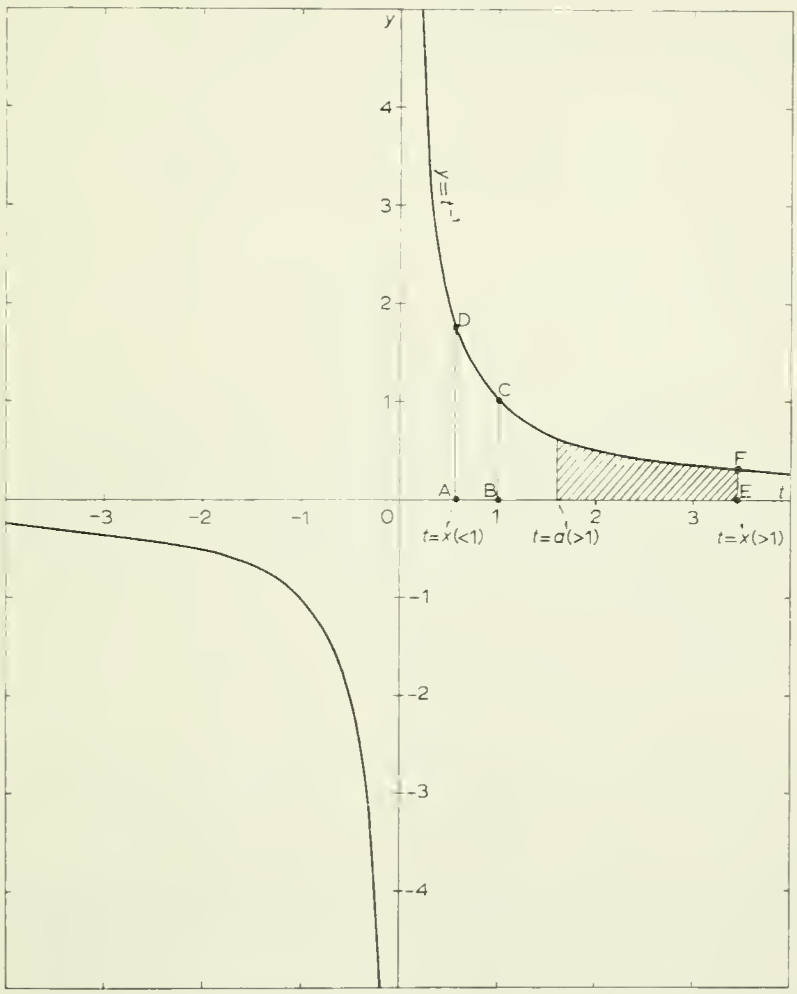

Fig. 4.1

and is the area under the rectangular hyperbola $y = t^{-1}$ between the ordinate $t = a$ and the variable ordinate $t = x$. This is the shaded area in fig. 4.1. The function $z(x)$ defined by eq. (4.1) has derivative

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}. \tag{4.2}$$

We define the *logarithmic function* or *natural logarithm* $\log x$ to be the function

$$\log x = \int_1^x \frac{\mathrm{d}t}{t}, \tag{4.3}$$

where the particular lower limit $a=1$ has been chosen to ensure that the logarithmic function obeys $\log 1=0$.

If we make the substitution $t=1/u$ in the integral in eq. (4.3) we obtain

$$\log x = \int_1^x \frac{\mathrm{d}t}{t} = -\int_1^{1/x} \frac{\mathrm{d}u}{u} = -\log\frac{1}{x}. \tag{4.4}$$

Using the result

$$\log x = -\int_1^{1/x} \frac{\mathrm{d}u}{u},$$

we see that as $x\to0$ the integral in this equation becomes $\int_1^\infty u^{-1}\,\mathrm{d}u$. By the integral test of Ch. 3 § 9, this integral behaves like $\sum_{n=1}^\infty n^{-1}$ and is therefore divergent. Thus as $x\to0$, $\log x$ becomes infinitely large and negative. Further referring to fig. 4.1 we see that when $0<x<1$, we have

$$\log x = \int_1^x \frac{\mathrm{d}t}{t} = -\int_x^1 \frac{\mathrm{d}t}{t},$$

and the latter integral without the negative sign is the positive value of the area ABCD.

Thus for $0<x<1$, $\log x$ is negative and as $x\to1$, AD moves up to BC and the magnitude of the area ABCD$\to0$. When $x=1$ the value of the integral in eq. (4.3) is obviously zero. When $x>1$ the integral in (4.3) is the magnitude of the area BEFC and steadily increases in numerical value as the ordinate FC moves farther away from BC, that is as x increases. Thus as x increases from a small positive number to $+\infty$ the value of $\log x$ steadily increases from $-\infty$ through the value zero at $x=1$, and tending to $+\infty$ as $x\to+\infty$.

When $x<0$, since there is a discontinuity in the integrand at $x=0$ the range of integration x to 1 must be divided into two parts at $x=0$ and since we have already shown that $\int_0^1 t^{-1}\,\mathrm{d}t$ is infinite, the integral when

x is negative, has no real value. Moreover since $\log x$ satisfies eq. (4.2) that is,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}, \tag{4.5}$$

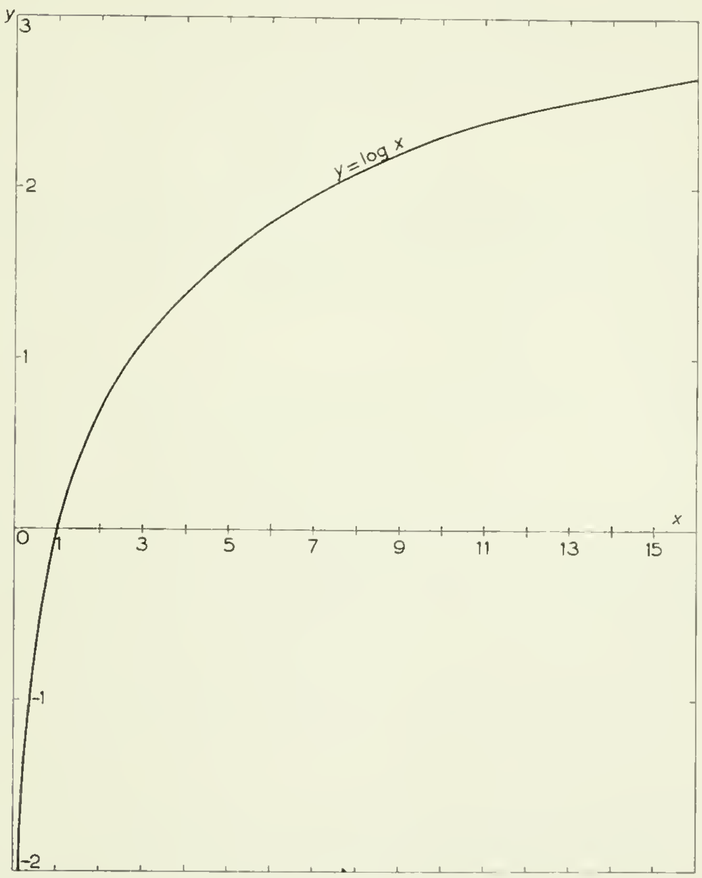

Fig. 4.2

then $\log x$ is a continuous function where it exists and is monotonic increasing when $x > 0$. This means that there is only one number x with a given logarithm; so if two numbers have equal logarithms they are equal.

From eq. (4.5) the derivative of $\log x$ is large when x is small and gradually decreases as x increases; as $x \to \infty$, $\mathrm{d}(\log x)/\mathrm{d}x \to 0$. From these results we deduce that the graph of $\log x$ is as it is shown in fig. 4.2.

Also from eq. (4.5) we see that we can write an *indefinite* integral of x^{-1} as

$$\int \frac{\mathrm{d}x}{x} = \log x,$$

provided $x>0$. If $x<0$, then writing $x=-u$, where u is positive, in the integral we get

$$\int \frac{\mathrm{d}x}{x} = \int \frac{-\mathrm{d}u}{-u} = \int \frac{\mathrm{d}u}{u} = \log u = \log(-x).$$

Thus when $x<0$ we have

$$\int \frac{\mathrm{d}x}{x} = \log |x|, \tag{4.6}$$

and in general an indefinite integral of x^{-1} is written in this form since it is true whether x is positive or negative.

§ 1.2. The addition theorem

The logarithmic function obeys the following fundamental law

$$\log ab = \log a + \log b. \tag{4.7}$$

In eq. (4.3) we make the substitution $t=au$ where a is a constant. We have $\mathrm{d}t/\mathrm{d}u=a$; when $t=1$, $u=1/a$, when $t=x$, $u=x/a=b$ (say). So $x=ab$; thus

$$\log ab = \log x = \int_1^x \frac{\mathrm{d}t}{t} = \int_{1/a}^b \frac{\mathrm{d}u}{u},$$

or

$$\log ab = \int_1^b \frac{\mathrm{d}u}{u} - \int_1^{1/a} \frac{\mathrm{d}u}{u} = \log b - \log(1/a) = \log b + \log a,$$

using eq. (4.4), proving the result.

For any number of arbitrary positive numbers $a_1, a_2, \ldots, a_n$ it follows that

$$\log(a_1\, a_2\, a_3 \ldots a_n) = \log a_1 + \log a_2 + \ldots + \log a_n, \tag{4.8}$$

and if $a_1=a_2=a_3=\ldots=a_n$, we have

$$\log a^n = n \log a, \tag{4.9}$$

n being a positive integer.

If we now put $a^{1/n}=\alpha$, so that $a=\alpha^n$, then using (4.9)

$$\log a = \log \alpha^n = n \log \alpha,$$

or

$$\log \alpha = \log a^{1/n} = \frac{1}{n} \log a. \tag{4.10}$$

If then m is a positive integer, we have

$$\frac{m}{n} \log a = m \log a^{1/n} = \log a^{m/n}. \tag{4.11}$$

The result

$$\log a^r = r \log a, \tag{4.12}$$

is thus established for all positive rational values of r; for $r=0$ it is correct since $\log 1=0$. For negative rational values of r we prove it as follows. When $r<0$, write $r=-s$ so that $s>0$. Then, using eqs. (4.4) and (4.12) we have

$$\log a^r = \log a^{-s} = \log\left(\frac{1}{a^s}\right) = -\log a^s = -s \log a,$$

since $s>0$. But $s=-r$ and so this gives

$$\log a^r = r \log a,$$

when r is a negative rational number.

The results given in eqs. (4.4), (4.7)–(4.12) are familiar properties of the common logarithm of a number.

§ 2. The number e

We now introduce a number, usually denoted by e, which is one of the fundamental constants of mathematical analysis.

We define e to be the number which is such that its logarithm is unity, or

$$\log \mathrm{e} = \int_1^{\mathrm{e}} \frac{\mathrm{d}t}{t} = 1. \tag{4.13}$$

Putting $a=\mathrm{e}$ in eq. (4.12) with r replaced by x, we have

$$\log \mathrm{e}^x = x \log \mathrm{e} = x, \tag{4.14}$$

whenever x is a positive or negative rational number.

Thus if we can express any number y in the form $y=\mathrm{e}^x$, then from eq. (4.14)

$$\log y = \log \mathrm{e}^x = x,$$

or, the logarithm of a number is the power to which one must raise e in order to get the number. This is essentially the same definition of a logarithm as that frequently used for the common logarithm; however instead of using a base 10, as for the common logarithm, we are using as a base the number e, defined in eq. (4.13). The direct relation between the logarithm of a number to the base e and the logarithm of a number to the base 10 will be seen in the next paragraph.

§ 3. The exponential function and its properties

In the preceding paragraph we have seen that if x is any positive or negative rational number and y is such that $y = e^x$, then

$$\log y = x.$$

We call e^x the exponential function of x and we shall frequently write it as "$\exp x$" for convenience of printing. This result shows that the logarithmic function and the exponential function are inverse functions, provided x is rational. We now define the exponential function $\exp x$ for all *real values* of the variable x as that function which is the inverse of the logarithmic function. In other words if

$$x = \log y, \tag{4.15}$$

then

$$y = e^x. \tag{4.16}$$

We have seen in § 1.1, noting that x and y have been interchanged, that as the number y increases steadily from 0 to $+\infty$, the value of the function $x = \log y$ increases steadily from $-\infty$ to $+\infty$, and the graph of $x = \log y$ given in fig. 4.3 is the same as that of fig. 4.2, but with the axes x and y interchanged. This means a reflection in the line OC bisecting the first quadrant, of the curve $y = \log x$ of fig. 4.2, shown in a broken line in fig. 4.3. But $x = \log y$ is the same as $y = \exp x$, so the unbroken line in fig. 4.3 is the graph of $y = \exp x$.

Since $\log y$ is a continuous function of y, it follows from Ch. 1 § 4.5 that $\exp x$ is a continuous function of x.

The derivative of the exponential function is found from the definition of the logarithm. For when

$$y = e^x, \tag{4.17}$$

then $x = \log y$ and $dx/dy = 1/y$. Thus

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = y. \tag{4.18}$$

These eqs. (4.17) and (4.18) express a very important property of the exponential function, namely that the derivative of the exponential function is the function itself.

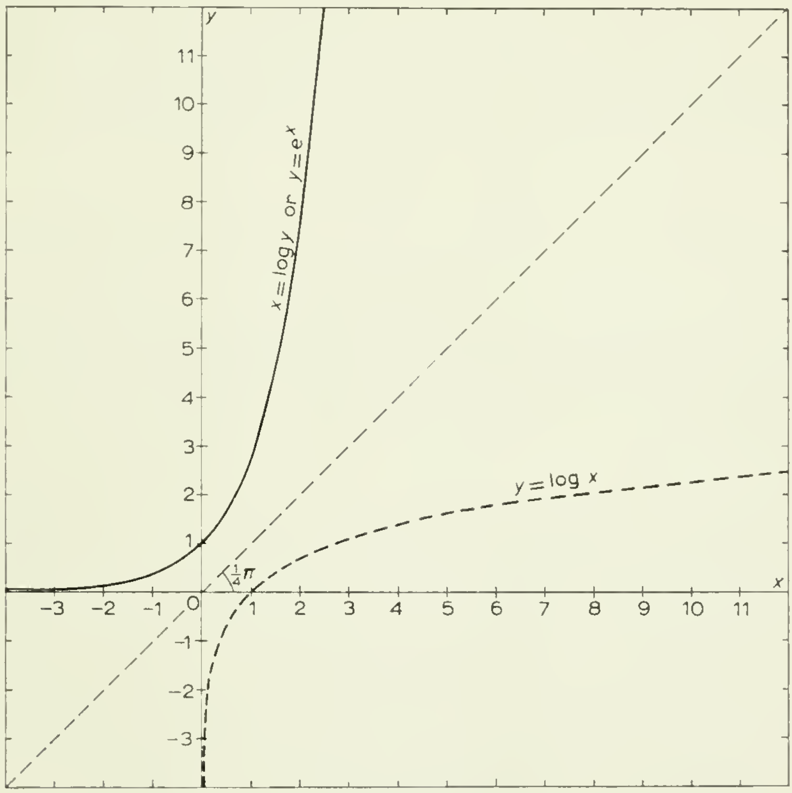

Fig. 4.3

More generally if a is any constant and

$$y = e^{ax}, \tag{4.19}$$

then using $t = ax$ and $\dfrac{dy}{dx} = \dfrac{dy}{dt}\,\dfrac{dt}{dx}$, we have

$$\frac{dy}{dx} = a\,e^{ax}. \tag{4.20}$$

Further, if α and β are any two real numbers, and we write

$$e^{\alpha} = a, \qquad e^{\beta} = b \quad \text{and} \quad e^{\alpha+\beta} = c,$$

we have

$$\alpha = \log a, \qquad \beta = \log b \quad \text{and} \quad \alpha + \beta = \log c.$$

But from eq. (4.7),

$$\log c = \alpha + \beta = \log a + \log b = \log ab.$$

Thus $c=ab$ or

$$e^{\alpha+\beta} = e^{\alpha} e^{\beta}, \tag{4.21}$$

and this is true when α, β are any real numbers. This is a familiar result in elementary algebra when α and β are rational numbers.

In particular when $\beta=-\alpha$ eq. (4.21) gives

$$e^{\alpha} e^{-\alpha} = e^{0} = 1,$$

so that

$$e^{-\alpha} = 1/e^{\alpha}. \tag{4.22}$$

The results contained in the eqs. (4.17)–(4.22) constitute the fundamental properties of the exponential function.

§ 3.1. The general function a^x

If a is any arbitrary positive number, then from the graph of $y=\log x$ (fig. 4.2) it is possible to find $\log a$. This means that we can write

$$a = \exp(\log a).$$

We then define the function a^x for all real values of x by the equation

$$a^x = \{\exp(\log a)\}^x.$$

Using $y=a^x$, we have

$$\log y = \log a^x = x \log a, \tag{4.23}$$

or

$$y = a^x = \exp(x \log a) = e^{x \log a}. \tag{4.24}$$

But by analogy with e^x, if we assume that $y=a^x$, then x is the logarithm of y to the base a, or

$$x = \log_a y. \tag{4.25}$$

Comparing eqs. (4.23) and (4.25) we have

$$\log_a y = \log_e y/\log_e a, \tag{4.26}$$

where to emphasize the distinction in the logarithms, the letter e has been inserted as the base in the natural logarithms in eq. (4.26). Putting $y=\mathrm{e}$ in eq. (4.26) we get

$$\log_a \mathrm{e} = 1/\log_{\mathrm{e}} a. \tag{4.27}$$

Putting $a=10$ in eqs. (4.26) and (4.27) we obtain the 'common logarithms', related to the natural logarithms by the results

$$\log_{10} \mathrm{e} = 1/\log_{\mathrm{e}} 10, \tag{4.28}$$

and then

$$\log_{10} y = \frac{\log_{\mathrm{e}} y}{\log_{\mathrm{e}} 10} = \log_{10} \mathrm{e} \log_{\mathrm{e}} y. \tag{4.29}$$

It is easy to pass from one system of logarithms to another, using eq. (4.29), once either $\log_{\mathrm{e}} 10$ or $\log_{10} \mathrm{e}$ has been calculated.

We have called the logarithms to the base e the 'natural logarithm' and if no base letter is inserted this is the logarithm that is meant. The natural logarithm is also called the *Napierian logarithm.*

To determine the derivative of a^x we use eqs. (4.20), (4.24) and the chain rule, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x \log a} \log a = a^x \log a. \tag{4.30}$$

Further from eqs. (4.21) and (4.24) if α and β are real numbers

$$a^\alpha a^\beta = \exp(\alpha \log a) \exp(\beta \log a) = \exp\{(\alpha + \beta) \log a\} = a^{\alpha+\beta},$$

whilst

$$(a^\alpha)^\beta = \exp(\beta \log a^\alpha) = \exp(\alpha\beta \log a) = a^{\alpha\beta},$$

so the law of indices, familiar enough when the α, β are rational, holds for all real numbers α, β.

§ 4. The exponential function represented as a limit

We are now in a position to establish some well-known limiting formulae for the exponential function. We begin with the definition of the derivative of any function $f(z)$ with respect to z, namely eq. (1.19):

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}. \tag{4.31}$$

Taking $f(z)=\log z$ for which, by definition of $\log z$ we have $f'(z)=1/z$, eq. (4.31) becomes

$$\frac{1}{z}=\lim_{h\to 0}\frac{\log(z+h)-\log z}{h}=\lim_{h\to 0}\left\{\frac{1}{h}\log\left(1+\frac{h}{z}\right)\right\}.$$

Putting $z=1/x$, this result becomes

$$x=\lim_{h\to 0}\frac{1}{h}\log(1+hx)=\lim_{h\to 0}\log(1+hx)^{1/h}. \tag{4.32}$$

Now the exponential function is a continuous function; so we have

$$\begin{aligned}\exp x&=\exp\{\lim_{h\to 0}\log(1+hx)^{1/h}\}\\ &=\lim_{h\to 0}\exp\log(1+hx)^{1/h}=\lim_{h\to 0}(1+hx)^{1/h}.\end{aligned} \tag{4.33}$$

Alternatively, since in eq. (4.31) h can tend to zero through positive or negative values, we have putting $h=1/\xi$ in eq. (4.32)

$$x=\lim\log\left(1+\frac{x}{\xi}\right)^{\xi},$$

where in the limit $\xi\to+\infty$ or $-\infty$. But again

$$\exp x=\exp\lim\log\left(1+\frac{x}{\xi}\right)^{\xi}=\lim\exp\log\left(1+\frac{x}{\xi}\right)^{\xi}=\lim\left(1+\frac{x}{\xi}\right)^{\xi},$$

as $\xi\to+\infty$ or $-\infty$. Thus

$$e^x=\lim_{\xi\to+\infty}\left(1+\frac{x}{\xi}\right)^{\xi}=\lim_{\xi\to-\infty}\left(1+\frac{x}{\xi}\right)^{\xi}.$$

In particular, if ξ takes integral values n only,

$$e^x=\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^{n}=\lim_{n\to-\infty}\left(1+\frac{x}{n}\right)^{n}=\lim_{n\to+\infty}\left(1-\frac{x}{n}\right)^{-n}. \tag{4.34}$$

The results in eqs. (4.33) and (4.34) are the well-known alternative ways of expressing the exponential function e^x. In particular if $x=1$, we have from eqs. (4.33) and (4.34) respectively.

$$e=\lim_{h\to 0}(1+h)^{1/h}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}. \tag{4.35}$$

★ § 4.1. THE NUMBER e AS AN INFINITE SERIES

The equation (4.35) can be used to express e as an infinite series. By the binomial theorem with n a positive integer, we have

$$\left(1+\frac{1}{n}\right)^n=\left\{1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\dots\binom{n}{r}\frac{1}{n^r}+\dots\right.$$
$$\left.+\binom{n}{n-1}\frac{1}{n^{n-1}}+\frac{1}{n^n}\right\}.$$

The rth term in this expansion is

$$\frac{n!}{r!(n-r)!n^r}=\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!n^r}$$
$$=\frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{r-1}{n}\right),$$

and is positive. As n increases each factor in this expression increases, so that the term itself increases. Also the number of terms in the series increases. Hence $(1+n^{-1})^n$ is a monotonic increasing sequence and so either tends to a finite limit or to $+\infty$ as $n\to\infty$. Using a method of comparison, we see that whatever value n may have, the series is such that

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\dots\left\{\frac{1}{r!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\right\}+\dots$$
$$<1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\frac{1}{r!}+\dots$$
$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\dots\frac{1}{2^r}+\dots.$$

This last expression is the sum of unity and a geometrical progression of common ratio $\frac{1}{2}$. Its sum to infinity is therefore $1+2=3$. The original expansion is obviously >2 and thus for all values of n

$$2<\left(1+\frac{1}{n}\right)^n<3.$$

Therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!} + \dots,$$

is a convergent series having a finite positive limit. This limit we have denoted by e, and we have proved that $2<\mathrm{e}<3$. ★

§ 5. Differentiation involving exponentials and logarithms

Rules for the differentiation of composite functions involving exponentials and logarithms will be illustrated by examples.

Example 1

If $y=\exp f(x)$, then putting $u=f(x)$, so that $y=\mathrm{e}^u$, we have by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^u \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^{f(x)} f'(x). \tag{4.36}$$

For instance, if

$$y = \mathrm{e}^{\sqrt{x}},$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{\sqrt{x}}}{2\sqrt{x}}.$$

Example 2

Similarly if $y=\log f(x)$, then putting $u=f(x)$, so that $y=\log u$, we have by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{f'(x)}{f(x)}. \tag{4.37}$$

For instance, if

$$y = \log(3x^2 - 2x + 1),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x-2}{3x^2-2x+1}.$$

Example 3

Differentiate $y=(x^2-2)(x+1)^{\frac{1}{2}}/(2x-5)^2$ with respect to x.

If two quantities are equal, their logarithms are equal; so

$$\log y = \log(x^2-2) + \tfrac{1}{2}\log(x+1) - 2\log(2x-5).$$

Differentiating both sides of this equation with respect to x, using eq. (4.37), we have

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2-2} + \frac{1}{2(x+1)} - \frac{4}{2x-5} = \frac{2x^3-25x^2-8x+6}{2(x^2-2)(x+1)(2x-5)}.$$

Therefore multiplying through by y, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^3 - 25x^2 - 8x + 6}{2(2x-5)^3\sqrt{(x+1)}}.$$

More generally if $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ are all differentiable functions of x, and

$$y = \frac{\prod_{r=1}^{m} u_r}{\prod_{r=1}^{n} v_r} = \frac{u_1 u_2 \cdots u_m}{v_1 v_2 \cdots v_n}, \tag{4.38}$$

then by equating the logarithms of these equal quantities, we have

$$\log y = \log \prod_{r=1}^{m} u_r - \log \prod_{r=1}^{n} v_r = \sum_{r=1}^{m} \log u_r - \sum_{r=1}^{n} \log v_r.$$

Differentiating both sides with respect to x we get

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{r=1}^{m} \frac{1}{u_r}\frac{\mathrm{d}u_r}{\mathrm{d}x} - \sum_{r=1}^{n} \frac{1}{v_r}\frac{\mathrm{d}v_r}{\mathrm{d}x}.$$

Multiplying through by y in the form (4.38) gives the value of $\mathrm{d}y/\mathrm{d}x$. This process is usually referred to as *logarithmic differentiation.*

A similar method may be used to find the derivative of a function, defined by u^v, where $u=f(x)$, $v=g(x)$. Writing $y=u^v$, we have

$$\log y = v \log u,$$

and then

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}\log u + \frac{v}{u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Example 4

When $y=x^x$, we have

$$\log y = x \log x,$$

and

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \log x + 1,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^x(\log x + 1).$$

Implicit functions involving exponentials and logarithms, or composite implicit functions may also be treated in a similar manner.

Example 5

Find $\mathrm{d}y/\mathrm{d}x$ when $\exp xy = x^3y$.

We have

$$xy = \log(x^3y) = 3\log x + \log y.$$

Therefore

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{3}{x} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(3 - xy)}{x(xy - 1)}.$$

EXERCISE 4.1

Differentiate with respect to x the functions given in Nos. 1–32, simplifying the answer where possible.

1. $\exp x^2$.

2. $\log\left\{\dfrac{\sqrt{(x^2+1)} - 1}{\sqrt{(x^2+1)} + 1}\right\}$.

3. $\sec\log\sqrt{(a^2 + x^2)}$.

4. $\log(\log x)$.

5. $x\exp(\sin 2x)$.

6. $(x^2 - x - 2)\,\mathrm{e}^{-2x}$.

7. $\log\{x + \sqrt{(a^2 + x^2)}\}$.

8. $\sec x\tan x + \log(\sec x + \tan x)$.

9. $\mathrm{e}^{-3x}\cos(2x - \frac{1}{4}\pi)$.

10. $\log\{(a\cos x + b)/(a + b\cos x)\}$.

11. $\log\{(a + b\tan x)/(a - b\tan x)\}$.

12. $\log(\sec^2 x + \sec x\tan x)$.

13. $\log\{(1 - \sin^2 x)/(1 + \sin^2 x)\}$.

14. $\log\{(4 - x^2)/(4 + x^2)\}$.

15. $\log\{(1 + \sqrt{x})/(1 - \sqrt{x})\}$.

16. $\mathrm{e}^{-2x}\cos^3 x$.

17. $\log(\sec 2x)$.

18. $\log\tan(\frac{1}{2}x + \frac{1}{4}\pi)$.

19. $\log[\log\{x + \sqrt{(x^2 + a^2)}\}]$.

20. $\left\{\dfrac{x^3(x-1)}{(x-2)^5}\right\}^{\frac{1}{2}}$.

21. $\dfrac{x - 1}{(x - 2)^2(x + 1)}$.

22. $x^2\,\mathrm{e}^x\sin 3x$.

23. $x^m\,\mathrm{e}^{-x^2}\sin^3 x$.

24. $\cos x\cdot\exp(x\sin x)$.

25. $x^{\log x}$.

26. $(1 + x)^x$.

27. $(1 + x^2)^x$.

28. $(\log x)^x$.

29. $(\sin x)^{\log x}$.

30. $(\sin x)^{\sin x}$.

31. $(\sin x)^{\cos x}$.

32. $x^{x\sin x}$.

33. If $z = \exp(x^2 + xy + y^2)\cos(x - y)$ show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3(x + y)z,$$

and that

$$\frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 9(x + y)^2 z + 6z.$$

34. Prove that $z = \exp\{-(ak^2 + b)y\} \sin(kx + \alpha)$ satisfies the relation

$$\frac{\partial z}{\partial y} = a \frac{\partial^2 z}{\partial x^2} - bz,$$

where a, b, α and k are constants

35. If $z = \exp(-cy) \cos(cx + \alpha)$, c, α being constants, prove that

$$\frac{\partial^2 z}{\partial x^2} = c \frac{\partial z}{\partial y}.$$

36. If $u = \mathrm{e}^{ax} \sin by$ show that

$$b^2 \frac{\partial^2 u}{\partial x^2} + a^2 \frac{\partial^2 u}{\partial y^2} = 0,$$

and verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

37. If $y = \mathrm{e}^{ax} \cos bx$ show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (a^2 + b^2)^{\frac{1}{2}} \cos(bx + \alpha),$$

where $\tan \alpha = b/a$. Hence by induction show that

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (a^2 + b^2)^{\frac{1}{2}n} \cos(bx + n\alpha).$$

Find the nth derivatives of the functions given in Nos. 38–46, using Leibniz' Theorem where necessary.

38. $\log(2x + 3)$.

39. $\log(x^2 - 1)$.

40. $\log(5x^2 + 6x + 1)$.

41. $\mathrm{e}^{3x} \cos 4x$.

42. $\mathrm{e}^{x} \sin x$.

43. $\mathrm{e}^{-2x}(\cos 3x + \sin 3x)$.

44. $\mathrm{e}^{-4x} \cos(3x + 2)$.

45. $x^2 \log(x + 1)$.

46. $x^3 \mathrm{e}^{-5x}$.

47. $x^2 \mathrm{e}^{3x} \cos 4x$.

48. If $y = (x \cos^{-1} x)/\sqrt{(1 - x^2)}$, show that

$$(x - x^3) \frac{\mathrm{d}y}{\mathrm{d}x} - y + x^2 = 0.$$

Hence show that for all integers n greater than 2 and $x = 0$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n(n-2)\frac{\mathrm{d}^{n-2}y}{\mathrm{d}x^{n-2}}.$$

49. Given $y = (\cos k\theta)/\cos\theta$, $x = \sin\theta$, prove

(i) $(1-x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} - xy + k\sin k\theta = 0,$

(ii) $(1-x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} + (k^2-1)y = 0,$

(iii) $(1-x^2)\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (2n+1)x\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} + (k^2-n^2)\dfrac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} = 0.$

50. Find $\mathrm{d}y/\mathrm{d}x$ when $x\,\mathrm{e}^y - y + 1 = 0$.

§ 6. The hyperbolic functions

In many practical problems the exponential function does not occur alone, but it appears in the following combinations

$$\tfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \qquad \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}).$$

For convenience we give these particular functions the special names hyperbolic cosine and hyperbolic sine, written as

$$\cosh x = \tfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \tag{4.39}$$

and

$$\sinh x = \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}), \tag{4.40}$$

respectively. This nomenclature is designed to suggest an analogy with the circular functions cosine and sine and also some relationship with the hyperbola. What this analogy is and also the relationship with the hyperbola will be explained in the course of this paragraph.

The graphs of these two functions are shown in fig. 4.4, the broken lines being the graphs of $y=\frac{1}{2}\mathrm{e}^x$, $y=\frac{1}{2}\mathrm{e}^{-x}$.

We see immediately that $\cosh x$ is an even function, while $\sinh x$ is an odd function. We notice also the following properties:

(i) $\cosh x \geqslant 1$ for all real values of x,
(ii) the least value of $\cosh x$ is when $x = 0$; $\cosh 0 = 1$,
(iii) $\sinh x < \cosh x$ for all real values of x,
(iv) $\sinh 0 = 0$.

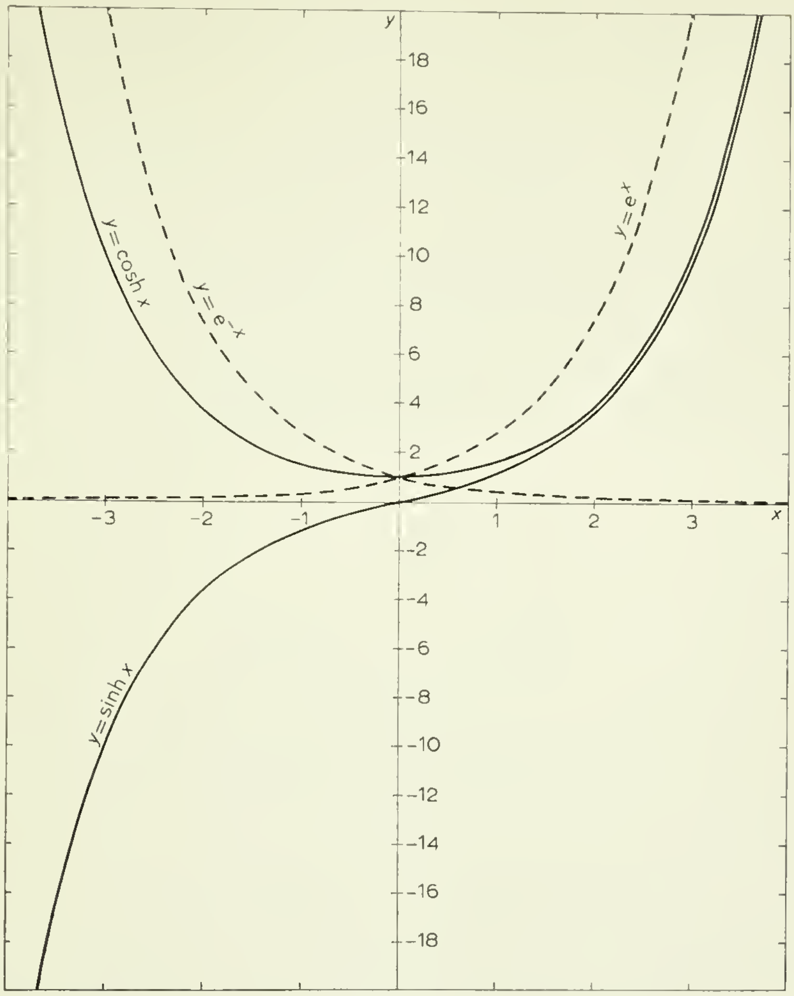
y
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
-14
-16
-18
-3
-2
-1
1
2
3
x
y = cosh x
y = e^{-x}
y = e^x
y = sinh x

Fig. 4.4

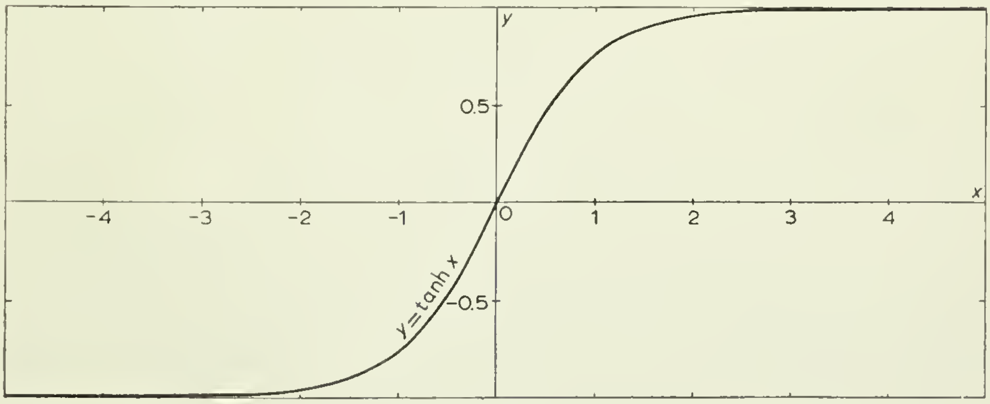
y
0.5
-0.5
x
-4
-3
-2
-1
0
1
2
3
4
y = tanh x

Fig. 4.5

Other hyperbolic functions are defined in terms of the hyperbolic cosine and sine in the same way as the circular functions; thus

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = 1/(\cosh x), \qquad \operatorname{cosech} x = 1/(\sinh x).$$

The graph of $\tanh x$ is shown in fig. 4.5, and this function exists for all values of x, as does $\operatorname{sech} x$. The functions $\coth x$ and $\operatorname{cosech} x$ are undefined at $x=0$, since they contain $\sinh x$ in the denominator.

From eqs. (4.39) and (4.40) we have immediately

$$\cosh x + \sinh x = e^x, \tag{4.41}$$

and

$$\cosh x - \sinh x = e^{-x}, \tag{4.42}$$

and hence by multiplication

$$\cosh^2 x - \sinh^2 x = 1. \tag{4.43}$$

Further by dividing this equation through by $\cosh^2 x$, we have

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \tag{4.44}$$

and dividing eq. (4.43) through by $\sinh^2 x$, we have

$$\coth^2 x - 1 = \operatorname{cosech}^2 x. \tag{4.45}$$

The following addition theorems also follow directly from the definitions:

$$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B, \tag{4.46}$$

$$\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B. \tag{4.47}$$

We have, in fact

$$\cosh(A + B) = \tfrac{1}{2}\{e^{A+B} + e^{-(A+B)}\} = \tfrac{1}{2}\{e^A e^B + e^{-A} e^{-B}\},$$

whilst similarly

$$\sinh(A + B) = \tfrac{1}{2}\{e^A e^B - e^{-A} e^{-B}\}.$$

Using eqs. (4.41), (4.42) with $x=A, B$ we obtain eqs. (4.46), (4.47). When $B=A$ these addition theorems (4.46), (4.47) give, using eq. (4.43),

$$\cosh 2A = \cosh^2 A + \sinh^2 A = 2\cosh^2 A - 1 = 1 + 2\sinh^2 A, \tag{4.48}$$

$$\sinh 2A = 2 \sinh A \cosh A. \tag{4.49}$$

The results (4.43)–(4.49) show immediately that there is an analogy between these hyperbolic functions and the circular functions, the only difference between these results and the corresponding results for the circular functions being certain sign changes.

A corresponding analogy holds for the following derivative properties:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} + \mathrm{e}^{-x}) = \tfrac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}) = \sinh x, \tag{4.50}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x; \tag{4.51}$$

further

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \mathrm{sech}^2\, x, \tag{4.52}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth x) = -\mathrm{cosech}^2\, x, \tag{4.53}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, x) = -\mathrm{sech}\, x \tanh x, \tag{4.54}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech}\, x) = -\mathrm{cosech}\, x \coth x. \tag{4.55}$$

We again notice that in eqs. (4.50)–(4.55) certain signs differ from those in the corresponding results for the circular functions.

At this point we can show the relationship between these functions and a hyperbola. The equation of a rectangular hyperbola in rectangular coordinates (x, y) is

$$x^2 - y^2 = a^2, \tag{4.56}$$

and its graph is shown in fig. 4.6. If we write

$$x = a \cosh \theta, \qquad y = a \sinh \theta, \tag{4.57}$$

then using eq. (4.43)

$$x^2 - y^2 = a^2(\cosh^2 \theta - \sinh^2 \theta) = a^2.$$

Thus the eq. (4.56) is satisfied by eqs. (4.57) for all real values of θ. We say that the rectangular hyperbola (4.56) can be *represented parametrically* by the eqs. (4.57), θ being the parameter.

Hence the analogy between the circular functions and the hyperbolic functions. The circular functions $\cos\theta$, $\sin\theta$ are such that

$$x = a\cos\theta, \qquad y = a\sin\theta,$$

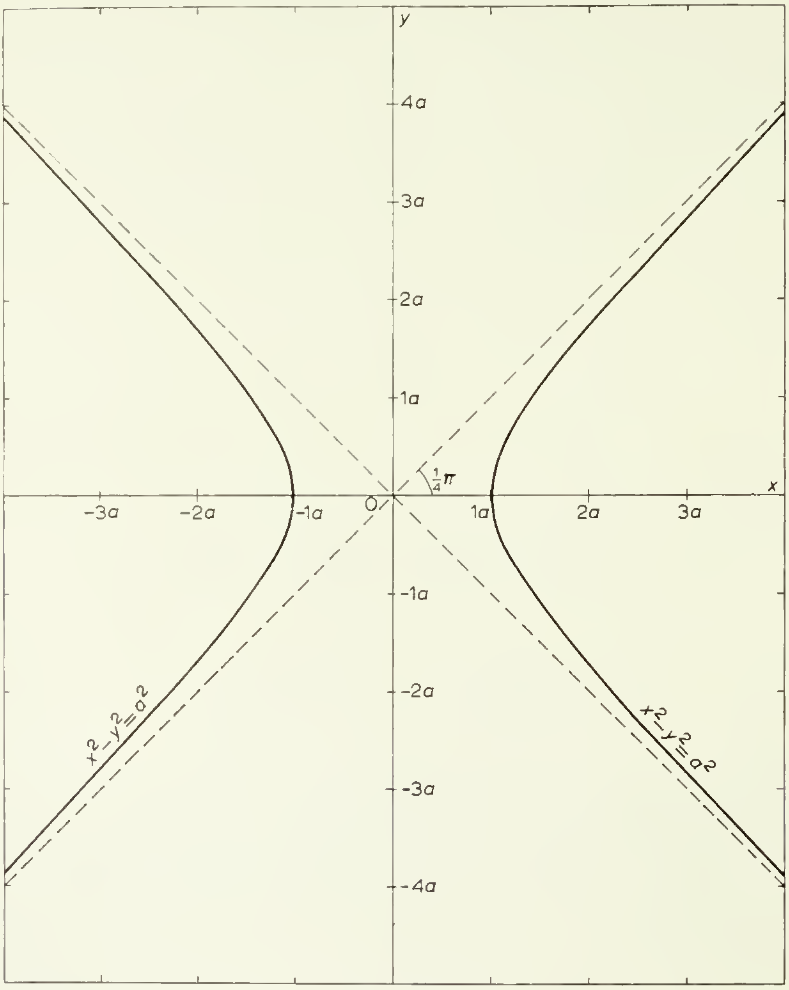

Fig. 4.6

is a parametric representation of the circle

$$x^2 + y^2 = a^2,$$

θ being the polar coordinate. In the parametric representation (4.57) for the hyperbola (4.56), θ is not directly associated with any angle.

Relations connecting the circular functions and the hyperbolic functions will be given in Ch. 7.

§ 6.1. The inverse hyperbolic functions

We define the inverse hyperbolic cosine to be such that

$$y = \cosh^{-1} x, \tag{4.58}$$

when $x = \cosh y$; the graph of this function is shown in fig. 4.7. We see that provided $x > 1$ then for every value of x there are two values of y

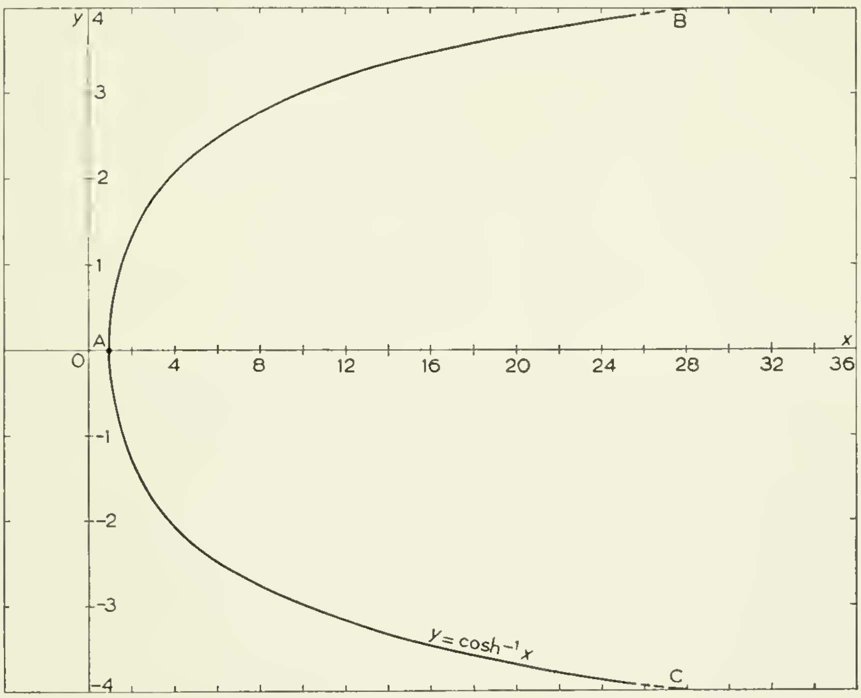

Fig. 4.7

which are equal in magnitude and opposite in sign. This result is more obvious analytically when we express the function $\cosh^{-1} x$ as a logarithm. We have in fact

$$x = \cosh y = \tfrac{1}{2}(e^y + e^{-y}).$$

Multiplying through by $2e^y$, we get

$$2x\, e^y = (e^y)^2 + 1,$$

or

$$(e^y)^2 - 2x\, e^y + 1 = 0;$$

this is a quadratic equation for e^y with roots given by

$$e^y = x \pm \sqrt{(x^2 - 1)},$$

and so

$$y = \cosh^{-1} x = \log\{x \pm \sqrt{(x^2 - 1)}\}. \tag{4.59}$$

As $x \geqslant 1$, the square root in the bracket is real; moreover since

$$\{x - \sqrt{(x^2 - 1)}\}\{x + \sqrt{(x^2 - 1)}\} = 1,$$

then

$$\log\{x - \sqrt{(x^2 - 1)}\} = \log\{x + \sqrt{(x^2 - 1)}\}^{-1} = -\log\{x + \sqrt{(x^2 - 1)}\}.$$

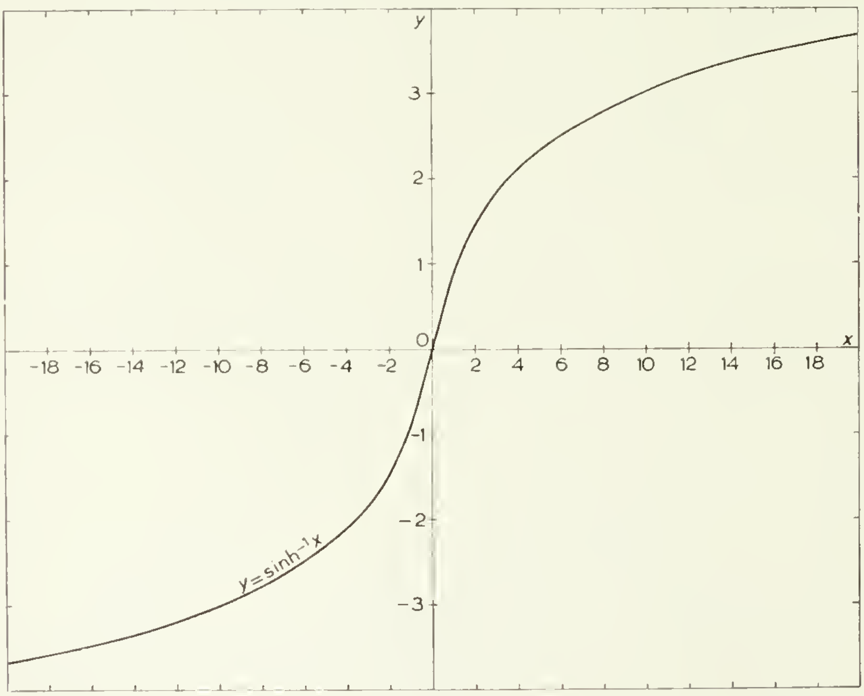

Fig. 4.8

So the two values of $\cosh^{-1} x$ corresponding to the alternate signs in eq. (4.59) are equal in magnitude and opposite in sign. They correspond to the two branches AB and AC of the curve in fig. 4.7.

The inverse hyperbolic sine is given by

$$y = \sinh^{-1} x,$$

when $x = \sinh y$. In a similar way we can show that

$$y = \log\{x \pm \sqrt{(x^2 + 1)}\},$$

but here, since $(x^2 + 1) > 1$, the negative sign gives the logarithm of a negative number which we have seen in § 1.1 has no real value. The logarithm of a negative number will be dealt with in Chs. 7 and 17.

Here we choose the real value to define $\sinh^{-1} x$, so

$$\sinh^{-1} x = \log\{x + \sqrt{(x^2 + 1)}\}, \tag{4.60}$$

and there is only one value for every value of x. Also notice that $\sinh^{-1} x$ is a monotonic increasing function. The graph of $\sinh^{-1} x$ is shown in fig. 4.8.

The functions $\tanh^{-1} x$ and $\coth^{-1} x$ may be defined in similar ways. For

$$y = \tanh^{-1} x,$$

we have

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1},$$

giving

$$e^{2y} = \frac{1 + x}{1 - x},$$

and

$$y = \tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x}. \tag{4.61}$$

Thus a real function $\tanh^{-1} x$ is defined only in the interval $-1 < x < 1$. This is necessary since $|x| = |\tanh y| < 1$ for real finite y (see fig. 4.5). Similarly

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x + 1}{x - 1}, \tag{4.62}$$

defining a real function only when $|x| > 1$. The graph of $\tanh^{-1} x$ is shown in fig. 4.9.

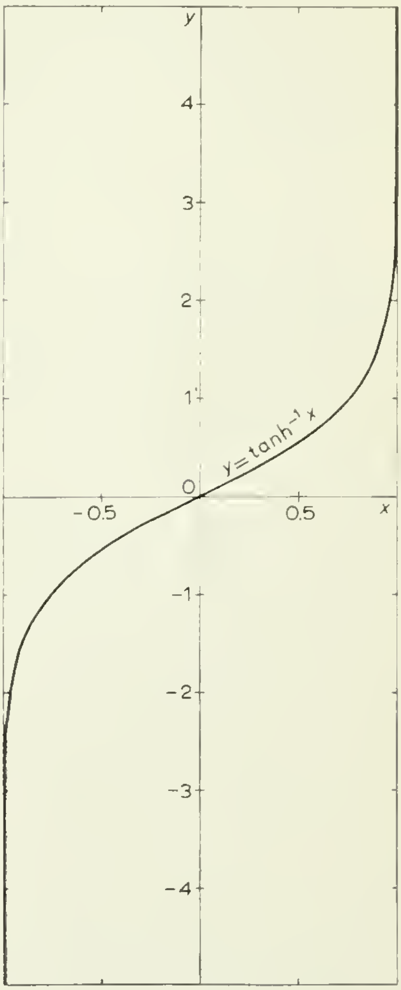

Fig. 4.9

§ 6.2. Derivatives of the inverse hyperbolic functions

The derivatives of these inverse functions are found either by the usual method for inverse functions or by using the logarithmic forms above. For example, if $y = \cosh^{-1} x$, then $x = \cosh y$, so that

$$\frac{dx}{dy} = \sinh y = \pm \sqrt{(\cosh^2 y - 1)} = \pm \sqrt{(x^2 - 1)}.$$

Thus

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \pm (x^2 - 1)^{-\frac{1}{2}}, \tag{4.63}$$

the alternate signs corresponding to the two branches of the curve $y = \cosh^{-1} x$.

When $y = \sinh^{-1} x$, we have $x = \sinh y$, so that

$$\frac{dx}{dy} = \cosh y = +\sqrt{(1 + \sinh^2 y)} = +\sqrt{(1 + x^2)},$$

the positive sign only being inserted since $\cosh y$ is positive for all values of y. Thus

$$\frac{dy}{dx} = (1 + x^2)^{-\frac{1}{2}}, \tag{4.64}$$

and the fact that this is always positive for all values of x may be seen from the graph in fig. 4.8.

It is left as an exercise for the reader to verify that

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}, \qquad |x| < 1; \tag{4.65}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2} = -\frac{1}{x^2 - 1}, \qquad |x| > 1; \tag{4.66}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{(1 - x^2)}}, \qquad |x| < 1; \tag{4.67}$$

$$\frac{d}{dx}(\operatorname{cosech}^{-1} x) = -\frac{1}{x\sqrt{(1 + x^2)}}. \tag{4.68}$$

EXERCISE 4.2

Prove the formulae given in Nos. 1–9.

1. $\sinh 2x = 2 \tanh x/(1 - \tanh^2 x)$.

2. $\sinh A + \sinh B = 2 \sinh \frac{1}{2}(A + B) \cosh \frac{1}{2}(A - B)$.

3. $\cosh A + \cosh B = 2 \cosh \frac{1}{2}(A + B) \cosh \frac{1}{2}(A - B)$.

4. $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.

5. $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$.

6. $\cosh^2 x \cos^2 x + \sinh^2 x \sin^2 x = \frac{1}{2}(\cosh 2x + \cos 2x)$.

7. $\tanh \frac{1}{2}A = (\cosh A - 1)/\sinh A$.

8. $\sinh^{-1} x = \cosh^{-1} \sqrt{(x^2 + 1)}$.

9. $\operatorname{sech}^{-1} x = \pm \log[\{1 + \sqrt{(1 - x^2)}\}/x]$.

10. If $u = \log \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$ prove that $\sinh u = \tan \theta$ and $\tanh \frac{1}{2}u = \tan \frac{1}{2}\theta$.

Differentiate with respect to x the functions given in Nos. 11–19, simplifying the answer where possible.

11. $(\cosh x - \cos x)/(\sinh x + \sin x)$.

12. $\log(\tanh x)$.

13. $\sin^{-1}(\tanh x)$.

14. $\sinh^{-1}(\tan x)$.

15. $\tanh^{-1}\{(x + a)/(1 + ax)\}$.

16. $\cosh^{-1}(\sec x^n)$.

17. $(\sinh x)^{\sinh x}$.

18. $\exp\{\exp(\tanh x)\}$.

19. $\sinh^{-1}\{\tanh(\log x)\}$.

20. If $y = (\sinh^{-1} x)/\sqrt{(1 + x^2)}$, prove that

$$(1 + x^2)\frac{dy}{dx} + xy = 1.$$

INTEGRATION. FURTHER RESULTS

§ 1. Introduction

In Ch. 2 we were unable to give a complete table of the recognised standard integrals since the logarithmic and exponential functions had not been defined and the corresponding derivatives were not known. Having defined these functions in Ch. 4, together with the hyperbolic functions, we are now able to tabulate a further number of standard integrals in the right hand column of Table 5.1, all of which follow from the corresponding derivative formula in the left hand column.
Again all these results may be extended to cover all the corresponding results when x is replaced by $kx+b$, where k, b are constants.

Example 1

$$\int \frac{\mathrm{d}x}{kx+b} = \frac{1}{k}\log(kx+b), \tag{5.1}$$

provided $kx+b>0$. In particular

$$\int \frac{\mathrm{d}x}{x-1} = \log(x-1),$$

provided $x-1>0$; if however $x-1<0$, then the integral is written as

$$-\int \frac{\mathrm{d}x}{1-x} = +\log(1-x),$$

taking $b=1$, $k=-1$ in the general result (5.1). Thus, in general, we can write

$$\int \frac{\mathrm{d}x}{kx+b} = \frac{1}{k}\log|kx+b|.$$

Example 2

$$\int \mathrm{e}^{kx+b}\,\mathrm{d}x = \frac{1}{k}\,\mathrm{e}^{kx+b}.$$

Example 3

$$\int \operatorname{sech}^2(kx+b)\,\mathrm{d}x = \frac{1}{k}\tanh(kx+b).$$

TABLE 5.1

$\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, \quad x > 0$	$\displaystyle\int \frac{1}{x}\,dx = \log \lvert x \rvert$
$\dfrac{d}{dx} e^x = e^x$	$\displaystyle\int e^x\,dx = e^x$
$\dfrac{d}{dx}(\sinh x) = \cosh x$	$\displaystyle\int \cosh x\,dx = \sinh x$
$\dfrac{d}{dx}(\cosh x) = \sinh x$	$\displaystyle\int \sinh x\,dx = \cosh x$
$\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	$\displaystyle\int \operatorname{sech}^2 x\,dx = \tanh x$
$\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$	$\displaystyle\int \operatorname{cosech}^2 x\,dx = -\coth x$
$\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$	$\displaystyle\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$
$\dfrac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$	$\displaystyle\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x$
$\dfrac{d}{dx}\left(\sinh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{(x^2 + a^2)}}$	$\displaystyle\int \frac{dx}{\sqrt{(x^2 + a^2)}} = \sinh^{-1}\frac{x}{a} = \log\left\{\frac{x + \sqrt{(x^2 + a^2)}}{a}\right\}$
$\dfrac{d}{dx}\left(\cosh^{-1}\dfrac{x}{a}\right) = \pm\dfrac{1}{\sqrt{(x^2 - a^2)}}, \quad x > a$	$\displaystyle\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1}\frac{x}{a}, \quad x > a$ $\displaystyle= \log\left\{\frac{x + \sqrt{(x^2 - a^2)}}{a}\right\}$
$\dfrac{d}{dx}\left(\tanh^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 - x^2}, \quad \lvert x \rvert < a$	$\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} = \frac{1}{2a}\log\frac{a + x}{a - x}, \quad \lvert x \rvert < a$
$\dfrac{d}{dx}\left(\coth^{-1}\dfrac{x}{a}\right) = -\dfrac{a}{x^2 - a^2}, \quad \lvert x \rvert > a$	$\displaystyle\int \frac{dx}{x^2 - a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a} = -\frac{1}{2a}\log\frac{x + a}{x - a}, \quad \lvert x \rvert > a$

Example 4

$$\int \frac{\mathrm{d}x}{\sqrt{\{(kx+b)^2-a^2\}}} = \frac{1}{k}\cosh^{-1}\frac{kx+b}{a} = \log\left\{\frac{kx+b+\sqrt{[(kx+b)^2-a^2]}}{a}\right\}.$$

provided $kx+b \geqslant a$. These and many similar results may be deduced from the standard integrals by a change of variable given by $t=kx+b$.

§ 1.1. Hyperbolic substitutions and other standard methods

In Ch. 2 § 1.3 we said that for integrals involving $\sqrt{(x^2-a^2)}$ and $\sqrt{(x^2+a^2)}$ the suggested substitutions were $x=a \sec \theta$ and $x=a \tan \theta$ respectively. Remembering the standard integrals involving these same square roots, we now point out that the substitutions $x=a \cosh \theta$ and $x=a \sinh \theta$ respectively in these integrals are often more suitable, leading to simpler forms of the integrals. The following examples will serve to illustrate the use of these substitutions; they also demonstrate methods of dealing with squares of hyperbolic functions in the same way as with squares of circular functions. Other examples will deal with integrals involving logarithms and exponentials, in particular with integrals involving the products of these functions with others, requiring the method of integration by parts to evaluate them.

Example 5

$$\int \sqrt{(x^2-a^2)}\,\mathrm{d}x.$$

Writing $x=a\cosh\theta$, so that $\sqrt{(x^2-a^2)}=a\sinh\theta$, $\mathrm{d}x=a\sinh\theta\,\mathrm{d}\theta$, the integral becomes

$$a^2\int \sinh^2\theta\,\mathrm{d}\theta = \tfrac{1}{2}a^2\int(\cosh 2\theta-1)\,\mathrm{d}\theta$$

using eq. (4.48); this becomes

$$\tfrac{1}{2}a^2\{\tfrac{1}{2}\sinh 2\theta-\theta\} = \tfrac{1}{2}a^2\{\sinh\theta\cosh\theta-\theta\} = \tfrac{1}{2}\left\{x\sqrt{(x^2-a^2)}-a^2\cosh^{-1}\frac{x}{a}\right\}.$$

Example 6

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \log|f(x)|,$$

writing $t=f(x)$ or $-f(x)$.

This result gives immediately as special cases

$$\int \frac{\mathrm{d}x}{x\log x} = \int \frac{x^{-1}\,\mathrm{d}x}{\log x} = \log|\log x|, \tag{5.2}$$

$$\int \tan x \, \mathrm{d}x = -\log|\cos x|, \tag{5.3}$$
$$\int \cot x \, \mathrm{d}x = \log|\sin x|, \tag{5.4}$$
$$\int \tanh x \, \mathrm{d}x = \log \cosh x, \tag{5.5}$$
$$\int \coth x \, \mathrm{d}x = \log|\sinh x|. \tag{5.6}$$

Example 7

$$\int \log x \, \mathrm{d}x.$$

Using integration by parts treating $\mathrm{d}u/\mathrm{d}x$ as unity and $v = \log x$, we get

$$\int \log x \, \mathrm{d}x = x \log x - \int \mathrm{d}x = x \log x - x.$$

The same method may be used for a polynomial in x multiplied by $\log x$.

Example 8

$$\int \tanh^{-1} x \, \mathrm{d}x.$$

Using the same method as in Example 7,

$$\int \tanh^{-1} x \, \mathrm{d}x = x \tanh^{-1} x - \int \frac{x}{1-x^2} \, \mathrm{d}x,$$

and using Example 6, this becomes

$$x \tanh^{-1} x + \tfrac{1}{2} \log(1 - x^2) = \tfrac{1}{2} x \log\left(\frac{1+x}{1-x}\right) + \tfrac{1}{2} \log(1 - x^2),$$

provided $|x| < 1$.

The same method can be used for all the other inverse hyperbolic functions.

Example 9

$$\int x \mathrm{e}^{ax} \, \mathrm{d}x.$$

Using integration by parts this becomes

$$\frac{1}{a} \mathrm{e}^{ax} x - \frac{1}{a} \int \mathrm{e}^{ax} \, \mathrm{d}x = \frac{x}{a} \mathrm{e}^{ax} - \frac{1}{a^2} \mathrm{e}^{ax}.$$

Example 10

$$\int \mathrm{e}^{ax} \cos bx \, \mathrm{d}x.$$

Here successive integration by parts must be used and like terms collected together. Using the notation

$$P \equiv \int \mathrm{e}^{ax} \cos bx \, \mathrm{d}x, \qquad Q \equiv \int \mathrm{e}^{ax} \sin bx \, \mathrm{d}x,$$

and writing $\mathrm{d}u/\mathrm{d}x = \mathrm{e}^{ax}$, $v = \cos bx$ in the formula for integration by parts, we get

$$P = \frac{1}{a} \mathrm{e}^{ax} \cos bx + \frac{b}{a} Q. \tag{5.7}$$

But in the same way

$$Q = \frac{1}{a} \mathrm{e}^{ax} \sin bx - \frac{b}{a} P; \tag{5.8}$$

substituting this value of Q in eq. (5.7) and collecting the terms in P, we get

$$(a^2 + b^2)P = e^{ax}(a \cos bx + b \sin bx). \tag{5.9}$$

Similarly by substituting for P from eq. (5.7) in eq. (5.8) we get

$$(a^2 + b^2)Q = e^{ax}(a \sin bx - b \cos bx). \tag{5.10}$$

Other methods of evaluating the integrals P and Q will be given in Ch. 7.

Similar methods may be used for products of the exponential and hyperbolic functions, but in these cases it is much simpler to express the hyperbolic functions in their exponential form.

Example 11

$$\int e^{ax} \cosh bx \, dx = \tfrac{1}{2}\int e^{ax}(e^{bx} + e^{-bx})\, dx,$$

and this is readily evaluated as

$$\frac{e^{ax}}{a^2 - b^2}\{\tfrac{1}{2}(a - b)e^{bx} + \tfrac{1}{2}(a + b)e^{-bx}\} = \frac{e^{ax}}{a^2 - b^2}\{a \cosh bx - b \sinh bx\},$$

a result analagous to eq. (5.9).

EXERCISE 5.1

Integrate with respect to x the functions in Nos. 1–6.

1. $\sqrt{(x^2 + 16)}$.

2. $(e^x - e^{-x})/(e^x + e^{-x})$.

3. $x \sinh^{-1} x$.

4. $x^2 \log x$.

5. $\dfrac{1}{x(1 + \log x)^2}$.

6. $\tanh^{-1}(2x + 1)$, $(-1 < x < 0)$.

Evaluate the integrals in Nos. 7–14.

7. $\displaystyle\int_3^6 \frac{dx}{\sqrt{(4x^2 - 9)}}$.

8. $\displaystyle\int_0^1 \frac{x + x^2}{\sqrt{(x^2 + 4)}}\, dx$.

9. $\displaystyle\int_2^\infty \frac{dx}{x(\log x)^2}$.

10. $\displaystyle\int_1^5 \log(2x + 3)\, dx$.

11. $\displaystyle\int_0^5 xe^{-x}\, dx$.

12. $\displaystyle\int_0^\infty (x^2 - 1)e^{-x}\, dx$.

13. $\displaystyle\int_0^{\frac{1}{2}\pi} e^{2x} \sin x\, dx$.

14. $\displaystyle\int_0^\infty e^{-2x} \cos^2 x\, dx$.

§ 2. Reduction formulae: A particular example

As an example of a reduction formula and its use we shall consider first the integral

$$I_{m,n} \equiv \int \sin^m \theta \cos^n \theta \, d\theta, \qquad (5.11)$$

where m, n are positive integers. In Ch. 2 § 1.4 we showed how to deal with integrals of this type when m, n were small integers; it would still be possible to deal with all integrals of this type by similar methods, using De Moivre's Theorem to express $\sin^m \theta$ and $\cos^n \theta$ in terms of sine and cosine of multiple values of θ (Ch. 7 § 4.3). However such methods would be tedious and for larger integers m and n we proceed as follows: write $I_{m,n}$ in the form

$$I_{m,n} = \int (\sin^m \theta \cos \theta) \cos^{n-1} \theta \, d\theta,$$

and use the method of integration by parts with $v = \cos^{n-1} \theta$ and $du/d\theta = \sin^m \theta \cos \theta$ so that $u = (\sin^{m+1} \theta)/(m+1)$. This gives for $n > 1$

$$I_{m,n} = \frac{1}{m+1} (\sin^{m+1} \theta \cos^{n-1} \theta) + \frac{n-1}{m+1} \int \sin^{m+2} \theta \cos^{n-2} \theta \, d\theta,$$

or, putting $\sin^2 \theta = 1 - \cos^2 \theta$ in the last integral and collecting terms,

$$(m+n) I_{m,n} = \sin^{m+1} \theta \cos^{n-1} \theta + (n-1) \int \sin^m \theta \cos^{n-2} \theta \, d\theta. \qquad (5.12)$$

The last integral in the result (5.12) is the same form of integral as in eq. (5.11) but with n replaced by $n-2$. Thus we can write it as $I_{m,n-2}$, and so

$$(m+n) I_{m,n} = \sin^{m+1} \theta \cos^{n-1} \theta + (n-1) I_{m,n-2}. \qquad (5.13)$$

Before proceeding we note that if we had used $du/d\theta = \sin \theta \cos^n \theta$ and $v = \sin^{n-1} \theta$, we would, by the same method, have found that for $m > 1$

$$(m+n) I_{m,n} = -\sin^{m-1} \theta \cos^{n+1} \theta + (m-1) I_{m-2,n}. \qquad (5.14)$$

Formulae such as (5.13) and (5.14) are known as reduction formulae. The integral $I_{m,n}$ is expressed in terms of $I_{m,n-2}$ in which the integer n is reduced by 2, or is expressed in terms of $I_{m-2,n}$ in which the integer m is reduced by 2. By repeating the process, using eq. (5.13) we can express $I_{m,n}$ as a series of terms similar to the first term in eq. (5.13) with the powers of n decreasing by 2 each time. This process ends with either an integral of the form $I_{m,1}$ when n is odd, or $I_{m,0}$ when n is even. When n

is odd, the integral to be evaluated after reduction is

$$I_{m,1} = \int \sin^m \theta \cos \theta \, d\theta = \frac{1}{m+1} \sin^{m+1} \theta; \tag{5.15}$$

the integral $I_{m,n}$ is therefore fully evaluated.

When n is even the integral to be evaluated is $I_{m,0}$ and the reduction formula (5.14) can be used with $n=0$, giving

$$mI_{m,0} = -\sin^{m-1} \theta \cos \theta + (m-1)I_{m-2,0}. \tag{5.16}$$

By repeating this result we get a series of terms similar to the first term in eq. (5.16) with the powers of m reducing by 2 each time and ending with the integral $I_{1,0}$ when m is odd or $I_{0,0}$ when m is even, both of which are trivial integrals.

It is possible therefore to give general formulae for the integral $I_{m,n}$ when the integral is indefinite, but in practical problems this is not usually required. Usually definite integrals are required and the limits of integration for θ are of the form $\frac{1}{2}r\pi$ to $\frac{1}{2}s\pi$ where r and s are integers. We will see later in this paragraph that all such definite integrals can be expressed in terms of a definite integral $I_{m,n}$ with limits 0 and $\frac{1}{2}\pi$. We therefore discuss now

$$I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m \theta \cos^n \theta \, d\theta; \tag{5.17}$$

with these particular limits the reduction formula (5.13) becomes, for $n>1$

$$(m+n)I_{m,n} = (n-1)I_{m,n-2}, \tag{5.18}$$

whilst the reduction formula (5.14) becomes, for $m>1$

$$(m+n)I_{m,n} = (m-1)I_{m-2,n}. \tag{5.19}$$

By writing $\varphi = \frac{1}{2}\pi - \theta$ in (5.17) it is also obvious that $I_{m,n} = I_{n,m}$.

Repeating the formula (5.18) with n replaced by $n-2$ successively, and inserting the successive values $I_{m,n-2}, I_{m,n-4}, \ldots$ in the original formula (5.18), we arrive at the result

$$I_{m,n} = \frac{(n-1)(n-3)\cdots 5\cdot 3\cdot 1}{(m+n)(m+n-2)\cdots(m+6)(m+4)(m+2)} I_{m,0}, \tag{5.20}$$

for n even, whilst

$$I_{m,n} = \frac{(n-1)(n-3)\cdots 6\cdot 4\cdot 2}{(m+n)(m+n-2)\cdots(m+7)(m+5)(m+3)} I_{m,1}, \tag{5.21}$$

for n odd. Using result (5.15) with the definite limits 0 to $\frac{1}{2}\pi$, we have $I_{m,1}=1/(m+1)$ for all values of m. Thus, for *n odd, m even or odd*

$$I_{m,n} = \frac{(n-1)(n-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+5)(m+3)}\,\frac{1}{(m+1)}. \tag{5.22}$$

To find the value when n is even we need to evaluate $I_{m,0}$. Using eq. (5.19) with $n=0$ we have for $m>1$

$$mI_{m,0} = (m-1)I_{m-2,0}. \tag{5.23}$$

Successive application of this equation gives for m even

$$I_{m,0} = \frac{(m-1)(m-3)\cdots 5\cdot 3\cdot 1}{m(m-2)\cdots 6\cdot 4\cdot 2}\,I_{0,0} \tag{5.24}$$

and for m odd,

$$I_{m,0} = \frac{(m-1)(m-3)\cdots 6\cdot 4\cdot 2}{m(m-2)\cdots 7\cdot 5\cdot 3}\,I_{1,0}. \tag{5.25}$$

But

$$I_{0,0} = \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta = \tfrac{1}{2}\pi, \tag{5.26}$$

and

$$I_{1,0} = \int_0^{\frac{1}{2}\pi} \sin\theta\,\mathrm{d}\theta = [-\cos\theta]_0^{\frac{1}{2}\pi} = 1. \tag{5.27}$$

Substitution of the results (5.24)–(5.27) in the equation (5.20) gives

$$I_{m,n} = \frac{(n-1)(n-3)\cdots 3\cdot 1\cdot(m-1)(m-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots(m+4)(m+2)m(m-2)\cdots 4\cdot 2}\,\frac{\pi}{2}, \tag{5.28}$$

for *n even, m even*, and

$$I_{m,n} = \frac{(n-1)(n-3)\cdots 3\cdot 1\cdot(m-1)(m-3)\cdots 4\cdot 2}{(m+n)(m+n-2)\cdots(m+4)(m+2)m(m-2)\cdots 3\cdot 1}, \tag{5.29}$$

for *n even, m odd.*

It is easy to see that the results (5.22), (5.28) and (5.29) can all be summed up in the formula

$$I_{m,n} = \frac{(n-1)(n-3)\cdots(m-1)(m-3)\cdots}{(m+n)(m+n-2)\cdots}\,A, \tag{5.30}$$

where the factors $(n-1)(n-3)\ldots$, $(m-1)(m-3)\ldots$, $(m+n)(m+n-2)\ldots$, all go down by 2 as far as they can, ending in 2 or 1 according to whether

they are even or odd, whilst $A=1$ except when m and n are both even when $A=\frac{1}{2}\pi$. The symmetry of this result in m and n verifies the fact that $I_{m,n}=I_{n,m}$.

Example 12

$$\int_0^{\frac{1}{2}\pi} \sin^5\theta\cos^9\theta\,\mathrm{d}\theta = \frac{4\cdot 2\cdot 8\cdot 6\cdot 4\cdot 2}{14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} = \frac{1}{210}.$$

Example 13

$$\int_0^{\frac{1}{2}\pi} \sin^8\theta\cos^4\theta\,\mathrm{d}\theta = \frac{7\cdot 5\cdot 3\cdot 1\cdot 3\cdot 1}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{7\pi}{2^{11}}.$$

As we have already mentioned, the formula (5.30) for $I_{m,n}$ can be used to evaluate integrals of the same function when the limits are not simply 0 to $\frac{1}{2}\pi$ but may be any range $\frac{1}{2}r\pi$ to $\frac{1}{2}s\pi$, where r and s are integers.

Example 14

$$\int_0^{\pi} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta = \int_0^{\frac{1}{2}\pi} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta + \int_{\frac{1}{2}\pi}^{\pi} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta.$$

In the second integral write $\theta=\frac{1}{2}\pi+\varphi$, so that $\mathrm{d}\theta=\mathrm{d}\varphi$ and when $\theta=\frac{1}{2}\pi$, $\varphi=0$, whilst when $\theta=\pi$, $\varphi=\frac{1}{2}\pi$. Also $\sin\theta=\cos\varphi$, $\cos\theta=-\sin\varphi$, so that the second integral becomes

$$(-1)^n\int_0^{\frac{1}{2}\pi} \cos^m\varphi\sin^n\varphi\,\mathrm{d}\varphi = (-1)^n I_{n,m} = (-1)^n I_{m,n}.$$

Thus altogether,

$$\int_0^{\pi} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta = \{1+(-1)^n\}I_{m,n}. \tag{5.31}$$

If n is odd the result is zero, if n is even the result is $2I_{m,n}$.

Example 15

$$\int_{\pi}^{\frac{3}{2}\pi} \sin^m\theta\cos^n\theta\,\mathrm{d}\theta.$$

In a similar way we write $\theta=\pi+\varphi$, so that the limits of φ are 0 and $\frac{1}{2}\pi$. Since then $\sin\theta=-\sin\varphi$ and $\cos\theta=-\cos\varphi$, the integral becomes $(-1)^{m+n}I_{m,n}$.

Similar changes of variable may be made for any range $\frac{1}{2}r\pi$ to $\frac{1}{2}(r+1)\pi$; a range $\frac{1}{2}r\pi$ to $\frac{1}{2}s\pi$ may then be divided into a series of such ranges.

We have discussed this particular reduction formula in detail so that the reader may understand in general how to make use of reduction

formulae. In addition the integral of the form $I_{m,n}$ in eq. (5.17) sometimes with either m or n zero, frequently occurs in physical applications. It does not always occur in the particular form given but can be reduced to such a form by a change of variable. For example, the algebraic integral

$$\int_0^1 x^m(1-x)^n \, dx,$$

is reduced by the substitution $x=\sin^2\theta$ to the form

$$2\int_0^{\frac{1}{2}\pi} \sin^{2m+1}\theta \cos^{2n+1}\theta \, d\theta = 2I_{2m+1,\,2n+1}.$$

Similarly by making the substitution $x=\sin\theta$ in the integral

$$\int_0^1 x^m(1-x^2)^n \, dx,$$

it becomes

$$\int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^{2n+1}\theta \, d\theta = I_{m,\,2n+1}.$$

Particular examples which will be required in later chapters are:

Example 16

$$\int_0^{\pi} \sin^{2n+1}\theta \, d\theta = 2\int_0^{\frac{1}{2}\pi} \sin^{2n+1}\theta \, d\theta,$$

using eq. (2.35). Then by the formula in eq. (5.30), we evaluate this integral as

$$\frac{2\cdot 2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 3\cdot 1} = \frac{2\{2n(2n-2)\cdots 4\cdot 2\}^2}{(2n+1)2n(2n-1)\cdots 4\cdot 3\cdot 2\cdot 1},$$

which can be written as

$$\frac{2\{2^n n!\}^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

Example 17

$$\int_{-1}^{+1} (1-\mu^2)^n \, d\mu = 2\int_0^1 (1-\mu^2)^n \, d\mu,$$

using eq. (2.29). With $\mu=\cos\theta$, $d\mu=-\sin\theta \, d\theta$ this becomes

$$\int_0^{\frac{1}{2}\pi} 2\sin^{2n+1}\theta \, d\theta = \frac{2^{2n+1}(n!)^2}{(2n+1)!},$$

from Example 16.

Example 18

A more general integral is

$$\int_{-1}^{+1} (1-\mu^2)^m \mu^{k-m}\,\mathrm{d}\mu,$$

where k is an integer. With $\mu = \cos\theta$ this becomes

$$\int_0^{\pi} \sin^{2m+1}\theta \cos^{k-m}\theta\,\mathrm{d}\theta.$$

Using eq. (5.31), this becomes

$$\{1+(-1)^{k-m}\} I_{2m+1,\,k-m} =$$
$$= \{1+(-1)^{k-m}\} \frac{2m(2m-2)\cdots 2\cdot(k-m-1)(k-m-3)\cdots}{(m+k+1)(m+k-1)\cdots}, \tag{5.32}$$

the power $2m+1$ being odd, the value of A in eq. (5.30) is unity.

§ 2.1. Reduction formulae: another example

Reduction formulae for many other types of integral can be found, and the integrals themselves can be evaluated by repeated application of the formulae as in § 2. One other example of a different type of reduction formula will be given here. Let

$$I_n \equiv \int \frac{\mathrm{d}x}{(a^2+x^2)^n},$$

when here n may be a positive integer or fraction. We determine the reduction formula by integration by parts with

$$\mathrm{d}u/\mathrm{d}x = 1 \quad \text{and} \quad v = (a^2+x^2)^{-n},$$

so that

$$I_n = \frac{x}{(a^2+x^2)^n} + n\int \frac{2x^2}{(a^2+x^2)^{n+1}}\,\mathrm{d}x.$$

Writing

$$\int \frac{x^2\,\mathrm{d}x}{(a^2+x^2)^{n+1}} = \int \frac{(x^2+a^2)-a^2}{(a^2+x^2)^{n+1}}\,\mathrm{d}x = I_n - a^2 I_{n+1},$$

we get

$$I_n = \frac{x}{(a^2+x^2)^n} + 2nI_n - 2na^2 I_{n+1}.$$

At first sight this may not look like a *reduction* formula, but after

rearranging the terms we see that we have

$$2na^2 I_{n+1} = \frac{x}{(a^2 + x^2)^n} + (2n - 1) I_n.$$

Replacing n by $n-1$, this gives

$$2(n - 1)a^2 I_n = \frac{x}{(a^2 + x^2)^{n-1}} + (2n - 3) I_{n-1}, \qquad (5.33)$$

and in this result $n > 1$.

If n is not very large, the indefinite integral I_n can be evaluated quickly.

Example 19

$$I_4 = \int \frac{\mathrm{d}x}{(a^2 + x^2)^4}.$$

By eq. (5.33) we get the following results:

$$2 \cdot 3 \cdot a^2 I_4 = \frac{x}{(a^2 + x^2)^3} + 5I_3,$$

$$2 \cdot 2 \cdot a^2 I_3 = \frac{x}{(a^2 + x^2)^2} + 3I_2,$$

and

$$2 \cdot 1 \cdot a^2 I_2 = \frac{x}{a^2 + x^2} + I_1.$$

Eq. (5.33) does not apply when $n = 1$, since then $n - 1 = 0$. For I_1 we have

$$I_1 = \int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a};$$

thus we get

$$I_4 = \frac{x}{3 \cdot 2a^2(a^2 + x^2)^3} + \frac{5x}{3 \cdot 2 \cdot 2^2 a^4 (a^2 + x^2)^2} + \frac{5x}{2^4 a^6 (a^2 + x^2)} + \frac{5}{2^4 a^7} \tan^{-1} \frac{x}{a}.$$

EXERCISE 5.2

Evaluate the integrals in Nos. 1–6.

1. $\int_0^{\frac{1}{2}\pi} \sin^5 x(1 + \cos x)\, \mathrm{d}x.$

2. $\int_0^{\pi} (\sin^2 x + \sin^3 x) \cos^4 x\, \mathrm{d}x.$

3. $\int_0^{\frac{1}{2}\pi} \cos 2x(1 - \sin x)^2\, \mathrm{d}x.$

4. $\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \sin^3 x \cos^6 x\, \mathrm{d}x.$

5. $\displaystyle\int_0^\infty \frac{dx}{(a^2 + x^2)^5}$.

6. $\displaystyle\int_0^\infty \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$.

7. Show that $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 3)^3} = \frac{\pi}{48\sqrt{3}}$.

8. By means of the substitution $x = y/(1 - y)$ show that

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1 + x)^{m+n}} = \int_0^1 y^{m-1}(1 - y)^{n-1}\,dy.$$

Deduce the value of

$$\int_0^\infty \frac{x^2\,dx}{(1 + x)^7}.$$

9. Show that if

$$I_n = \int \frac{x^n}{\sqrt{(1 + x^2)}}\,dx,$$

then for $n > 1$

$$nI_n = x^{n-1}\sqrt{(1 + x^2)} - (n - 1)I_{n-2}.$$

Evaluate

$$\int \frac{x^5}{\sqrt{(1 + x^2)}}\,dx.$$

10. If

$$I_n = \int_0^a (a^2 - x^2)^n\,dx,$$

prove that for $n > 0$

$$I_n = \frac{2na^2}{2n + 1} I_{n-1}.$$

11. If

$$I_n = \int_0^\infty x^n e^{-x}\,dx,$$

prove that for $n \geqslant 1$, $I_n = nI_{n-1}$. Hence show that

$$I_n = n!.$$

12. If

$$I_n = \int \sec^n \theta\,d\theta,$$

obtain the reduction formula

$$(n - 1)I_n = \sec^{n-2}\theta \tan\theta + (n - 2)I_{n-2}.$$

Hence evaluate

$$\int_0^{\frac{1}{4}\pi} \sec^6 \theta \, d\theta.$$

13. Obtain a reduction formula for the integral

$$I_{m,n} = \int \cos^m x \sin nx \, dx,$$

in the form

$$(m + n)I_{m,n} = -\cos^m x \cos nx + mI_{m-1,n-1}.$$

Prove that, if n is a positive integer >1

$$\int_0^{\frac{1}{2}\pi} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}.$$

14. Prove that when n is an integer, and $l \geqslant 2$ then

$$\int_0^{2\pi} x^l \sin nx \, dx = -\frac{(2\pi)^l}{n} - \frac{l(l-1)}{n^2} \int_0^{2\pi} x^{l-2} \sin nx \, dx.$$

Hence evaluate

$$\int_0^{2\pi} x^6 \sin nx \, dx.$$

15. If

$$I_n = \int_0^{\frac{1}{2}\pi} \cos x (\sinh x)^n \, dx,$$

obtain the reduction formula

$$(1 + n^2)I_n = (\sinh \tfrac{1}{2}\pi)^n - n(n-1)I_{n-2},$$

for $n \geqslant 2$. Evaluate I_5.

§ 3. Integration by partial fractions

In this paragraph we shall deal with integrals of rational functions in the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials in x. To facilitate the evaluation of such integrals we consider first an integral of the form

$$\int \frac{P(x)}{ax^2 + bx + c} \, dx, \tag{5.34}$$

where a, b and c are constants. If $P(x)$ is a polynomial of degree higher than one then we must first divide out the numerator by the denominator until the remainder is of degree one in the form $lx+m$, say. The resulting

quotient $q(x)$ say, will be a polynomial in x which can always be integrated: thus

$$\int \frac{P(x)\,\mathrm{d}x}{ax^2+bx+c} = \int q(x)\,\mathrm{d}x + \int \frac{lx+m}{ax^2+bx+c}\,\mathrm{d}x,$$

and we have to evaluate an integral of the form

$$\int \frac{lx+m}{ax^2+bx+c}\,\mathrm{d}x. \tag{5.35}$$

Example 20

$$\int \frac{x^3-x^2-3}{x^2-x-2}\,\mathrm{d}x = \int x\,\mathrm{d}x + \int \frac{2x-3}{x^2-x-2}\,\mathrm{d}x.$$

The method of dealing with an integral of the form (5.35) then depends on whether the denominator will factorise into linear factors. If $b^2 \geqslant 4ac$ the denominator will factorise and we can always use the method of partial fractions, which is assumed to be known to the reader. The integral in Example 20 can be evaluated in this way. We write

$$\frac{2x-3}{x^2-x-2} = \frac{2x-3}{(x-2)(x+1)} = \frac{1}{3}\left\{\frac{1}{x-2} + \frac{5}{x+1}\right\};$$

so the integral becomes

$$\frac{1}{3}\int\left\{\frac{1}{x-2} + \frac{5}{x+1}\right\}\mathrm{d}x = \tfrac{1}{3}\log|x-2| + \tfrac{5}{3}\log|x+1|.$$

If $b^2 = 4ac$ so that the denominator is a perfect square the partial fractions take a special form, but the method of evaluation is essentially the same.

Example 21

$$\int \frac{4x+5}{(2x+1)^2}\,\mathrm{d}x = \int\left\{\frac{2}{2x+1} + \frac{3}{(2x+1)^2}\right\}\mathrm{d}x,$$

and therefore has the value

$$\log|2x+1| - \frac{3}{2(2x+1)}.$$

If $b^2 < 4ac$ the denominator cannot be factorised into real factors, so partial fraction methods cannot be used. We rearrange the numerator of

the integrand to give

$$\int \frac{(lx+m)\,\mathrm{d}x}{ax^2+bx+c} = \frac{l}{2a}\int \frac{(2ax+b)\,\mathrm{d}x}{ax^2+bx+c} + \frac{2am-bl}{2a}\int \frac{\mathrm{d}x}{ax^2+bx+c}. \qquad (5.36)$$

The integrand of the first integral on the right in (5.36) has been chosen to have the form $f'(x)/f(x)$ where $f(x)=ax^2+bx+c$, and the coefficient $(l/2a)$ has been chosen to include the whole of the term lx in this integral. It is easy to check that the constant in front of the second integral ensures the correctness of eq. (5.36). The first term is then

$$\frac{l}{2a}\int \frac{f'(x)\,\mathrm{d}x}{f(x)} = \frac{l}{2a}\log|f(x)|,$$

or

$$\frac{l}{2a}\log|ax^2+bx+c|.$$

The integral in the second term is

$$\int \frac{\mathrm{d}x}{ax^2+bx+c},$$

where ax^2+bx+c cannot be factorised into real factors; we use the method of completing the square in the denominator, and write

$$\int \frac{\mathrm{d}x}{ax^2+bx+c} = \int \frac{\mathrm{d}x}{a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right\}} = \int \frac{4a\,\mathrm{d}x}{(2ax+b)^2+4ac-b^2}.$$

Using the change of variable $t=2ax+b$, the integral can be evaluated as

$$\frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$$

In terms of the letters l, m, a, b, c this result appears more complicated than it is in particular examples, as we shall now see.

Example 22

$$\int \frac{1-x}{x^2-x+1}\,\mathrm{d}x = -\frac{1}{2}\int \frac{(2x-1)\,\mathrm{d}x}{x^2-x+1} + \frac{1}{2}\int \frac{\mathrm{d}x}{x^2-x+1}$$
$$= -\tfrac{1}{2}\log|x^2-x+1| + 2\int \frac{\mathrm{d}x}{(2x-1)^2+3},$$

and the last integral is a standard form and has the value

$$\frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}.$$

§ 3.1. Integrals of the form $\int P(x)/Q(x)\,\mathrm{d}x$

Having evaluated integrals of the form (5.35) we can now proceed to integrals of the form

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x. \tag{5.37}$$

Here again, if $P(x)$ is a polynomial of degree higher than or equal to that of $Q(x)$, we can divide $P(x)$ by $Q(x)$ to give a quotient which is a polynomial in x and can therefore be integrated, and a remainder which is a polynomial of lower degree than $Q(x)$. We shall assume that this division has already been done; we therefore need to deal only with integrals in which $P(x)$ is of a degree lower than $Q(x)$.

We now proceed by factorising $Q(x)$ into a set of linear and quadratic factors. A theorem in algebra states (see Barnard and Child [1947]):

A polynomial in x with real coefficients can be resolved into factors which are linear or quadratic functions of x with real coefficients.

We shall assume that the quadratic factors cannot be further factorised into real linear factors. Having factorised $Q(x)$ in this way the rules for partial fractions can be applied.

All such integrals may therefore be reduced to the sum of integrals of the forms

$$\int \frac{A\,\mathrm{d}x}{(ax+b)^r}, \tag{5.38}$$

and

$$\int \frac{(lx+m)\,\mathrm{d}x}{(ax^2+bx+c)^r}, \tag{5.39}$$

where r takes any integral value. Integrals of the form (5.38) are standard integrals or can be dealt with by writing $t=ax+b$. Integrals of the form (5.39) when $r=1$ have been dealt with in § 3, whilst if $r>1$ the method is essentially the same except that the two integrals involved in the evaluation are

$$\int \frac{(2ax+b)\,\mathrm{d}x}{(ax^2+bx+c)^r}, \tag{5.40}$$

and

$$\int \frac{B\,\mathrm{d}x}{(ax^2+bx+c)^r}. \tag{5.41}$$

The integral (5.40) can be evaluated by writing $t=ax^2+bx+c$. The integral (5.41) can be evaluated by writing $2ax+b=(4ac-b^2)^{\frac{1}{2}}\tan\theta$ and completing the square in the denominator, the integral then reduces to one of the form

$$\int \cos^{2r-2}\theta\,\mathrm{d}\theta,$$

which can be evaluated by a reduction formula if necessary.

Example 23

$$\int \frac{49\,\mathrm{d}x}{(1+2x)(1-x+x^2)^2}. \tag{5.42}$$

Writing

$$\frac{49}{(1+2x)(1-x+x^2)^2} = \frac{A}{1+2x} + \frac{Bx+C}{1-x+x^2} + \frac{Dx+E}{(1-x+x^2)^2},$$

we find that $A=16$, $B=-8$, $C=12$, $D=-14$, $E=21$; thus the integral becomes

$$\int \frac{16\,\mathrm{d}x}{1+2x} - \int \frac{(8x-12)\,\mathrm{d}x}{1-x+x^2} - \int \frac{14x-21}{(1-x+x^2)^2}\,\mathrm{d}x. \tag{5.43}$$

The first integral in (5.43) is simply $8\log|1+2x|$. The second integral in (5.43) is

$$-4\int \frac{(2x-1)\,\mathrm{d}x}{1-x+x^2} + 8\int \frac{\mathrm{d}x}{1-x+x^2}$$

$$= -4\log|1-x+x^2| + 32\int \frac{\mathrm{d}x}{(2x-1)^2+3}$$

$$= -4\log|1-x+x^2| + \frac{16}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

The third integral in (5.43) is

$$-7\int \frac{(2x-1)\,\mathrm{d}x}{(1-x+x^2)^2} + 14\int \frac{\mathrm{d}x}{(1-x+x^2)^2} = \frac{7}{(1-x+x^2)} + 224\int \frac{\mathrm{d}x}{\{(2x-1)^2+3\}^2}.$$

This last integral is evaluated by writing $(2x-1)=\sqrt{3}\tan\theta$; then

$$2\,\mathrm{d}x = \sqrt{3}\sec^2\theta\,\mathrm{d}\theta \quad \text{and} \quad (2x-1)^2+3 = 3\sec^2\theta$$

so that it becomes

$$\frac{112\sqrt{3}}{9}\int \cos^2\theta\,\mathrm{d}\theta = \frac{56\sqrt{3}}{9}(\theta+\sin\theta\cos\theta),$$

or in terms of x

$$\frac{56\sqrt{3}}{9}\tan^{-1}\frac{2x-1}{\sqrt{3}} + \frac{14(2x-1)}{3(1-x+x^2)}.$$

Thus altogether the complete integral (5.42) is

$$4\log\left|\frac{(1+2x)^2}{1-x+x^2}\right|+\frac{104\sqrt{3}}{9}\tan^{-1}\frac{2x-1}{\sqrt{3}}+\frac{7}{1-x+x^2}+\frac{14(2x-1)}{3(1-x+x^2)}.$$

EXERCISE 5.3

Evaluate the integrals in Nos. 1–10.

1. $\displaystyle\int_2^5 \frac{(x-1)\,dx}{2x^2-4x+1}$.

2. $\displaystyle\int \frac{(5x+7)\,dx}{6x^2-5x-6}$.

3. $\displaystyle\int \frac{x^4\,dx}{(x-1)^2}$.

4. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+x+2)}$.

5. $\displaystyle\int_0^\infty \frac{(x+1)\,dx}{(x^2+1)(x^2+4)}$.

6. $\displaystyle\int_0^{\frac{1}{2}a} \frac{dx}{(x+a)(x-a)^2}$.

7. $\displaystyle\int \frac{x^5\,dx}{x^4-5x^2+4}$.

8. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$.

9. $\displaystyle\int_4^5 \frac{(x+2)(x-3)\,dx}{(2x+1)^2(x-2)}$.

10. $\displaystyle\int \frac{x^4\,dx}{x^3+27}$.

§ 4. Integration of irrational functions

In this paragraph we shall deal with certain integrals involving $\sqrt{(ax+b)}$ or $\sqrt{(ax^2+bx+c)}$.

Dealing with $\sqrt{(ax+b)}$ first, let us write $X=\sqrt{(ax+b)}$. Suppose then that the integrand of the required integral is a rational function of the two variables x and X, which we will denote by $S(x, X)$. We mean that $S(x, X)$ may be the ratio of two polynomials involving x and X. By the change of variable $ax+b=t^2$ we have $a\,dx=2t\,dt$ and $x=(t^2-b)/a$, so that $S(x, X)=S\{(t^2-b)/a, t\}$; the integral of $S(x, X)$ becomes

$$\frac{1}{a}\int S\{(t^2-b)/a, t\}2t\,dt,$$

which is the integral of a rational function of t with respect to t, and can be evaluated by the methods of § 3.

Example 24

$$\int \frac{\mathrm{d}x}{1 - \sqrt{x}}.$$

Writing $\sqrt{x}=t$, $\mathrm{d}x=2t\,\mathrm{d}t$, we get for the integral

$$\int \frac{2t\,\mathrm{d}t}{1-t} = -2\int \mathrm{d}t + 2\int \frac{\mathrm{d}t}{1-t},$$

which gives immediately

$$-2t - 2\log|1-t| = -2\sqrt{x} - 2\log|1-\sqrt{x}|.$$

Example 25

$$\int x\sqrt{(1+x)}\,\mathrm{d}x.$$

Writing $\sqrt{(1+x)}=t$, $\mathrm{d}x=2t\,\mathrm{d}t$, we get

$$\int (t^2-1)\,t\,2t\,\mathrm{d}t = \int (2t^4-2t^2)\,\mathrm{d}t,$$

which gives immediately

$$\tfrac{2}{5}(1+x)^{\frac{5}{2}} - \tfrac{2}{3}(1+x)^{\frac{3}{2}}.$$

§ 4.1. Integrals involving $\sqrt{(ax^2+bx+c)}$

For integrals of this type we note first that we can already deal with integrals of the form

$$\int (ax^2+bx+c)^{-\frac{1}{2}}\,\mathrm{d}x, \tag{5.44}$$

and

$$\int (ax^2+bx+c)^{\frac{1}{2}}\,\mathrm{d}x. \tag{5.45}$$

We write

$$ax^2+bx+c = a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\}$$

$$= \frac{1}{4a}\{(2ax+b)^2 + 4ac - b^2\},$$

and then a substitution $2ax+b=t$ will reduce the integral (5.44) to one of the standard forms

$$\int \frac{\mathrm{d}t}{\sqrt{(t^2-\alpha^2)}}, \quad \int \frac{\mathrm{d}t}{\sqrt{(t^2+\alpha^2)}}, \quad \int \frac{\mathrm{d}t}{\sqrt{(\alpha^2-t^2)}},$$

α being a constant. By the same substitution the integral (5.45) will reduce to corresponding forms with the roots in the numerator. All such integrals have been dealt with by the appropriate trigonometric or hyperbolic substitution in Ch. 2 § 1.3 and § 1.1 of this chapter.

Example 26

$$\int \frac{dx}{\sqrt{(x^2+2x+3)}} = \int \frac{dx}{\sqrt{\{(x+1)^2+2\}}};$$

with $x+1=t$, this reduces to the standard form

$$\int \frac{dt}{\sqrt{(t^2+2)}} = \sinh^{-1}\frac{t}{\sqrt{2}} = \log\frac{t+\sqrt{(t^2+2)}}{\sqrt{2}},$$

or in terms of x,

$$\log\frac{x+1+\sqrt{(x^2+2x+3)}}{\sqrt{2}}.$$

Example 27

$$\int \sqrt{(5x^2+2x-7)}\,dx = \frac{1}{\sqrt{5}}\int \sqrt{(25x^2+10x-35)}\,dx = \frac{1}{\sqrt{5}}\int \sqrt{\{(5x+1)^2-36\}}\,dx.$$

Writing $5x+1=t$, the integral will reduce to the form

$$\int \sqrt{(t^2-36)}\,dt,$$

and then the further substitution $t=6\cosh\theta$ must be made. This is usually done directly by writing $5x+1=6\cosh\theta$, $5\,dx=6\sinh\theta\,d\theta$, and then the integral is

$$\frac{18}{5\sqrt{5}}\int 2\sinh^2\theta\,d\theta = \frac{18}{5\sqrt{5}}\int(\cosh 2\theta-1)\,d\theta = \frac{18}{5\sqrt{5}}(\sinh\theta\cosh\theta-\theta);$$

in terms of x this becomes

$$\tfrac{1}{10}\left\{(5x+1)\sqrt{(5x^2+2x-7)} - \frac{36}{\sqrt{5}}\cosh^{-1}\frac{5x+1}{6}\right\}.$$

Example 28

$$\int \sqrt{(1+2x-3x^2)}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{(3+6x-9x^2)}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{\{4-(3x-1)^2\}}\,dx,$$

and here the appropriate substitution is either $3x-1=2\sin\theta$ or $3x-1=2\cos\theta$. The integral is then evaluated as in Example 27, except that trigonometric functions are involved instead of the hyperbolic functions.

We proceed now to consider the following integrals

$$\int \frac{(lx+m)\,dx}{\sqrt{(ax^2+bx+c)}}, \tag{5.46}$$

and

$$\int (lx+m)\sqrt{(ax^2+bx+c)}\,dx. \tag{5.47}$$

In both these integrals we proceed by similar methods. We write for

example

$$\int \frac{(lx+m)\,dx}{\sqrt{(ax^2+bx+c)}} = \frac{l}{2a}\int \frac{(2ax+b)\,dx}{\sqrt{(ax^2+bx+c)}} + \left(m-\frac{lb}{2a}\right)\int \frac{dx}{\sqrt{(ax^2+bx+c)}}, \tag{5.48}$$

where again, as in § 3 we have arranged the linear factor $2ax+b$ in the first integral to be the derivative of ax^2+bx+c, and the coefficient $l/2a$ has been chosen to include the whole of the term lx in this integral. It is again easy to check that the constant term in front of the second integral ensures the correctness of eq. (5.48). The first term in eq. (5.48) can then be evaluated by writing $t=ax^2+bx+c$, whilst the second is of the form (5.44). The integral (5.47) is dealt with in a similar way leading to an integral of the form (5.45).

Example 29

$$\int \frac{(1+x)\,dx}{\sqrt{(1-x-x^2)}} = -\frac{1}{2}\int \frac{-(2x+1)\,dx}{\sqrt{(1-x-x^2)}} + \frac{1}{2}\int \frac{dx}{\sqrt{(1-x-x^2)}}.$$

Putting $u=1-x-x^2$ in the first integral on the right gives

$$-\frac{1}{2}\int \frac{du}{\sqrt{u}} + \frac{1}{2}\int \frac{dx}{\sqrt{\{\frac{5}{4}-(x+\frac{1}{2})^2\}}} = -\sqrt{u} + \tfrac{1}{2}\sin^{-1}\frac{2x+1}{\sqrt{5}}$$
$$= -\sqrt{(1-x-x^2)} + \tfrac{1}{2}\sin^{-1}\frac{2x+1}{\sqrt{5}}.$$

Example 30

$$\int (x+1)\sqrt{(3x^2-2x+1)}\,dx$$
$$= \tfrac{1}{6}\int (6x-2)\sqrt{(3x^2-2x+1)}\,dx + \tfrac{4}{3}\int \sqrt{(3x^2-2x+1)}\,dx. \tag{5.49}$$

Putting $u=3x^2-2x+1$ in the first integral on the right in eq. (5.49) gives

$$\tfrac{1}{6}\int \sqrt{u}\,du + \frac{4}{3\sqrt{3}}\int \sqrt{\{(3x-1)^2+2\}}\,dx.$$

The first integral here is simply $\frac{1}{9}u^{\frac{3}{2}}=\frac{1}{9}(3x^2-2x+1)^{\frac{3}{2}}$: writing $3x-1=\sqrt{2}\sinh\theta$, so that $3\,dx=\sqrt{2}\cosh\theta\,d\theta$ and $\sqrt{\{(3x-1)^2+2\}}=\sqrt{2}\cosh\theta$, the second integral is

$$\frac{4}{9\sqrt{3}}\int 2\cosh^2\theta\,d\theta = \frac{4}{9\sqrt{3}}\int (1+\cosh 2\theta)\,d\theta = \frac{4}{9\sqrt{3}}(\theta+\sinh\theta\cosh\theta),$$

or in terms of x, the whole of the integral (5.49) is

$$\tfrac{1}{9}(3x^2-2x+1)^{\frac{3}{2}} + \frac{4}{9\sqrt{3}}\sinh^{-1}\frac{3x-1}{\sqrt{2}} + \tfrac{2}{9}(3x-1)\sqrt{(3x^2-2x+1)}.$$

We deal next with an integral of the form

$$\int \frac{\mathrm{d}x}{(lx+m)\sqrt{(ax^2+bx+c)}}. \tag{5.50}$$

A standard method for evaluating an integral of this type is to write $lx+m=1/u$; then since $x=(u^{-1}-m)/l$ the quadratic factor ax^2+bx+c will become in terms of u a factor $(Au^2+Bu+C)/u^2$ where A, B, C are all constants. Thus since also $\mathrm{d}x=-\mathrm{d}u/lu^2$ the evaluation of the integral will reduce to the evaluation of an integral of the form

$$\int \frac{\mathrm{d}u}{\sqrt{(Au^2+Bu+C)}},$$

which has already been dealt with.

Example 31

$$\int \frac{\mathrm{d}x}{(x+1)\sqrt{(1+2x-x^2)}}.$$

Write $x+1=1/u$, so that $\mathrm{d}x=-\mathrm{d}u/u^2$. Also $1+2x-x^2=(-1+4u-2u^2)/u^2$, so that the integral becomes

$$-\int \frac{\mathrm{d}u}{\sqrt{(-1+4u-2u^2)}} = -\int \frac{\mathrm{d}u}{\sqrt{\{1-2(u-1)^2\}}} = -\frac{1}{\sqrt{2}}\sin^{-1}(u-1)\sqrt{2},$$

or again in terms of x, this is

$$\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{x+1}.$$

Many other integrals involving $\sqrt{(ax^2+bx+c)}$ may be reduced to one or other of the forms dealt with in this paragraph and thereby evaluated. In general, if $S(x, X)$ is a rational function of the variables x, X where $X=\sqrt{(ax^2+bx+c)}$, then an integral of $S(x, X)$ can be evaluated by an appropriate trigonometric or hyperbolic substitution. Such integrals will be mentioned later in § 5.3.

§ 4.2. Evaluation of the integrals of certain irrational functions by special substitutions

Certain particular integrals of the type treated in § 4.1 may be evaluated more simply by special substitutions. These will be illustrated by examples.

Example 32

$$\int \left(\frac{a - x}{a + x}\right)^{\frac{1}{2}} \mathrm{d}x.$$

This integral may be evaluated by writing it as

$$\int \frac{(a - x)\,\mathrm{d}x}{\sqrt{(a^2 - x^2)}},$$

and proceeding as in Example 29. A simpler method is to make the substitution $x = a \cos 2\theta$, so that $\mathrm{d}x = -2a \sin 2\theta \, \mathrm{d}\theta = -4a \sin\theta \cos\theta \, \mathrm{d}\theta$, and

$$\sqrt{\{(a - x)/(a + x)\}} = \sqrt{\{2a \sin^2\theta/2a \cos^2\theta\}} = \sin\theta/\cos\theta.$$

Then the integral becomes

$$-2a \int 2\sin^2\theta \, \mathrm{d}\theta = -2a \int (1 - \cos 2\theta)\,\mathrm{d}\theta$$
$$= -2a(\theta - \sin\theta\cos\theta) = -a\cos^{-1}(x/a) + \sqrt{(a^2 - x^2)}.$$

Example 33

$$\int \left(\frac{b - x}{a + x}\right)^{\frac{1}{2}} \mathrm{d}x.$$

The substitution $b - x = (a + x)\tan^2\theta$ or $x = b\cos^2\theta - a\sin^2\theta$ reduces this integral to

$$-(b + a)\int 2\sin^2\theta\,\mathrm{d}\theta = -(b + a)\{\theta + \tfrac{1}{2}\sin 2\theta\},$$

and remembering that $\sin 2\theta = 2\tan\theta/(1 + \tan^2\theta)$ the result in terms of x is

$$-(a + b)\tan^{-1}\left(\frac{b - x}{a + x}\right)^{\frac{1}{2}} - (b - x)^{\frac{1}{2}}(a + x)^{\frac{1}{2}}.$$

Similar substitutions may also be found in the following cases:

Example 34

In $\displaystyle\int \left(\frac{x - a}{x + a}\right)^{\frac{1}{2}} \mathrm{d}x$, put $x = a\cosh 2\theta$.

Example 35

In $\displaystyle\int \left(\frac{x \pm b}{x \pm a}\right)^{\frac{1}{2}} \mathrm{d}x$, put $x \pm b = (x \pm a)\tanh^2\theta$.

Example 36

In $\displaystyle\int \left(\frac{b + x}{a - x}\right)^{\frac{1}{2}} \mathrm{d}x$, put $b + x = (a - x)\tan^2\theta$.

EXERCISE 5.4

Evaluate the integrals in Nos. 1–10.

1. $\displaystyle\int x^2\sqrt{(2-x)}\,\mathrm{d}x.$

2. $\displaystyle\int \frac{x^2}{\sqrt{(2x-1)}}\,\mathrm{d}x.$

3. $\displaystyle\int_0^6 \frac{x^2+2}{\sqrt{(x+3)}}\,\mathrm{d}x.$

4. $\displaystyle\int_0^\infty \frac{\sqrt{x}\,\mathrm{d}x}{1+x^2}.$

5. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(3+2x-x^2)}}.$

6. $\displaystyle\int \sqrt{(2x^2-7x+5)}\,\mathrm{d}x.$

7. $\displaystyle\int_0^a \frac{x^2+ax+a^2}{\sqrt{(x^2+a^2)}}\,\mathrm{d}x.$

8. $\displaystyle\int x\sqrt{(x^2+6x+109)}\,\mathrm{d}x.$

9. $\displaystyle\int_{-3}^{-2} \frac{(3x+2)\,\mathrm{d}x}{\sqrt{(1-4x-x^2)}}.$

10. $\displaystyle\int_2^3 \frac{\mathrm{d}x}{(x-1)\sqrt{(x^2+x+1)}}.$

11. Show that

$$\int_0^\infty \frac{\mathrm{d}x}{(x+1)\sqrt{(x^2+x+1)}} = \frac{\pi}{3\sqrt{3}}.$$

12. Prove that, if a and b are positive,

$$\int_b^\infty \frac{\mathrm{d}x}{(x+a)\sqrt{(x-b)}} = \frac{\pi}{\sqrt{(a+b)}}.$$

Evaluate the integrals in Nos. 13, 14.

13. $\displaystyle\int_a^{2a} x\left(\frac{x-a}{x+a}\right)^{\frac{1}{2}}\mathrm{d}x.$

14. $\displaystyle\int_0^2 (1+x)\left(\frac{2-x}{2+x}\right)^{\frac{1}{2}}\mathrm{d}x.$

§ 5. Integrals of other non-rational functions

Frequently we have to integrate functions involving the transcendental functions e^x, $\log x$, circular and hyperbolic functions. We have already

dealt with the integration of quite a number of simple functions of this type, but in general a suitable substitution is the only means of transforming such functions into rational functions which can be integrated.

§ 5.1. Rational functions of e^x

Let $S(e^x)$ denote a rational function of the variable quantity e^x. We wish to integrate such a function with respect to x. We make the substitution $u=e^x$ or $x=\log u$, so that $dx=du/u$ and then we have

$$\int S(e^x)\,dx = \int \frac{S(u)}{u}\,du. \tag{5.51}$$

The integrand in the integral on the right in eq. (5.51) is a rational function of u and may therefore be integrated by the methods of § 3.

Example 37

$$\int \frac{e^x\,dx}{1+2e^x+e^{2x}}.$$

By the substitution $u=e^x$, this integral becomes

$$\int \frac{du}{1+2u+u^2} = \int \frac{du}{(1+u)^2} = -\frac{1}{1+u},$$

or in terms of x, simply $-(1+e^x)^{-1}$.

Example 38

$$\int \operatorname{sech} x\,dx = \int \frac{2\,dx}{e^x+e^{-x}},$$

and with the substitution $u=e^x$, this becomes

$$\int \frac{2\,du}{1+u^2} = 2\tan^{-1} u = 2\tan^{-1} e^x.$$

Thus we have the result

$$\int \operatorname{sech} x\,dx = 2\tan^{-1} e^x. \tag{5.52}$$

Example 39

In the same way as in Example 38 it may be shown that

$$\int \operatorname{cosech} x\,dx = -2\tanh^{-1} e^x = -\log\left|\frac{1+e^x}{1-e^x}\right|. \tag{5.53}$$

§ 5.2. A RATIONAL FUNCTION OF THE VARIABLES SIN x, COS x

Let $S(\cos x, \sin x)$ denote a rational function of the variables $\sin x$, $\cos x$. The substitution

$$t = \tan \tfrac{1}{2}x, \quad \text{with} \quad \mathrm{d}t = \tfrac{1}{2}\sec^2 \tfrac{1}{2}x \, \mathrm{d}x \quad \text{or} \quad \mathrm{d}x = 2\,\mathrm{d}t/(1+t^2), \tag{5.54}$$

gives

$$\int S(\cos x, \sin x)\,\mathrm{d}x = \int S\left\{\frac{1-t^2}{1+t^2}, \quad \frac{2t}{1+t^2}\right\}\frac{2\,\mathrm{d}t}{1+t^2},$$

and again the integrand in the right hand integral is a rational function of t. For example, consider

$$\int \frac{\mathrm{d}x}{a + b\cos x}, \qquad a \neq b. \tag{5.55}$$

Using eq. (5.54) we have for the integral

$$\int \frac{2\,\mathrm{d}t}{a(1+t^2) + b(1-t^2)} = \frac{2}{a-b}\int \frac{\mathrm{d}t}{t^2 + (a+b)/(a-b)}.$$

If $(a+b)/(a-b) = +\beta^2$, this becomes

$$\frac{2}{(a-b)\beta}\tan^{-1}\frac{t}{\beta} = \frac{2}{(a-b)\beta}\tan^{-1}\left(\frac{1}{\beta}\tan \tfrac{1}{2}x\right). \tag{5.56}$$

If $(a+b)/(a-b) = -\beta^2$, the integral becomes

$$\frac{-2}{(a-b)\beta}\tanh^{-1}\frac{t}{\beta} = \frac{1}{(a-b)\beta}\log\left|\frac{\tan \frac{1}{2}x - \beta}{\tan \frac{1}{2}x + \beta}\right|. \tag{5.57}$$

Example 40

$$\int \frac{\mathrm{d}x}{3 + 2\cos x}.$$

With the substitution $t = \tan \frac{1}{2}x$, this becomes

$$\int \frac{2\,\mathrm{d}t}{5+t^2} = \frac{2}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}} = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan \tfrac{1}{2}x\right).$$

Example 41

$$\int_{-\pi}^{+\pi} \frac{(1-\alpha^2)\,\mathrm{d}x}{1 - 2\alpha\cos x + \alpha^2} \qquad (0 < \alpha < 1).$$

Here, since $\cos x$ is an even function, the integrand is an even function, and using the result in Ch. 2 § 4, the integral is equal to

$$2(1-\alpha^2)\int_0^\pi \frac{dx}{(1+\alpha^2)-2\alpha\cos x}.$$

This integral is of the form (5.55) with $a=1+\alpha^2$, $b=-2\alpha$, so that

$$\frac{a+b}{a-b}=\frac{(1-\alpha)^2}{(1+\alpha)^2}=+\beta^2$$

where $\beta=(1-\alpha)/(1+\alpha)$. The value of the integral is therefore, using eq. (5.56)

$$4\left[\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\tfrac{1}{2}x\right)\right]_0^\pi = 4\tan^{-1}\left\{\frac{1+\alpha}{1-\alpha}\tan\tfrac{1}{2}\pi\right\}=2\pi.$$

Other particular examples of the integral (5.55) are $\int \sec x\,dx$ and $\int \operatorname{cosec} x\,dx$. We have

$$\int \sec x\,dx = \int \frac{dx}{\cos x}, \tag{5.58}$$

or using $t=\tan\frac{1}{2}x$ this becomes

$$\int \frac{2\,dt}{1-t^2} = 2\tanh^{-1} t = \log\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right|,$$

which may be written in the forms

$$\log|\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x)| \quad \text{or} \quad \log|\sec x+\tan x|, \tag{5.59}$$

the latter form following after some elementary trigonometry. Similarly we can show that

$$\int \operatorname{cosec} x\,dx = \log|\tan\tfrac{1}{2}x| = -\log|\operatorname{cosec} x+\cot x|. \tag{5.60}$$

Having derived these results in this way, we can immediately see another simpler way of verifying them. In the integral (5.58) we multiply the integrand $\sec x$ by $(\sec x+\tan x)$ in the numerator and denominator; we get

$$\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx.$$

This is of the form

$$\int \frac{f'(x)\,dx}{f(x)},$$

with $f(x) = \sec x + \tan x$; the result is therefore

$$\log|\sec x + \tan x|.$$

Similarly we can show that

$$\int \operatorname{cosec} x \,\mathrm{d}x = -\log|\operatorname{cosec} x + \cot x|, \tag{5.61}$$

$$\int \operatorname{cosech} x \,\mathrm{d}x = -\log|\operatorname{cosech} x + \coth x|. \tag{5.62}$$

It is left as an exercise for the reader to verify that there is no corresponding result for $\int \operatorname{sech} x \,\mathrm{d}x$. The value of this integral is given by eq. (5.52).

In certain cases the rational function of $\cos x$, $\sin x$ denoted by $S(\cos x, \sin x)$ may be arranged as a function of $\tan x$. The appropriate substitution is then

$$t = \tan x, \quad \text{with} \quad \mathrm{d}x = \mathrm{d}t/(1 + t^2). \tag{5.63}$$

Example 42

$$\int \frac{\cos x \,\mathrm{d}x}{\cos x + \sin x} = \int \frac{\mathrm{d}x}{1 + \tan x}.$$

Using the substitution (5.63) the integral becomes

$$\int \frac{\mathrm{d}t}{(1 + t)(1 + t^2)} = \frac{1}{2} \int \frac{\mathrm{d}t}{1 + t} - \frac{1}{2} \int \frac{(t - 1)\,\mathrm{d}t}{1 + t^2},$$

using partial fractions. The methods of §§ 3 and 3.1 will lead to the result

$$\tfrac{1}{2} \log|1 + t| - \tfrac{1}{4} \log(1 + t^2) + \tfrac{1}{2} \tan^{-1} t,$$

or in terms of x this becomes

$$\tfrac{1}{2} \log|\sin x + \cos x| + \tfrac{1}{2}x.$$

Rational functions of $\cosh x$ and $\sinh x$ are transformed into rational functions of t with the substitutions $t = \tanh \frac{1}{2}x$ or $t = \tanh x$, by the methods of this paragraph.

§ 5.3. Integrals involving $\sqrt{(ax^2+bx+c)}$

Certain integrals involving $\sqrt{(ax^2+bx+c)}$ were discussed in § 4.1. In general if we write $X = \sqrt{(ax^2+bx+c)}$ and denote by $S(x, X)$ any rational function of the variables x, X then the function $S(x, X)$ may be reduced to a rational function of either $(\sin\theta, \cos\theta)$ or a rational function

of (sinh θ, cosh θ) by the appropriate trigonometric or hyperbolic substitution. The integral can then be evaluated either by the methods of § 5.2 or simpler methods.

xample 43

$$\int_0^1 (x^2 + 1)\sqrt{(1 - x^2)}\, dx.$$

Write $x = \sin\theta$ with $dx = \cos\theta\, d\theta$, so that the integral becomes

$$\int_0^{\frac{1}{2}\pi} (1 + \sin^2\theta)\cos^2\theta\, d\theta$$

which is easily evaluated as

$$\tfrac{1}{2}[\theta + \sin\theta\cos\theta]_0^{\frac{1}{2}\pi} + \tfrac{1}{4}\cdot\tfrac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 44

$$\int_1^\infty \frac{dx}{x^3\sqrt{(x^2 - 1)}}.$$

Put $x = \cosh\theta$ with $dx = \sinh\theta\, d\theta$; the integral becomes

$$\int_{x=1}^{x=\infty} \operatorname{sech}^3\theta\, d\theta.$$

To evaluate this integral we write $du/d\theta = \operatorname{sech}^2\theta$, $v = \operatorname{sech}\theta$ in the method of integration by parts, and omitting the limits of integration for the present, we get

$$\int \operatorname{sech}^3\theta\, d\theta = \operatorname{sech}\theta\tanh\theta + \int \operatorname{sech}\theta\tanh^2\theta\, d\theta.$$

Putting $\tanh^2\theta = 1 - \operatorname{sech}^2\theta$ and collecting like terms, we get

$$2\int \operatorname{sech}^3\theta\, d\theta = \operatorname{sech}\theta\tanh\theta + \int \operatorname{sech}\theta\, d\theta = \operatorname{sech}\theta\tanh\theta + 2\tan^{-1} e^\theta, \qquad (5.64)$$

using eq. (5.52). Putting the result in terms of x, we have

$$\int_1^\infty \frac{dx}{x^3\sqrt{(x^2 - 1)}} = \left[\frac{\sqrt{(x^2 - 1)}}{2x^2} + \tan^{-1}\{x + \sqrt{(x^2 - 1)}\}\right]_1^\infty = \tfrac{1}{2}\pi - \tfrac{1}{4}\pi = \tfrac{1}{4}\pi.$$

Although in this chapter we have attempted to give methods of integration for many of the functions that occur in physical problems there are nevertheless many quite simple functions which, being continuous, have integrals, but whose integrals cannot be expressed in finite terms by known elementary functions. One such integral is $\int (1 - x^4)^{-\frac{1}{2}}\, dx$; this is a special case of the integral $\int \{(1 - x^2)(1 - k^2x^2)\}^{-\frac{1}{2}}\, dx$, which with the

substitution $x = \sin\theta$ becomes $\int (1 - k^2 \sin^2\theta)^{-\frac{1}{2}}\,d\theta$. This integral is known as an *elliptic integral*, the name arising from the fact that such integrals arise in the problem of determining the length of the arc of an ellipse. Other functions whose indefinite integrals cannot be expressed in terms of known elementary functions are $\exp(-x^2)$, $(\sin x)/x$ and $\sin x^2$. Certain of these integrals, including elliptic integrals, are of such importance that their values for particular limits of integration and for certain values of k in the case of elliptic integrals, have been tabulated, but the consideration of these integrals is beyond the scope of this book.

EXERCISE 5.5

Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{e^{2x} + e^{-2x} + 2}$.

2. $\displaystyle\int \frac{e^x + 2}{e^x - e^{-x}}\,dx$.

3. $\displaystyle\int_0^\infty \frac{dx}{(1 + e^x)(1 + e^{-x})}$.

4. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{d\theta}{4 + 3\cos\theta}$.

5. $\displaystyle\int_0^{\pi} \frac{d\theta}{5 + 4\cos\theta}$.

6. $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\sin\theta(1 + \sin\theta)}$.

7. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\,dx$.

8. $\displaystyle\int_{\frac{1}{2}\pi}^{\frac{2}{3}\pi} \frac{dx}{\sin x(3\sin x + \cos x)}$.

9. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{2\cos x + 3\sin x + 3}$.

10. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1 + 2\cos x}{(2 + \cos x)^2}\,dx$.

11. $\displaystyle\int \sec^3\theta\,d\theta$.

12. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{7 + \cos 2x}$.

13. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{3\sin x + 4\cos x + 5}{2\sin x + \cos x + 2}\,dx$.

14. $\displaystyle\int \frac{dx}{(x + x^2)\sqrt{(x^2 + 1)}}$ $(x > 0)$.

15. $\displaystyle\int_0^{2a} x^5\sqrt{(2ax - x^2)}\,dx$.

16. $\displaystyle\int_0^{4} \frac{x^2 + 1}{\sqrt{(x^2 + 9)}}\,dx$.

§ 6. Convergence of infinite integrals

In Ch. 2 § 3 we dealt with integrals

$$\int_a^b f(x)\,\mathrm{d}x,$$

in which either the limits of integration were infinite or the function $f(x)$ was discontinuous at some point in the range $a \leqslant x \leqslant b$. However in that chapter it was assumed that an indefinite integral

$$F(x) = \int f(x)\,\mathrm{d}x,$$

could be evaluated. When $F(x)$ cannot be found in terms of known functions it is still possible to determine whether such integrals are convergent.

We shall assume that $f(x)$ is positive; a negative $f(x)$ can be dealt with by considering $\int_b^a -f(x)\,\mathrm{d}x$. As in Ch. 2 § 3 we can reduce the different forms of infinite integral to either

$$\text{(i)} \lim_{A\to\infty} \int_a^A f(x)\,\mathrm{d}x, \qquad \text{(ii)} \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)\,\mathrm{d}x \quad \text{or} \quad \text{(iii)} \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^{b} f(x)\,\mathrm{d}x.$$

Suppose a function $\varphi(x)$ can be found whose indefinite integral is known, and k is a finite positive constant such that

$$f(x) \leqslant k\varphi(x),$$

for all values of x in the given range. Then by the definition of a definite integral as the limit of a sum and by the comparison test for series, we know that

$$\int_a^A f(x)\,\mathrm{d}x \leqslant k \int_a^A \varphi(x)\,\mathrm{d}x, \tag{5.65}$$

and similarly for the integrals in (ii) and (iii). The limit of the integral on the left hand side of (5.65) will exist provided the limit on the right hand side exists, and this latter can be tested by the methods of Ch. 2 § 3. The same applies to integrals of the form (ii) and (iii).

Example 45

To show that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,\mathrm{d}x \tag{5.66}$$

converges provided $0 < a < 1$.

The integral (5.66) has an infinite upper limit and also when $a<1$ the integrand becomes infinite at the lower limit. We therefore divide the integral into two parts and consider

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{x^{a-1}}{1+x}\,\mathrm{d}x, \tag{5.67}$$

where ε is positive, and

$$\lim_{A\to\infty}\int_{1}^{A}\frac{x^{a-1}}{1+x}\,\mathrm{d}x. \tag{5.68}$$

For values of x in the range $\varepsilon<x<1$, we have

$$\frac{x^{a-1}}{1+x}<x^{a-1}$$

and so

$$\int_{\varepsilon}^{1}\frac{x^{a-1}}{1+x}\,\mathrm{d}x<\int_{\varepsilon}^{1}x^{a-1}\,\mathrm{d}x=\left[\frac{x^a}{a}\right]_{\varepsilon}^{1},$$

and this last expression has a finite value as $\varepsilon\to 0$ provided $a>0$. Thus the integral (5.67) is convergent.

For values of x in the range $1\leqslant x\leqslant A$, we have

$$\frac{x^{a-1}}{1+x}\leqslant\frac{1}{x(1+x)},$$

provided $a<1$. Thus

$$\int_{1}^{A}\frac{x^{a-1}}{1+x}\,\mathrm{d}x\leqslant\int_{1}^{A}\frac{\mathrm{d}x}{x(1+x)}=\left[\log\frac{x}{x+1}\right]_{1}^{A}=\log\frac{2A}{A+1},$$

and since $\lim_{A\to\infty}\log\{2A/(A+1)\}=\log 2$, then the integral (5.68) is convergent. The result follows.

Another example of the convergence of an integral of this kind will be given in Ch. 19 § 1.

FURTHER THEOREMS CONCERNING FUNCTIONS OF ONE VARIABLE. EXPANSION IN SERIES

§ 1. Introduction

The theorems which we will prove in the beginning of this chapter are concerned with continuous and differentiable functions; they are of great importance for any subsequent development of the analysis of functions of one variable and also for the study of functions of more than one variable. In later sections of the chapter we shall see that they are of frequent application in practical problems. The proofs of the theorems although not difficult in themselves are based on the fundamental properties of continuous functions. These properties of continuous functions are themselves dependent on an abstract theorem usually known as the Heine–Borel theorem. The difficulty of proving this theorem is one of the main obstacles to giving a rigorous *elementary* account of this part of the subject.

We feel, in fact, that rigorous proofs of the Heine–Borel theorem and others immediately dependent upon it are outside the scope of a book of this description, but that it is sufficient to state, without proof, those theorems on continuous functions on which the chapter is based. We will see that these theorems are 'intuitively obvious'.

§ 1.1. Bounds of a function

In Ch. 3 § 4.1 we defined bounds of a sequence (α_n). Similar ideas can now be introduced for a function $\varphi(x)$. The values assumed by $\varphi(x)$ for different values of x, say $x_0(=a), x_1, x_2,\ldots, x_n(=b)$ in the interval (a, b) will form a sequence of terms $\varphi_0, \varphi_1,\ldots, \varphi_n$ where $\varphi_n=\varphi(x_n)$. If there is a number K such that $\varphi(x)\leqslant K$ for all values of x in the interval (a, b) we say that $\varphi(x)$ is bounded above and the least value of K which satisfies this condition is called the *least upper bound* U. Thus if U is the least upper bound of $\varphi(x)$ then

$$\varphi(x) \leqslant U, \tag{6.1}$$

for all x in (a, b) and for any small positive number ε

$$\varphi(x) > U - \varepsilon, \tag{6.2}$$

for at least one value of x in (a, b).

Similarly we define the *greatest lower bound* L of $\varphi(x)$ in (a, b) to be such that

$$\varphi(x) \geqslant L, \tag{6.3}$$

for all x in (a, b), and

$$\varphi(x) < L + \varepsilon, \tag{6.4}$$

for at least one value of x in (a, b).

The two theorems on continuous functions which we will require, but which we will not prove are as follows:

THEOREM 1. *If $\varphi(x)$ is continuous in (a, b) then it is bounded in (a, b).*

THEOREM 2. *If $\varphi(x)$ is continuous in (a, b) and U, L are its least upper and greatest lower bounds, then there is at least one value of $x=\xi$, such that $a \leqslant \xi \leqslant b$ where*

$$\varphi(\xi) = U, \tag{6.5}$$

and at least one value of $x=\eta$ such that $a \leqslant \eta \leqslant b$ where

$$\varphi(\eta) = L. \tag{6.6}$$

We say that the function $\varphi(x)$ *attains its bounds* in the closed interval.

§ 1.2. ROLLE'S THEOREM

Let $f(x)$ be a function subject to the following conditions:

(i) $f(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$,

(ii) $f'(x)$ exists in the open interval $a < x < b$,

(iii) $f(a) = f(b)$;

then there is a point c, such that $a < c < b$ where $f'(c) = 0$.

This theorem is obvious geometrically as shown in fig. 6.1. If $f(x)$ decreases initially as x increases from a, then in order for it to return to its original value $f(a)$ when $x=b$, $f(x)$ must increase for some value of x in (a, b). If $x=c$ is the point where the increase begins, the tangent to the curve at this point will be parallel to the x-axis, so that $f'(c)=0$. This presupposes that there is no sharp bend or kink in the curve at $x=c$ as shown in the curves (α) and (β) in fig. 6.2; so the existence of $f'(x)$ for $a<x<b$ is essential. It does not matter if the curve has a sharp bend at $x=a$ or $x=b$.

To prove the result analytically, we require Theorems 1 and 2 of § 1.1. Since $f(x)$ is continuous, it is bounded and attains its bounds. Then also since $f(a)=f(b)$ the bounds U and L could both be equal to $f(a)$ and the function be constant throughout the interval, that is $f(x) \equiv f(a)$; the theorem is then trivial.

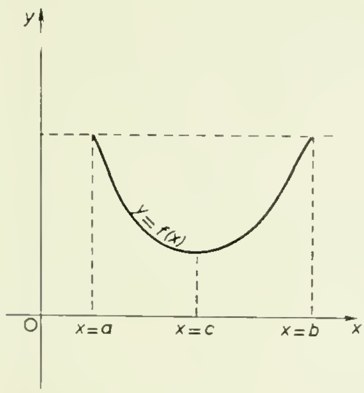

Fig. 6.1

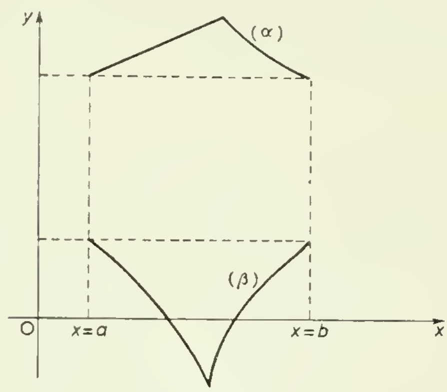

Fig. 6.2

If the bounds are not both equal to $f(a)$, then suppose the least upper bound U is different from $f(a)$ and attained at the point $x=c$ where $c \neq a$ or b; then we can show that $f'(c)=0$. From eq. (6.5) the value of the upper bound U is $f(c)$ and therefore from eq. (6.1) there must exist a range of positive values of h for which

$$f(c+h) \leqslant f(c),$$

or

$$f(c+h) - f(c) \leqslant 0;$$

hence dividing through by h, we get

$$\frac{f(c+h) - f(c)}{h} \leqslant 0. \tag{6.7}$$

But similarly from eq. (6.1), there must exist a range of positive values of h for which

$$f(c-h) \leqslant f(c),$$

or

$$f(c-h) - f(c) \leqslant 0,$$

and dividing through by $-h$, we get

$$\frac{f(c-h) - f(c)}{-h} \geqslant 0. \tag{6.8}$$

Now as $h \to 0$, the expressions on the left hand sides of eqs. (6.7) and (6.8) both tend to the value of $f'(c)$, the derivative of $f(x)$ at $x=c$; this derivatives exists since $a<c<b$. So from eq. (6.7) $f'(c) \leqslant 0$, whilst from eq. (6.8) $f'(c) \geqslant 0$. These two results can be satisfied simultaneously only if

$$f'(c) = 0.$$

The result can be derived in a similar way using eqs. (6.6) and (6.3) if the greatest lower bound L is the one that is different from $f(a)$.

§ 1.3. Mean value theorem

Let $f(x)$ be a function which satisfies the following conditions:

(i) $f(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$,

(ii) $f'(x)$ exists in the open interval $a<x<b$;

then there is a point c in the open interval $a<c<b$ where

$$f(b) - f(a) = (b - a)f'(c). \tag{6.9}$$

The conditions (i) and (ii) in this theorem are exactly the same as in Rolle's theorem, but the condition (iii) of Rolle's theorem is not satisfied. However we use Rolle's theorem to prove it in the following way.

Define a function $\psi(x)$ given by

$$\psi(x) = f(x) - Sx, \tag{6.10}$$

where S is a constant, so chosen that $\psi(a) = \psi(b)$; S is therefore given by

$$f(a) - Sa = f(b) - Sb,$$

or

$$S = \frac{f(b) - f(a)}{b - a}. \tag{6.11}$$

Then, since $f(x)$ and x are continuous functions, so is $\psi(x)$; also since

$$\psi'(x) = f'(x) - S, \tag{6.12}$$

$\psi'(x)$ exists for $a<x<b$; so $\psi(x)$ satisfies conditions (i), (ii) and (iii) of Rolle's theorem. Thus there must be a point c such that $a<c<b$ where

$$\psi'(c) = 0;$$

using eqs. (6.10) and (6.11) this becomes

$$f'(c) = S = \frac{f(b) - f(a)}{b - a},$$

which may be written in the form (6.9).

Again the geometrical interpretation is obvious from fig. 6.3. The expression

$$S = \frac{f(b) - f(a)}{b - a},$$

is the slope of the chord AB, whilst $f'(c)$ is the slope of the tangent at the point C where $x=c$. Thus the theorem states that there must be a point such as C on the curve between A and B, where the tangent is parallel to the chord AB.

It is often convenient to express the mean value theorem in a slightly different form. If we put $b=a+h$, then any point c between a and b can be written as $c=a+\theta h$ where $0<\theta<1$. The result (6.9) then becomes

$$f(a + h) = f(a) + hf'(a + \theta h). \qquad (6.13)$$

This is the form in which the theorem is most often quoted.

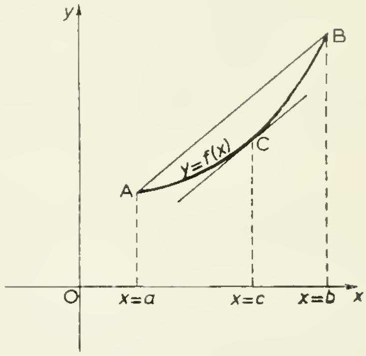

Fig. 6.3

Example 1

Show that for the function

$$f(x) = 2x^3 + x^2 - 4x - 2, \qquad (6.14)$$

the conditions of Rolle's theorem are satisfied for the interval $(-\frac{1}{2}, \sqrt{2})$. Verify the conclusion of this theorem for this function.

We have from eq. (6.14)

$$f'(x) = 6x^2 + 2x - 4. \qquad (6.15)$$

Since $f(x)$ is a polynomial it is continuous in the given range; from eqs. (6.15), $f'(x)$ exists in the open interval $)-\frac{1}{2}, \sqrt{2}($. Also $f(-\frac{1}{2})=0$ and $f(\sqrt{2})=0$, so that the three conditions of Rolle's theorem are satisfied. Thus there must be some point c such that $-\frac{1}{2}<c<\sqrt{2}$ where $f'(c)=0$. Putting $x=c$ in eq. (6.15) we find that c must satisfy the equation

$$6c^2 + 2c - 4 = 0,$$

or

$$2(3c + 1)(c - 2) = 0,$$

giving $c=-\frac{1}{3}$ or 2. The value $c=-\frac{1}{3}$ lies in the required range $-\frac{1}{2}<c<\sqrt{2}$.

Given functions $f(x)$ satisfying the required conditions, as in Example 1, it is possible to find values of c satisfying either Rolle's theorem or the

mean value theorem for specific values of a and b. However the mean value theorem is more often used to prove inequalities as in the following example.

Example 2

Use the mean value theorem to show that, if $x>1$

$$x - 1 > \log x > \frac{x-1}{x}.$$

Let $f(x)=\log x$, so that

$$f'(x) = 1/x. \tag{6.16}$$

Then the conditions of the mean value theorem are satisfied when $x>1$. We can therefore use the theorem in the form (6.9) with $b=x$ and $a=1$, it gives

$$\log x = \log 1 + (x - 1)f'(c),$$

where $1<c<x$. But $f'(c)=1/c$ from eq. (6.16), and therefore since $\log 1=0$ we have

$$\log x = \frac{x-1}{c},$$

or

$$c = \frac{x-1}{\log x}.$$

But $1<c<x$ and so we get

$$1 < \frac{x-1}{\log x} < x,$$

which becomes, since $x-1>0$,

$$\frac{1}{x-1} < \frac{1}{\log x} < \frac{x}{x-1},$$

or

$$x - 1 > \log x > \frac{x-1}{x}$$

§ 1.4. Cauchy's formula

Cauchy's formula is a generalisation of the mean value theorem, dealing with two functions instead of one.

Let $f(x)$ and $\varphi(x)$ be two functions satisfying the following conditions:

(i) $f(x)$ and $\varphi(x)$ are continuous in the closed interval (a, b),
(ii) $f'(x)$ and $\varphi'(x)$ exist in the open interval $)a, b($,
(iii) $\varphi(a)\neq\varphi(b)$,
(iv) $f'(x)$ and $\varphi'(x)$ do not vanish together;

then there is a point c such that $a<c<b$ where

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(c)}{\varphi'(c)}. \tag{6.17}$$

This reduces to the mean value theorem when $\varphi(x)\equiv x$, and the conditions on $\varphi(x)$ are then satisfied.

Again we choose a function $\psi(x)$ which satisfies all the conditions of Rolle's theorem. Let

$$\psi(x)=f(x)-A\varphi(x),$$

then to ensure that $\psi(a)=\psi(b)$ we can choose

$$A=\frac{f(a)-f(b)}{\varphi(a)-\varphi(b)}, \tag{6.18}$$

since $\varphi(a)\neq\varphi(b)$. Then by Rolle's theorem

$$\psi'(c)=f'(c)-A\varphi'(c)=0, \tag{6.19}$$

for some c in $]a, b[$. In eq. (6.19), $\varphi'(c)\neq 0$, since if $\varphi'(c)=0$ then $f'(c)=0$ and condition (iv) would not be satisfied. Thus eq. (6.19) can be divided through by $\varphi'(c)$ to give the result in the form (6.17) using the value of A in (6.18).

The formula (6.17) due to Cauchy is used mainly to prove a rule referred to as l'Hospital's rule for the evaluation of indeterminate forms.

§ 2. Indeterminate forms

In Ch. 1 we found that certain ratios of functions $f(x)/\varphi(x)$, on substitution of a particular value $x=a$, took the form $0/0$ or ∞/∞. When the ratios took these particular forms, it was only possible to evaluate their limits as $x\to a$ by special methods. Ratios taking the form $0/0$ or ∞/∞ are often referred to as *indeterminate forms*. We note, of course, that these expressions $0/0$ or ∞/∞ are purely symbolic; they mean that we require the value of $\lim_{x\to a}\{f(x)/\varphi(x)\}$ when $\lim_{x\to a} f(x)$ and $\lim_{x\to a}\varphi(x)$ are both zero or both infinite. Several types of limit need to be considered in detail.

Type I

Suppose that $f(x)$ and $\varphi(x)$ are such that $f(a)=\varphi(a)=0$, but $f'(a)$ and $\varphi'(a)$ are both finite and one of them, say $\varphi'(a)$, is non-zero. Then

$$\frac{f(x)}{\varphi(x)}=\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)}=\frac{\{f(x)-f(a)\}/(x-a)}{\{\varphi(x)-\varphi(a)\}/(x-a)},$$

and therefore

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\frac{\lim_{x\to a}\left\{\dfrac{f(x)-f(a)}{x-a}\right\}}{\lim_{x\to a}\left\{\dfrac{\varphi(x)-\varphi(a)}{x-a}\right\}}=\frac{f'(a)}{\varphi'(a)}.$$

Since $\varphi'(a)\neq 0$, $f'(a)/\varphi'(a)$ is zero or finite. If $f'(a)$ is the non-zero derivative, then $\lim_{x\to a}\{\varphi(x)/f(x)\}$ is zero or finite.

Example 3

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$$

Example 4

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{\sin x}{\cos x}=0.$$

Type 2

Suppose that $f(a)=\varphi(a)=0$, $f'(x)$ and $\varphi'(x)$ exist near but not necessarily at $x=a$. If $f'(x)/\varphi'(x)$ tends to a finite limit l as $x\to a$ then

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=l. \tag{6.20}$$

This is known as *l'Hospital's rule.*

To prove it under the conditions given, we note that $f'(a)$ and $\varphi'(a)$ do not vanish together; we can therefore use Cauchy's formula for the interval (a, x) and then make $x\to a$. We have for some value c in the interval $a<c<x$

$$\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)}=\frac{f'(c)}{\varphi'(c)};$$

since $f(a)=\varphi(a)=0$ and $c\to a$ as $x\to a$, we get from this result

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)}=\lim_{c\to a}\frac{f'(c)}{\varphi'(c)},$$

noting that $\varphi'(a)\neq 0$. This last limit is the same as

$$\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=l,$$

and thus l'Hospital's rule is established.

Type 3

If both the ratios $f(x)/\varphi(x)$ and $f'(x)/\varphi'(x)$ assume the indeterminate form 0/0 when $x=a$, then eq. (6.20) can be applied again to $f'(x)/\varphi'(x)$ to give

$$\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\lim_{x\to a}\frac{f''(x)}{\varphi''(x)},$$

provided the latter limit exists.

In general, if $f^{(n)}(a)$ and $\varphi^{(n)}(a)$ both vanish for $n<m$, but

$$\lim_{x\to a} f^{(m)}(a)/\varphi^{(m)}(a)$$

is finite or zero, which means that $\varphi^{(m)}(a)\neq 0$, then repeated application of l'Hospital's rule gives the result

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f^{(m)}(a)}{\varphi^{(m)}(a)}.$$

If $f^{(m)}(a)\neq 0$ and $\varphi^{(m)}(a)=0$, the value of the $\lim_{x\to a}\varphi^{(m)}(a)/f^{(m)}(a)$ is finite or zero.

Example 5

$$\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos 4x}=\lim_{x\to 0}\frac{\sin 2x}{2\sin 4x}=\lim_{x\to 0}\frac{2\cos 2x}{8\cos 2x}=\frac{1}{4}.$$

Example 6

$$\lim_{x\to 0}\frac{x-\sin x}{x^2}=\lim_{x\to 0}\frac{1-\cos x}{2x}=\lim_{x\to 0}\frac{\sin x}{2}=0.$$

Type 4

Suppose now that $f(x)$ and $\varphi(x)$ are two functions each of which tends to infinity as $x\to a$. Suppose that $f'(x)$ and $\varphi'(x)$ exist near, but not necessarily at $x=a$. If $f'(x)/\varphi'(x)$ tends to a finite limit l as $x\to a$, then the same rules may be applied as were used when $f(x)$ and $\varphi(x)$ both tended to zero as $x\to a$. In fact, since we can write

$$\frac{f(x)}{\varphi(x)}=\frac{\{\varphi(x)\}^{-1}}{\{f(x)\}^{-1}},$$

then the latter ratio is of the form 0/0; we may therefore use the previous

results to give

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \lim_{x\to a}\frac{\{\varphi(x)\}^{-1}}{\{f(x)\}^{-1}} = \lim_{x\to a}\frac{\varphi'(x)}{\{\varphi(x)\}^2}\,\frac{\{f(x)\}^2}{f'(x)} = \lim_{x\to a}\frac{\varphi'(x)}{f'(x)}\left\{\lim_{x\to a}\frac{f(x)}{\varphi(x)}\right\}^2;$$

therefore

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \left\{\lim_{x\to a}\frac{\varphi'(x)}{f'(x)}\right\}^{-1}. \tag{6.21}$$

But if $\lim_{x\to a}\{f'(x)/\varphi'(x)\}=l$ then $\lim_{x\to a}\{\varphi'(x)/f'(x)\}=l^{-1}$ and so substituting in eq. (6.21), we have

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = l.$$

Example 7

$$\lim_{x\to 0+}\frac{\log x}{x^{-1}} = \lim_{x\to 0+}\left(\frac{x^{-1}}{-x^{-2}}\right) = \lim_{x\to 0+}(-x) = 0.$$

Type 5

If the ratio $f(x)/\varphi(x)$ assumes either of the indeterminate forms 0/0 or ∞/∞ as x tends to infinity, similar rules may also be used to evaluate $\lim_{x\to\infty} f(x)/\varphi(x)$. Alternatively, we may write $x=t^{-1}$ and evaluate

$$\lim_{t\to 0+}\{f(t^{-1})/\varphi(t^{-1})\}.$$

Example 8

If $n>0$,

$$\lim_{x\to\infty}\frac{\log x}{x^n} = \lim_{x\to\infty}\frac{x^{-1}}{nx^{n-1}} = 0.$$

Thus $\log x$ tends to infinity with x more slowly that any positive power of x.

Example 9

$$\lim_{x\to\infty}\frac{\frac{1}{2}\pi - \tan^{-1} x}{\sin(1/x)} = \lim_{t\to 0+}\frac{\frac{1}{2}\pi - \tan^{-1}(1/t)}{\sin t},$$

which, using l'Hospital's rule becomes

$$\lim_{t\to 0+}\frac{1}{(1+t^2)\cos t} = 1.$$

Other indeterminate forms may be reduced to one of the standard types by simply rearranging the function.

Type 6

The limit $\lim_{x\to a} f(x)\varphi(x)$ where $f(x)\to 0$, $\varphi(x)\to\infty$ as $x\to a$ may be evaluated by writing it as $\lim_{x\to a} [f(x)/\{\varphi(x)\}^{-1}]$ which takes the form 0/0 at $x=a$.

Example 10

$$\lim_{x\to\frac{1}{2}\pi} (\pi^2 - 4x^2) \tan x = \lim_{x\to\frac{1}{2}\pi} \frac{\pi^2 - 4x^2}{\cot x},$$

which, using eq. (6.20), becomes

$$\lim_{x\to\frac{1}{2}\pi} \frac{-8x}{-\operatorname{cosec}^2 x} = 4\pi.$$

Type 7

Suppose $y=f^\varphi$, where $f\equiv f(x)\to 0+$, and $\varphi\equiv\varphi(x)\to 0$ as $x\to a$. We have

$$\log y = \varphi \log f = \varphi/\{\log f\}^{-1};$$

this last expression is of the form 0/0 at $x=a$. If then $\varphi \log f\to l$ as $x\to a$, then $y\to e^l$ since the exponential function is continuous. Thus

$$\lim_{x\to a} f^\varphi = e^l.$$

Similar methods can be used if $f\to\infty$, $\varphi\to 0$; also if $f\to 1$ or $\log f\to 0$ and $\varphi\to\infty$ as $x\to a$.

Example 11

To evaluate $\lim_{x\to 0+} x^x$. Let $y=x^x$, then

$$\log y = x \log x,$$

and

$$\lim_{x\to 0+} (x \log x) = \lim_{x\to 0+} \frac{\log x}{x^{-1}},$$

which has been evaluated in Example 7 as zero. Therefore $\lim_{x\to 0+} x^x=e^0=1$.

Example 12

To evaluate $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$. Let $y=\left(1+\frac{x}{n}\right)^n$, then

$$\log y = n \log\left(1+\frac{x}{n}\right),$$

and

$$\lim_{n\to\infty} n \log\left(1+\frac{x}{n}\right) = \lim_{n\to\infty} \left\{\frac{\log\left(1-\frac{x}{n}\right)}{n^{-1}}\right\}.$$

Using l'Hospital's rule, this becomes

$$\lim_{n\to\infty} \frac{xn^{-2}}{(1+xn^{-1})n^{-2}} = x.$$

Thus $\lim_{n\to\infty} (\log y) = x$ and therefore

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

Type 8

The function $f(x)-\varphi(x)$ where $f(x)\to\infty$, $\varphi(x)\to\infty$ as $x\to a$, can always be written as

$$\frac{\varphi^{-1}-f^{-1}}{(f\varphi)^{-1}},$$

where $f\equiv f(x)$ and $\varphi\equiv\varphi(x)$; this is now of the form 0/0. Usually it may be rearranged more easily in the form 0/0.

Example 13

$$\lim_{x\to 0} (\operatorname{cosec} x - \cot x) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = 0.$$

EXERCISE 6.1

Use the mean value theorem in the form of eq. (6.9) with $f(x)=\log x$ and the suggested values for a and b to prove the inequalities in Nos. 1, 2.

1. $\dfrac{x}{1+x} < \log(1+x) < x;\qquad b=1+x,\quad a=1,\quad x>0.$

2. $x < \log\dfrac{1}{1-x} < \dfrac{x}{1-x};\qquad b=1,\quad a=1-x,\quad 0<x<1.$

Evaluate the limits in Nos. 3–14.

3. $\displaystyle\lim_{x\to\frac{1}{4}\pi} \frac{1-\tan x}{1-\sqrt{2}\sin x}.$

4. $\displaystyle\lim_{x\to 0} \frac{\tan mx}{\tan nx}.$

5. $\displaystyle\lim_{x\to 0} \frac{\log(1+x)-x}{x^2}.$

6. $\displaystyle\lim_{x\to 0} \frac{\sinh x - \sin x}{x(1-\cos x)}.$

7. $\displaystyle\lim_{x\to 1} \frac{(x-1)^2}{\cos^2 \frac{1}{2}\pi x}.$

8. $\displaystyle\lim_{x\to 1} \frac{\log(1+\log x)}{x-1}.$

9. $\displaystyle\lim_{x\to 0} \frac{e^{\pi x}-1}{x(e^{\pi x}+1)}.$

10. $\displaystyle\lim_{x\to 0} \left(\operatorname{cosec}^2 x - \frac{1}{x^2}\right).$

11. $\lim_{x\to\pi} (x - \pi) \cot x.$

12. $\lim_{n\to\infty} \left(\cos \dfrac{x}{n}\right)^{n^2}.$

13. $\lim_{x\to 0} (\cos x)^{1/x}.$

14. $\lim_{n\to\infty} \left(\dfrac{n^2}{\alpha^2} - \cot^2 \dfrac{\alpha}{n}\right).$

15. Prove that $\lim_{n\to\infty} n^{1/n} = 1$; hence prove that $\lim_{x\to 0} x \log x = 0$.

16. Show that

$$\lim_{x\to 0} (\cos x + 2 \sin x)^{\cot x} = e^2.$$

17. Show that

$$\lim_{x\to\infty} \left(\frac{x + a}{x - a}\right)^x = e^{2a}.$$

18. Evaluate

$$\lim_{x\to\infty} x^2 \left[1 - x \sin \frac{1}{x}\right].$$

19. Evaluate

$$\lim_{n\to\infty} \left[\frac{n^2 + 2n - 3}{n^2 + 2n + 1}\right]^n.$$

§ 3. Taylor's theorem

Taylor's theorem allows us to approximate to a given function $f(a+h)$ by a polynomial in h of any given order n. We may regard it as a generalisation of the mean value theorem in the form of eq. (6.13) which is

$$f(a + h) = f(a) + hf'(a + \theta h), \qquad 0 < \theta < 1 \tag{6.22}$$

in that Taylor's theorem in its most widely used form is written

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h), \qquad 0 < \theta < 1. \tag{6.23}$$

In its original form, the mean value theorem states that when $f(x)$ satisfies certain conditions in (a, b), there exists a point c $(a<c<b)$ such that

$$f'(c) = \{f(b) - f(a)\}/(b - a).$$

This form is obtained from (6.22) by putting $a+h=b$ and $a+\theta h=c$. The analogous form of eq. (6.23), putting 'intermediate point' terms on one

side is

$$\frac{h^n}{n!} f^{(n)}(c) = R_n(a, b), \tag{6.24}$$

where

$$R_n(a, b) = f(b) - \sum_{r=0}^{n-1} \frac{(b-a)^r}{r!} f^{(r)}(a). \tag{6.25}$$

Then $R_n(a, b)$ is the difference between the function and its polynomial approximation. We apply Rolle's theorem to prove that this 'remainder' has the form stated in eq. (6.24). We define a function $\varphi(x)$ by

$$\varphi(x) = R_n(x, b) - \left(\frac{b-x}{b-a}\right)^p R_n(a, b), \tag{6.26}$$

where p is a constant not less than 1, to be chosen later, and $R_n(x, b)$ is given by eq. (6.25) with a replaced by x:

$$R_n(x, b) = f(b) - \sum_{r=0}^{n-1} \frac{(b-x)^r}{r!} f^{(r)}(x). \tag{6.27}$$

This function $\varphi(x)$ satisfies the conditions of Rolle's theorem if

(i) $f(x)$ and all its derivatives up to and including $f^{(n-1)}(x)$ are continuous in $a \leq x \leq b$,
(ii) $f^{(n)}(x)$ exists in $a<x<b$,
(iii) $\varphi(a)=\varphi(b)$.

Since it is obvious from eqs. (6.25) and (6.26) that $\varphi(a)=\varphi(b)=0$, Rolle's theorem can be applied provided $f(x)$ satisfies (i) and (ii). Thus there is a point c $(a<c<b)$ where $\varphi'(c)=0$. But from eq. (6.26)

$$\varphi'(x) = \frac{\mathrm{d}}{\mathrm{d}x} R_n(x, b) + \frac{p(b-x)^{p-1}}{(b-a)^p} R_n(a, b),$$

and this becomes

$$\varphi'(x) = -\sum_{r=0}^{n-1} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(b-x)^r}{r!} f^{(r)}(x)\right] + \frac{p(b-x)^{p-1}}{(b-a)^p} R_n(a, b),$$

which is

$$\varphi'(x) = \sum_{r=1}^{n-1} \frac{(b-x)^{r-1}}{(r-1)!} f^{(r)}(x) - \sum_{r=0}^{n-1} \frac{(b-x)^r}{r!} f^{(r+1)}(x) + \frac{p(b-x)^{p-1}}{(b-a)^p} R_n(a, b).$$

Changing $r-1$ to r in the first summation, this becomes

$$\varphi'(x) = \sum_{r=0}^{n-2} \frac{(b-x)^r}{r!} f^{(r+1)}(x) - \sum_{r=0}^{n-1} \frac{(b-x)^r}{r!} f^{(r+1)}(x) + \frac{p(b-x)^{p-1}}{(b-a)^p} R_n(a, b).$$

In this result, every term of the first summation is cancelled by the corresponding term of the second, the only one left being the last term in the second summation; thus

$$\varphi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{p(b-x)^{p-1}}{(b-a)^p} R_n(a, b).$$

This derivative is zero for some value $x=c$ where $a<c<b$; so

$$\varphi'(c) = -\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) + \frac{p(b-c)^{p-1}}{(b-a)^p} R_n(a, b) = 0, \tag{6.28}$$

Ir

$$R_n(a, b) = \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c). \tag{6.29}$$

of we put $p=n$, we have what is known as Lagrange's form of the remainder, that is

$$R_n(a, b) = \frac{(b-a)^n}{n!} f^{(n)}(c). \tag{6.30}$$

Thus by eq. (6.25), $f(b)$ can be expanded as

$$f(b) = \sum_{r=0}^{n-1} \frac{(b-a)^r}{r!} f^{(r)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(c), \qquad a < c < b.$$

Writing $b=a+h$, $c=a+\theta h$, so that $0<\theta<1$, this becomes

$$f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h), \tag{6.31}$$

which was the form given in eq. (6.23).

Of course eq. (6.29) applies for all $p \geqslant 1$ and by giving p different values we get other forms of the remainder. When $p=1$, we have Cauchy's form of the remainder given by

$$R_n(a, b) = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c),$$

or

$$R_n(a, b) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h), \tag{6.32}$$

and if we use this result we know that for some θ in $)0,1($

$$f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^r(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h); \qquad (6.33)$$

we note that the value of θ in this result is not necessarily the same as the value of θ in eq. (6.31).

§ 3.1. Taylor's series

Suppose that $f(x)$ is a function that has derivatives of all orders in an interval $(a-\eta, a+\eta)$ about the point $x=a$. Then provided $h<\eta$, we have by Taylor's theorem, using the Lagrange form of the remainder as in eq. (6.31)

$$f(a+h) = \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) + R_n,$$

where

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h), \qquad 0 < \theta < 1. \qquad (6.34)$$

Now since $f(x)$ has derivatives of all orders, we may let n have as large a value as we please. If then as $n \to \infty$, $R_n \to 0$, we may write

$$f(a+h) = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{h^r}{r!} f^{(r)}(a) = \sum_{r=0}^{\infty} \frac{h^r}{r!} f^{(r)}(a), \qquad (6.35)$$

which is an infinite series expansion of $f(a+h)$ in powers of h. This is known as *Taylor's series*. The validity of this series as a true representation of the function $f(a+h)$ can be tested by the usual methods of Ch. 3 for convergent series, or can be tested by showing that $R_n \to 0$ as $n \to \infty$.

§ 3.2. Maclaurin's series

When $a=0$ in eq. (6.35) for Taylor's series it reduces to

$$f(h) = \sum_{r=0}^{\infty} \frac{h^r}{r!} f^{(r)}(0),$$

which is known as *Maclaurin's series*. It is usually written with h replaced by x in the form

$$f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0), \qquad (6.36)$$

and gives the functions $f(x)$ as a power series in x, provided $f(x)$ has derivatives of all orders in an interval $(-\eta, +\eta)$ about the point $x=0$, and $x<\eta$. The Lagrange form of the remainder R_n now becomes

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \qquad 0 < \theta < 1, \tag{6.37}$$

and for the series (6.36) to be valid, $R_n \to 0$ as $n \to \infty$.

We will now consider examples of certain well-known series.

Example 14

The exponential series

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ and in particular $f^{(n)}(0) = 1$. Thus $f(x)$ has derivatives of all orders and satisfies the conditions for Maclaurin's series; so using eq. (6.36) we have

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}, \tag{6.38}$$

with R_n, given by eq. (6.37), as

$$R_n = \frac{x^n}{n!} e^{\theta x}, \qquad 0 < \theta < 1.$$

We have seen in Ch. 3 Example 20 that this series (6.38) is a convergent series for all finite values of x. We note that since $0<\theta<1$ then $e^{\theta x}<e^x$ for all finite values of x and e^x is independent of n. The convergence of this series implies that $R_n \to 0$ as $n \to \infty$; this means that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0. \tag{6.39}$$

The series (6.38) is known as the power series for the exponential function. Note that when $x=1$, we have the series given in Ch. 4 § 4.1 for the number e.

Example 15

The sine and cosine series

If $f(x) = \sin x$, then in Ch. 1 Example 32, we saw that $f^{(n)}(x) = \sin(x + \frac{1}{2}n\pi)$. Thus $f(x)$ has derivatives of all orders and satisfies the condition for Maclaurin's series. Also, when n is even, $f^{(n)}(0) = 0$, whilst when n is odd $f^{(n)}(0) = (-1)^{\frac{1}{2}(n-1)}$. Therefore, using eq. (6.36) we can write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \tag{6.40}$$

provided the series is convergent. The value of R_n, from eq. (6.37), is

$$R_n = \frac{x^n}{n!} \sin(\theta x + \tfrac{1}{2}n\pi), \qquad 0 < \theta < 1.$$

In this result $|\sin(\theta x+\frac{1}{2}n\pi)|<1$ for all values of x and so again $R_n \to 0$ as $n\to\infty$ if

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

This result has been shown to be true in eq. (6.39). Thus the series (6.40) is convergent for all finite values of x.

The cosine series is similarly

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots, \tag{6.41}$$

and is also convergent for all finite values of x.

Example 16

The binomial series

If $f(x)=(1+x)^m$ where m is any rational number, we have

$$f^{(n)}(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n},$$

and

$$f^{(n)}(0) = m(m-1)\cdots(m-n+1).$$

Hence the conditions for Maclaurin's series are satisfied, so using eq. (6.36) we have

$$(1+x)^m = 1 + mx + \binom{m}{2}x^2 + \ldots \binom{m}{n-1}x^{n-1} + \ldots, \tag{6.42}$$

where $\binom{m}{r} = m(m-1)\cdots(m-r+1)/r!$, coinciding with the well-known binomial coefficient when m is a positive integer. When m is a positive integer the series terminates and has $m+1$ terms. When m is not a positive integer the series is infinite. In Ch. 3 Example 19 we have shown that this series is convergent when $|x|<1$. To examine convergence we can also study the remainder R_n;

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \binom{m}{n} x^n (1+\theta x)^{m-n}, \tag{6.43}$$

for $0<\theta<1$. With this form of the remainder it is only possible to show that $R_n\to 0$ as $n\to\infty$ if $0<x<1$. With these restrictions on θ and x, we have

$$(1+\theta x)^{m-n} < 1, \quad \text{if} \quad n > m,$$

and then $\binom{m}{n}x^n \to 0$ as $n\to\infty$. If however $-1<x<0$, with $0<\theta<1$, we have $1+\theta x<1$ and $(1+\theta x)^{m-n}>1$ when $n>m$. Thus it cannot be shown from the form (6.43) for R_n, that $R_n\to 0$ as $n\to\infty$. This type of difficulty is the main reason for using different forms of remainder R_n. The Cauchy form of remainder for the binomial expansion is

$$R_n = \binom{m}{n} n(1-\theta)^{n-1}(1+\theta x)^{m-n},$$

and it can be shown quite easily that this remainder R_n does tend to zero as $n\to\infty$ provided $|x|<1$, and for all values of m.

Example 17

The logarithmic series
If we take $f(x) = \log x$ this function does not satisfy the conditions of Taylor's theorem in any range which includes $x=0$, since $\log x$ and its derivatives are all discontinuous at $x=0$; so we cannot expand $\log x$ as a Maclaurin series in powers of x. However we can expand $\log(a+x)$ in such a series provided $a>0$, and $x>-a$. Since $\log(a+x) = \log a + \log(1+x/a)$ we deal with the function $\log(1+x)$ for $x>-1$.

Taking then $f(x) = \log(1+x)$ we have

$$f'(x) = (1+x)^{-1}, \qquad f''(x) = -(1+x)^{-2}, \qquad f'''(x) = +2!(1+x)^{-3},$$

and so by the method of induction

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}.$$

Thus $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ and so from eq. (6.36) with $f(0)=0$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots + \frac{(-1)^n x^{n-1}}{n-1} + \dots, \tag{6.44}$$

with

$$R_n = \frac{(-1)^{n-1}x^n}{n(1+\theta x)^n}, \qquad 0 < \theta < 1.$$

With this form for R_n we can see that for $0 \leqslant x \leqslant 1$, $R_n \to 0$ as $n \to \infty$.

When $-1 < x < 0$, we have again to resort to Cauchy's form of remainder given by

$$R_n = \frac{(-1)^{n-1}x^n(1-\theta)^{n-1}}{(1+\theta x)^n},$$

and writing $x=-y$, with $0<y<1$, this becomes

$$R_n = -\frac{y^n}{1-\theta y}\left(\frac{1-\theta}{1-\theta y}\right)^{n-1}.$$

In this result $(1-\theta)/(1-\theta y) < 1$, and so since $1-\theta y$ is independent of n, we have $R_n \to 0$ as $n \to \infty$. Thus the logarithmic series (6.44) is convergent if $-1 < x \leqslant 1$. We leave it as an exercise for the reader to prove this result using the methods of Ch. 3 § 12.

Before leaving these Maclaurin series expansions for given functions of x, we note that certain indeterminate forms may be evaluated by expanding some of the functions involved, in a power series.

Example 18

To evaluate $\lim_{x\to\infty} e^x x^{-n}$ for all values of n.

If $n<0$, $x^n \to 0$ as $x \to \infty$ and since $e^x \neq 0$ then $e^x x^{-n} \to \infty$ as $x \to \infty$.

If $n=0$, $e^x x^{-n} = e^x$ and $e^x \to \infty$ as $x \to \infty$.

If $n>0$, expand $\exp x$ in the power series given in eq. (6.38), and let k be the

least integer which is greater than n. Then, since a series of positive terms is greater than any one of its terms,

$$e^x = \sum_{r=0}^{k} \frac{x^r}{r!} + \frac{x^{k+1}}{(k+1)!} + \ldots > \frac{x^k}{k!}.$$

Therefore

$$\frac{e^x}{x^n} > \frac{x^{k-n}}{k!},$$

where $k > n$. Thus as $x \to \infty$, $x^{k-n}/k! \to \infty$; so $e^x x^{-n} \to \infty$ as $x \to \infty$ for all values of n. Therefore $\exp x$ tends to infinity with x more rapidly than any positive power of x. (Compare this result with Example 8.)

EXERCISE 6.2

In the questions in this exercise, the range of values of x given for each series, is the range for which the series is convergent.

Use Maclaurin's series to prove the results in Nos. 1–5.

1. $\log(1 + x) + e^{-x} = 1 + \frac{1}{6}x^3 - \frac{5}{24}x^4 + \ldots,\qquad (-1 < x \leqslant 1).$

2. $\log(1 + x + x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \ldots,\qquad (-1 \leqslant x < 1).$

3. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots,\qquad (-\frac{1}{2}\pi < x < \frac{1}{2}\pi).$

4. $\log \sec(x + \frac{1}{4}\pi) = \frac{1}{2}\log 2 + x + x^2 + \frac{2}{3}x^3 + \frac{2}{3}x^4 + \ldots,\qquad (-\frac{3}{4}\pi < x < \frac{1}{4}\pi).$

5. $\sec x \tan x = x + \frac{5}{6}x^3 + \frac{61}{120}x^5 + \ldots,\qquad (-\frac{1}{2}\pi < x < \frac{1}{2}\pi).$

Find the first three non-vanishing terms in the Maclaurin expansions of the functions $f(x)$ given in Nos. 6–9.

6. $f(x) = \sin(\sin x)$, (all finite values of x).

7. $f(x) = e^{1-\cos x}$, (all finite values of x).

8. $f(x) = \log(\operatorname{sech} x)$, (all finite values of x).

9. $f(x) = x \cot x$, $(-\pi < x < \pi)$.

10. Using the methods of Exercise 4.1, No. 37, find the nth derivative of $e^{4x}\sin(3x+2)$ and hence expand this function as a power series in x for all finite values of x.

11. Show that, for all finite values of x

$$e^{-x}\cos x = 1 - x + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 + \tfrac{1}{30}x^5 + \ldots.$$

12. Expand $\log(1+\sin x)$ in powers of x as far as the term in x^5, $(-\frac{1}{2}\pi < x < \frac{1}{2}\pi)$.

13. If $f(x) = (\sin^{-1} x)^2$, $x < 1$, prove that $(1 - x^2)f''(x) - xf'(x) = 2$. Hence by Leibniz' theorem, show that

$$(1 - x^2)f^{(n+1)}(x) - (2n - 1)xf^{(n)}(x) - (n - 1)^2 f^{(n-1)}(x) = 0,$$

and find the value of $f^{(n)}(0)$. Deduce the general term in the expansion of $(\sin^{-1} x)^2$ in powers of x.

14. If $f(x) = \sin\{\log(1 + x)\}$, $|x| < 1$, prove that

$$(1 + x)^2 f''(x) + (1 + x)f'(x) + f(x) = 0;$$

and that $f^{(n+2)}(0) + (2n + 1)f^{(n+1)}(0) + (n^2 + 1)f^{(n)}(0) = 0$.

Deduce the expansion of $f(x)$ in powers of x as far as the term in x^5.

15. Using Taylor's theorem in the form

$$f(b) - f(a) = (b - a)f'(a) + \frac{(b - a)^2}{2!} f''\{a + \theta(b - a)\},$$

show that

$$\log(n + 1) - \log n = \frac{1}{n} - \frac{1}{2(n + \theta)^2}, \qquad 0 < \theta < 1.$$

Deduce that

$$\left(\sum_{r=1}^{n} \frac{1}{r}\right) - \log n,$$

has a finite limit as $n \to \infty$. This limit is known as *Euler's constant* and usually denoted by γ. Its value is approximately 0.577.

16. Using the mean value theorem, show that if

$$\text{(i)} \quad f(0) = 0 \qquad \text{(ii)} \quad f'(x) > 0 \quad \text{for all} \quad x > 0,$$

then $f(x) > 0$ for all $x > 0$.

If

$$f(x) = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 \ldots + \frac{(-1)^n x^{2n+1}}{2n + 1},$$

and

$$g(x) = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 \ldots + \frac{(-)1^{n-1}x^{2n-1}}{2n - 1} + \frac{(-1)^n x^{2n+1}}{(2n + 1)(1 + x^2)},$$

find and simplify the derivatives of $f(x) - \tan^{-1} x$ and $g(x) - \tan^{-1} x$.

Hence show that, for all x, $\tan^{-1} x$ lies between $f(x)$ and $g(x)$.

17. If a is real and $\neq 0, 1, 2, \ldots$, and N is any positive integer $> a$, prove that for all $x > 0$, $(1 + x)^a$ lies between

$$\sum_{n=0}^{N-1} a_n x^n \quad \text{and} \quad \sum_{n=0}^{N} a_n x^n,$$

where

$$a_0 = 1; \qquad a_n = a(a - 1)\cdots(a - n + 1)/n!, \qquad (n = 1, 2, 3, \ldots).$$

§ 4. Increasing and decreasing functions

Let a variable y be given as a function of the variable x by the equation

$$y = f(x),$$

and suppose that $f(x)$ is a differentiable function of x in any given range of values of x. Then by definition of the derivative we have

$$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = f'(x),$$

or

$$\frac{\delta y}{\delta x} = f'(x) + \varepsilon,$$

where ε is a function of x which may be positive or negative, but is such that $\varepsilon\to 0$ as $\delta x\to 0$. Suppose that $f'(x)>0$, then since $\varepsilon\to 0$ as $\delta x\to 0$, we can choose δx to be so small that $f'(x)+\varepsilon>0$ even when ε is negative. This means that δy will have the same sign as δx for all sufficiently small values of δx. Thus if δx is positive and sufficiently small, $\delta y=f(x+\delta x)-f(x)>0$ or $f(x+\delta x)>f(x)$. A function which satisfies the inequality $f(x+\delta x)>f(x)$ for small positive δx is said to be an *increasing function* of x, or, more simply, is *increasing with* x.

Similarly if $f'(x)<0$, δy is negative when δx is small and positive, so that $f(x+\delta x)<f(x)$ for small positive δx. The function is then said to be *decreasing with* x. Thus altogether we have

(i) $f'(x)>0$, $f(x)$ is increasing with x,
(ii) $f'(x)<0$, $f(x)$ is decreasing with x.

Example 19

Prove that the function

$$\tfrac{1}{2}x^2 + x - (1 + x)\log(1 + x),$$

increases with x in the range $0<x<\infty$.

We have

$$f'(x) = x - \log(1 + x),$$

and by the method of Example 2 we can prove that

$$x > \log(1 + x),$$

for $0<x<\infty$. Thus $f'(x)$ is positive, and so $f(x)$ is increasing with x.

§ 4.1. Maximum and minimum values of a function

We define a maximum value of a differentiable function $f(x)$ to be a value which is greater than the values of the function in the immediate neighbourhood, on either side.

In fig. 6.4 this means that the points P_1, P_2, P_3 on the graph $y=f(x)$, are points where the function has a maximum value. More precisely, the function $f(x)$ is said to have a maximum value at the point $x=a$, if

$$f(a+h) < f(a), \tag{6.45}$$

for all sufficiently small values of h whether positive or negative. Thus if $f(x)$ has a maximum value at $x=a$, then whatever the value of $\delta x=h$, whether it be positive or negative, the value of $\delta y=f(a+h)-f(a)<0$.

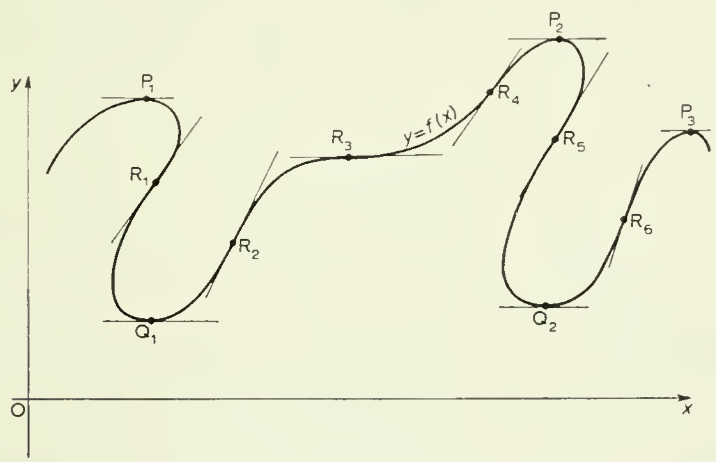

Fig. 6.4

Thus, using the results of § 4, $f'(a)$ cannot be positive or negative; therefore $f'(a)=0$ at a maximum.

Further, consider the value of the integral

$$\int_b^a f'(x)\,\mathrm{d}x, \tag{6.46}$$

when $b<a$. We have

$$\int_b^a f'(x)\,\mathrm{d}x = f(a) - f(b) > 0, \tag{6.47}$$

using eq. (6.45) with h negative, b being sufficiently near to a. But if $f'(x)$ is negative for all values of x in the range $b \leqslant x < a$, we know that the value of the integral (6.46) is negative. This would lead to a contradiction of eq. (6.47); thus $f'(x)$ is positive in any sufficiently small range $b \leqslant x < a$.

Similarly we can show that for any sufficiently small range $a<x\leqslant c$ the value of $f'(x)$ must be negative.

These results are intuitively obvious from fig. 6.5 remembering that $f'(x)$ is the gradient of the tangent at any point x on the curve $y=f(x)$. Thus, if $f(x)$ has a maximum value at the point $x=a$, then $f'(a)=0$ and $f'(x)$ changes sign from positive to negative as the value of x increases through the value $x=a$.

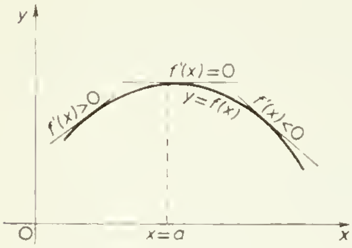

Fig. 6.5

Similarly we define a minimum value of a function as one which is less than the values in the immediate neighbourhood on either side, such as points Q_1, Q_2 in fig. 6.4. More precisely, $f(x)$ is said to have a minimum value at the point $x=a$, if

$$f(a+h) > f(a), \tag{6.48}$$

for all sufficiently small values of h, whether positive or negative. Here

$$\delta y = f(a+h) - f(a) > 0$$

for all values of $\delta x=h$, whether positive or negative; therefore $f'(x)$ cannot be positive or negative, and $f'(a)=0$.

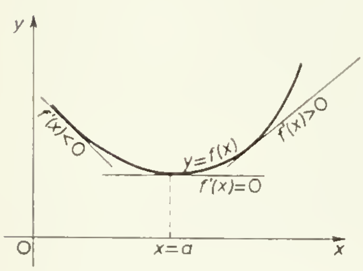

Fig. 6.6

By the same reasoning we used in discussing a maximum value, we can show that $f(x)$ has a minimum value at $x=a$ if $f'(a)=0$ and $f'(x)$ changes from negative to positive as the value of x increases through the value $x=a$. This is illustrated in fig. 6.6.

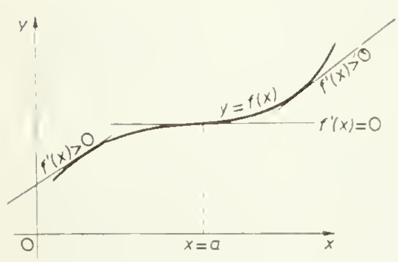

Fig. 6.7

Thus we see that a *necessary* condition for a maximum or minimum value of a function $f(x)$ at the point $x=a$, is that $f'(a)=0$. Note however that this condition is not *sufficient*, since other conditions are also required. A simple geometrical illustration will help to show this.

The function $f(x)$ shown in fig. 6.7 curves in such a way that its tangent is parallel to the axis of x at the point $x=a$, that is, $f'(a)=0$. However for $h<0$, $f(a+h)<f(a)$; while for $h>0$, $f(a+h)>f(a)$; thus remembering the conditions (6.45) and (6.46) we see that $f(x)$ has neither a maximum nor a minimum value at the point $x=a$. Such a point is called a *point of inflexion*. The curve $y=x^3$ shown in fig. 6.8 has a point

of inflexion of precisely this kind at the origin. Here $f'(x)=3x^2$ and is zero at $x=0$, but is positive for all other values of x whether x is positive or negative. The point R_3 on the curve in fig. 6.4 is a point of inflexion of this type. In general points such as $P_1, P_2, \ldots, Q_1, Q_2, \ldots$, and R_3 in

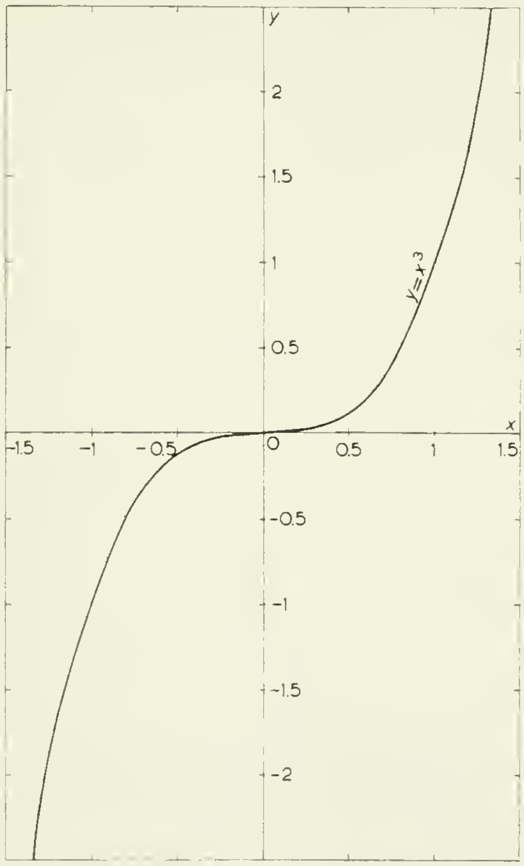

Fig. 6.8

fig. 6.4 where $f'(x)=0$ are called *stationary points*, while the points $P_1, P_2, \ldots, Q_1, Q_2, \ldots$, where $f'(x)$ changes sign are called *turning* points.

For the present we shall assume that the further conditions required for maximum and minimum values are those already given, namely,

(i) maximum value at $x=a$, $f'(a)=0$, $f'(x)$ changes from positive to negative as x increases through the value $x=a$;

(ii) minimum value at $x=a$, $f'(a)=0$, $f'(x)$ changes sign from negative to positive as x increases through the value $x=a$.

If $f'(a)=0$, but $f'(x)$ does not change sign as x increases through the value $x=a$, then the point $x=a$ may be a point of inflexion. More general conditions for a point of inflexion will be given later.

Example 20

To find the points where the function

$$f(x) = 2x^3 - 3x^2 - 36x + 10,$$

has maximum and minimum values. We have

$$f'(x) = 6x^2 - 6x - 36 = 6(x + 2)(x - 3),$$

and therefore $f'(x)=0$ when $x=-2$ or $x=3$.

When $x=-2-h$ $(h>0)$,

$$f'(x) = 6(-h)(-h - 5)$$

and so $f'(x)$ is positive.

When $x=-2+h$ $(h>0)$,

$$f'(x) = 6h(h - 5)$$

and for small values of $h(<5)$, $f'(x)$ is negative.

Thus $f(x)$ has a maximum value at $x=-2$.

When $x=3-h$ $(h>0)$,

$$f'(x) = 6(5 - h)(-h)$$

and for sufficiently small values of $h(<5)$, $f'(x)$ is negative.

When $x=3+h$ $(h>0)$,

$$f'(x) = 6(5 + h)h$$

and is essentially positive.

Thus $f(x)$ has a minimum value at $x=3$.

Example 21

To find the rectangle of the greatest area, having a given perimeter.

Denoting the perimeter by $2a$, the lengths of two adjacent sides may be taken as x and $a-x$; hence we have to find the maximum value of the function

$$f(x) = x(a - x).$$

We have

$$f'(x) = a - 2x,$$

and therefore $f'(x)=0$ when $x=\frac{1}{2}a$, and obviously changes in sign from positive to negative as x increases through the value $x=\frac{1}{2}a$.

Thus $f(x)$ has a maximum value when $x=\frac{1}{2}a$. So the rectangle with the greatest area, for a given perimeter, is a square.

Example 22

A particle moves in a parabola under the action of gravity only. The initial velocity is V at an angle α with the horizontal. Find the greatest height of the particle.

If y denotes the height above the initial point at any instant t,

$$y = Vt \sin \alpha - \tfrac{1}{2}gt^2,$$

and

$$\frac{dy}{dt} = V \sin \alpha - gt.$$

Thus $dy/dt=0$ when $t=(V \sin \alpha)/g$ and obviously changes from positive to negative as increases through this value. Thus y has a maximum value when

$$t = (V \sin \alpha)/g$$

and this value is

$$y = (V^2 \sin^2 \alpha)/2g.$$

We note that dy/dt measures the upward component of the velocity, and the maximum height is reached when this velocity changes from positive or upwards, to negative or downwards.

Example 23

The velocity v of waves of lengths λ on deep water is proportional to

$$\left(\frac{\lambda}{a} + \frac{a}{\lambda}\right)^{\frac{1}{2}},$$

where a is a certain linear magnitude; prove that the velocity has a minimum value when $\lambda=a$. Let

$$v = k\left(\frac{\lambda}{a} + \frac{a}{\lambda}\right)^{\frac{1}{2}},$$

where k is a constant of proportionality. Then

$$\frac{dv}{d\lambda} = \frac{k(\lambda^2 - a^2)}{2a\lambda^2\left(\dfrac{\lambda}{a} + \dfrac{a}{\lambda}\right)^{\frac{1}{2}}},$$

and $dv/d\lambda=0$ when $\lambda=\pm a$. Here $\lambda=-a$ has no meaning since waves cannot have a negative wave length. Thus we consider only $\lambda=+a$. As λ increases through the value $\lambda=+a$, $dv/d\lambda$ obviously changes sign from negative to positive, and therefore v has a minimum value when $\lambda=a$.

§ 4.2. Concavity and convexity

If $y=f(x)$ is the equation of a curve, then $dy/dx=f'(x)$ measures the gradient of the tangent at any point on the curve. Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right),$$

measures the derivative of dy/dx and therefore from § 4, if d^2y/dx^2 is positive, dy/dx is increasing, whilst if d^2y/dx^2 is negative, dy/dx is decreasing.

When the gradient of an arc of a curve is increasing with x, as in fig. 6.9 then the arc bends upwards and thus lies above the tangent at any point on the arc; the arc is then said to be *concave upwards* or *convex downwards*. A curve is said to be concave upwards at any point P, when in the immediate neighbourhood of P, it lies wholly above its tangent at P. Thus at P, $\mathrm{d}y/\mathrm{d}x$ must be increasing and therefore $\mathrm{d}^2y/\mathrm{d}x^2$ is positive.

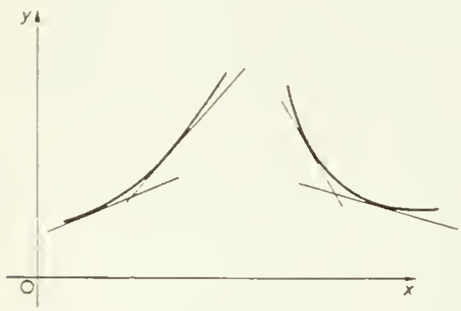

Fig. 6.9

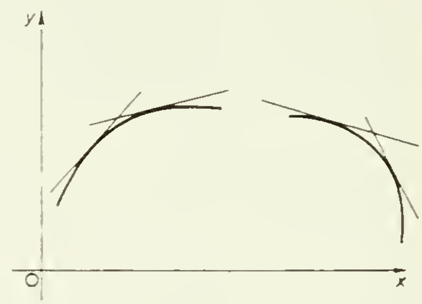

Fig. 6.10

Similarly, as shown in fig. 6.10, a curve is said to be *convex upwards* or *concave downwards* at a point P when in the immediate neighbourhood of P, the curve lies below its tangent at P. In this case, $\mathrm{d}y/\mathrm{d}x$ is decreasing, so that $\mathrm{d}^2y/\mathrm{d}x^2$ is negative at P.

At points where $\mathrm{d}^2y/\mathrm{d}x^2 = f''(x) = 0$ and *changes sign*, the curve must be changing from convex upwards to convex downwards or vice-versa. Therefore the curve cannot lie wholly above or below its tangent at such points on the curve. Thus the curve crosses its tangent at such points and these points are *points of inflexion*. Note that at such points, it is not necessary that $\mathrm{d}y/\mathrm{d}x = 0$, that is the tangent is not necessarily parallel to the axis of x. However, it is necessary that $\mathrm{d}^2y/\mathrm{d}x^2$ is zero and changes sign at such points. Points of inflexion are shown at $R_1, R_2, R_3, \ldots, R_6$ on the curve in fig. 6.4, but it is only at the point R_3 that $\mathrm{d}y/\mathrm{d}x = 0$. In fig. 6.8 the curve $y = x^3$ has a point of inflexion at $x = 0$, which is also a stationary value.

Since $f''(x)$ must change sign at the given point, say $x = a$, then as x increases through the value $x = a$, $f''(x)$ must be either increasing or decreasing and therefore *its* derivative $f'''(x)$ must be positive or negative. Thus the necessary and sufficient condition for $f(x)$ to have a point of inflexion at $x = a$ is

$$f''(a) = 0 \quad \text{and} \quad f'''(a) \neq 0. \tag{6.49}$$

§ 4.3. Further tests for maximum and minimum values

Suppose $f(x)$ is a function which has derivatives up to the nth order at $x=a$. Suppose that $f^{(r)}(a)=0$ for all values of r up to and including $r=n-1$. Then by Taylor's theorem, using Lagrange's form of remainder,

$$f(a+h)=f(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h), \qquad 0<\theta<1,$$

and thus

$$f(a+h)-f(a)=\frac{h^n}{n!}f^{(n)}(a+\theta h). \tag{6.50}$$

But, using eq. (6.45), $f(x)$ has a maximum value at $x=a$, if

$$f(a+h)-f(a)<0, \tag{6.51}$$

whilst using eq. (6.48), $f(x)$ has a minimum value at $x=a$, if

$$f(a+h)-f(a)>0, \tag{6.52}$$

for all sufficiently small values of h whether positive or negative in both eqs. (6.51) and (6.52). If now $f^{(n)}(a)$ has a finite non-zero value and $f^{(n)}(x)$ is continuous at $x=a$, then for sufficiently small values of h and with $0<\theta<1$, the sign of $f^{(n)}(a+\theta h)$ is the same as $f^{(n)}(a)$. Therefore if $f(a+h)-f(a)$ is to be of constant sign for *all* small h, whether positive or negative, n must be even. Also, if n is even, and using eqs. (6.50) and (6.51) $f(x)$ will have a maximum value at $x=a$ if $f^{(n)}(a)$ is negative, whilst using eqs. (6.50) and (6.52) $f(x)$ will have a minimum value at $x=a$, if $f^{(n)}(a)$ is positive. Further if n is odd, the function $f(x)$ cannot have a maximum or minimum value at $x=a$. Therefore the general conditions are: if $f(x)$ is to have a maximum or a minimum value at $x=a$, the first derivative $f^{(n)}(a)$ which does not vanish must have n even; if then $f^{(n)}(a)$ is negative, $f(x)$ has a maximum value, whilst if $f^{(n)}(a)$ is positive, $f(x)$ has a minimum value.

Example 24

To find the points on the curve

$$y=\cos x-\tfrac{1}{3}\cos 3x,$$

in the range $0\leqslant x\leqslant\pi$, where y has maximum or minimum values, and also to find the points of inflexion.

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin x-\sin 3x=2\sin x\cos 2x,$$

and $dy/dx = 0$, when either

(i) $\sin x = 0$, $\quad x = 0$ or π, $\qquad$ or (ii) $\cos 2x = 0$, $\quad x = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$.

Further

$$\frac{d^2y}{dx^2} = -\cos x + 3\cos 3x = 2\cos x(6\cos^2 x - 5).$$

Fig. 6.11

Testing the values of x given in (i) and (ii) for maximum and minimum values, we have

$$x = 0, \qquad \frac{d^2y}{dx^2} = +2, \quad \text{minimum value;}$$

$$x = \tfrac{1}{4}\pi, \qquad \frac{d^2y}{dx^2} = -2\sqrt{2}, \quad \text{maximum value;}$$

$$x = \tfrac{3}{4}\pi, \qquad \frac{d^2y}{dx^2} = +2\sqrt{2}, \quad \text{minimum value;}$$

$$x = \pi, \qquad \frac{d^2y}{dx^2} = -2, \quad \text{maximum value.}$$

We note that it is also very easy to study the changes of sign of $dy/dx = f'(x)$ at these points.

The points of inflexion are given by $d^2y/dx^2 = 0$ or

$$2 \cos x(6 \cos^2 x - 5) = 0,$$

so that either $\cos x = 0$, $x = \frac{1}{2}\pi$, or $\cos x = \pm(\frac{5}{6})^{\frac{1}{2}}$ so that in the given range $x \approx 0.42$ radians or $x \approx \pi - 0.42$ radians. Also $d^3y/dx^3 = -\sin x + 9 \sin 3x$, and it is easy to verify that for these values of x, $d^3y/dx^3 \neq 0$. Thus these are points of inflexion; we note that they occur between the points where $f(x)$ has maximum and minimum values. The graph of the function is shown in fig. 6.11.

In general, when $f(x)$ is a continuous, single valued function in the given range, the maximum and minimum values occur alternatively as x increases, and the points of inflexion occur between them, as in this example.

Example 25

The equation of a plane curve is

$$y^3 + x^3 - 9xy + 1 = 0, \tag{6.53}$$

and (x_1, y_1) is a point on the curve where $dy/dx = 0$. Prove that at (x_1, y_1)

$$\frac{d^2y}{dx^2} = \frac{18}{27 - x_1^3}.$$

Prove also that the stationary values of y occur at the points where $x = (27 \pm 3\sqrt{78})^{\frac{1}{3}}$, and determine which of these gives a maximum value of y and which a minimum.

For the curve in eq. (6.53), the slope dy/dx is found by the usual method for implicit functions given in Ch. 1 § 4.6: differentiating throughout with respect to x

$$3y^2 \frac{dy}{dx} + 3x^2 - 9y - 9x \frac{dy}{dx} = 0, \tag{6.54}$$

and therefore $dy/dx = 0$ when $x = x_1$, $y = y_1$ if

$$3x_1^2 - 9y_1 = 0. \tag{6.55}$$

Since (x_1, y_1) must also lie on the curve, these coordinates must satisfy eq. (6.53):

$$y_1^3 + x_1^3 - 9x_1y_1 + 1 = 0. \tag{6.56}$$

Differentiating eq. (6.54) again with respect to x, we have

$$3y^2 \frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 6x - 18\frac{dy}{dx} - 9x\frac{d^2y}{dx^2} = 0.$$

Substituting $x = x_1$, $y = y_1$, $dy/dx = 0$, we get

$$3y_1^2 \frac{d^2y}{dx^2} + 6x_1 - 9x_1 \frac{d^2y}{dx^2} = 0,$$

or

$$\frac{d^2y}{dx^2} = \frac{2x_1}{3x_1 - y_1^2}.$$

Using eq. (6.55), this becomes

$$\frac{d^2y}{dx^2} = \frac{18}{27 - x_1^3}. \tag{6.57}$$

The stationary values of y are given for values (x_1, y_1) satisfying eqs. (6.55) and (6.56) simultaneously. Eliminating y_1, we get

$$x_1^6 - 54x_1^3 + 27 = 0,$$

or

$$x_1^3 = 27 \pm 3\sqrt{78}.$$

Using eq. (6.57) we see that when

(i) $x_1^3 = 27 + 3\sqrt{78}$, d^2y/dx^2 is negative and therefore y has a maximum value,
(ii) $x_1^3 = 27 - 3\sqrt{78}$, d^2y/dx^2 is positive and therefore y has a minimum value.

For this curve, although we know that

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x},$$

from eq. (6.54) it is not easy to see how the sign of dy/dx is affected as x increases through the given values, since the value of y^2-3x is not easily determined. Thus the value of d^2y/dx^2 is more easily found here. In most simple cases however, when $d^2y/dx^2 \neq 0$, the original criteria given in § 4.1, are usually the easier to apply in practice.

Example 26

Test the function $f(x) = \cos x - 1 - \frac{1}{2}x^2$ for a maximum or minimum value at $x=0$. We have

$$\begin{aligned}
f'(x) &= -\sin x - x, & x &= 0, & f'(0) &= 0;\\
f''(x) &= -\cos x + 1, & x &= 0, & f''(0) &= 0;\\
f^{(3)}(x) &= -\sin x, & x &= 0, & f^{(3)}(0) &= 0;\\
f^{(4)}(x) &= \cos x, & x &= 0, & f^{(4)}(0) &= 1.
\end{aligned}$$

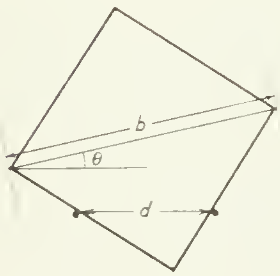

Fig. 6.12

Thus the first derivative which does not vanish is $f^{(n)}(0)$ with $n=4$, and since n is even and $f^{(4)}(0)$ is positive, the point $x=0$ is a point where $f(x)$ has a minimum value.

Example 27

A square plate of weight W rests in a vertical plane between two smooth pegs at the same horizontal level, as shown in fig. 6.12. Let d be the distance between the pegs and b the length of the diagonal of the square plate. If θ is the angle that the diagonal makes with the horizontal, it can be shown that the potential energy V of the plate in this position is given by

$$V = \tfrac{1}{2}W(b\cos\theta - d\cos 2\theta) + \text{const.}$$

For stable equilibrium V must have a minimum value for variations of θ. Differentiating with respect to θ, we get

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \tfrac{1}{2}W \sin\theta(4d\cos\theta - b),$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = \tfrac{1}{2}W\{\cos\theta(4d\cos\theta - b) - 4d\sin^2\theta\},$$

$$\frac{\mathrm{d}^3V}{\mathrm{d}\theta^3} = \tfrac{1}{2}W\{b\sin\theta - 8d\sin 2\theta\}.$$

Because of symmetry we need only consider angles in the first quadrant, that is, the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

If $b > 4d$, the only real value of θ for which $\mathrm{d}V/\mathrm{d}\theta = 0$ is $\theta = 0$, and since when $\theta = 0$, $\mathrm{d}^2V/\mathrm{d}\theta^2 < 0$, V is a maximum, and the position is unstable.

If $b < 4d$, $\mathrm{d}V/\mathrm{d}\theta = 0$ when either $\theta = 0$ or $\cos\theta = b/4d$. When $\theta = 0$, $\mathrm{d}^2V/\mathrm{d}\theta^2 > 0$, so this is a stable position. When $\theta = \cos^{-1}(b/4d)$, $\mathrm{d}^2V/\mathrm{d}\theta^2 < 0$, so this is an unstable position.

If $b = 4d$, $\mathrm{d}V/\mathrm{d}\theta$, $\mathrm{d}^2V/\mathrm{d}\theta^2$, $\mathrm{d}^3V/\mathrm{d}\theta^3$ all vanish for $\theta = 0$, and $\mathrm{d}^4V/\mathrm{d}\theta^4$ is negative. Thus V is a maximum and the position is unstable.

EXERCISE 6.3

1. Show that $x - \sin x$ is an increasing function of x throughout any interval of values of x. Deduce that $x - \sin x > 0$ if $x > 0$.

2. Show that $\tan x - x$ increases as x increases from $x = -\frac{1}{2}\pi$ to $x = \frac{1}{2}\pi$.

3. Show that $\tan^{-1} x - x/(1 + x^2)$ is an increasing function for all values of x, except $x = 0$, and that its maximum rate of increase is $\frac{1}{2}$.

4. Show that $(\sin\theta)/\theta$ decreases as θ increases from 0 to $\frac{1}{2}\pi$.

Find the maximum and minimum values of the functions given in Nos. 5–10.

5. $(x - 1)^2(x + 1)^{-3}$.

6. $2x^3 - 3x^2 - 36x + 10$.

7. $x(x^2 + 1)(x^4 - x^2 + 1)^{-1}$.

8. $x(a^2 + x^2)^{-\frac{3}{2}}$.

9. $(x^3 + x)\,\mathrm{e}^{-x^2}$.

10. $\cos x(1 + \frac{1}{3}\tan^2 x)$, $(-\frac{1}{2}\pi < x < \frac{1}{2}\pi)$

11. Given $y = \mathrm{e}^{-\alpha t} - \mathrm{e}^{-\beta t}$, $(\alpha < \beta)$ find for what value of t the value of y is a maximum, Evaluate the maximum value of y when $\alpha = 1$, $\beta = 3$.

12. Show that $y = x^3\,\mathrm{e}^{-4x}$ has a maximum ordinate and find it. Find also the points of inflexion.

13. Find the turning points and points of inflexion on the curve

$$y = 4 \sin x - \sin 2x,$$

for values of x in the range $0 \leqslant x \leqslant \pi$.

14. Find the maximum and minimum points and points of inflexion on the curve

$$y = 4x^3 + 9x^2 + 6x + 1.$$

Trace the curve.

15. Examine the curve

$$y = \sin^2 x \cos^3 x,$$

for turning points and points of inflexion in the range $0 \leqslant x \leqslant \pi$.

16. If $f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$, divide the interval $(0, \pi)$ into subintervals in each of which $f(x)$ is increasing or decreasing, indicating the sense of variation in each subinterval. Prove that $f(x) > 0$ for $0 < x < \pi$ and find where $f(x)$ attains its greatest value in the interval.

17. Find and discriminate the maximum and minimum values of the function

$$y = \exp(-\tfrac{1}{2}x) \sin(x + \tfrac{1}{6}\pi),$$

in the range $0 \leqslant x \leqslant 2\pi$.

18. Find and discriminate the maximum and minimum values of r on the curve

$$r = a\left(\cos^3 \theta + \frac{3\sqrt{3}}{4} \sin \theta\right),$$

between the values $\theta = 0$ and $\theta = \pi$.

19. A rectangular sheet of tin is 5 ft long and 28 in wide; four equal squares are removed from the corners and the sides are then turned up so as to form an open rectangular box; find the size of the pieces that must be cut out in order that the box may have the greatest volume.

20. A sector, angle θ, is cut from a circular disc of tin plate and the remainder is bent in the form of a conical funnel. Find the value of θ for which the volume of the funnel will be a maximum, and find this maximum volume in terms of the radius a of the disc.

21. The load which an aeroplane, flying at a constant speed can support is proportional to $\sin 2\theta/(1+\sin^2 \theta)$ where θ is the acute angle which the plane makes with the horizontal. Find the angle at which maximum lifting capacity occurs.

22. P is a point on the circle whose equation is

$$(x - h)^2 + (y - h)^2 = a^2,$$

where $h > a$, and PM, PN are the perpendiculars on the coordinate axes. Find the positions of P for which the area of the triangle PMN is a maximum or a minimum, and show that there are two positions of each or one of each according as h is less or greater than $a\sqrt{2}$.

CHAPTER 7

COMPLEX NUMBERS

§ 1. Introduction

In the process of building up our knowledge of arithmetic and algebra we can recall that every extension of the idea of a number first presented itself as the solution to some fundamental dilemma. For example, for simple addition of the natural numbers or integers 1, 2, 3,... we needed only the positive numbers, but as soon as subtraction was thought of, negative numbers had to be introduced. Similarly for multiplication the natural positive and negative integers were sufficient, but division forced us to introduce the idea of fractions, and irrational numbers were needed to describe geometrical lengths. The simple problem of solving all quadratic equations and equations of higher degree, has led to the introduction of *complex numbers*.

The general quadratic equation

$$ax^2 + 2bx + c = 0, \tag{7.1}$$

has no real solution when $b^2 < ac$. If we solve such an equation, for example

$$x^2 + 6x + 13 = 0,$$

we are led to the formal result $x = -3 \pm \sqrt{(-4)}$ which is meaningless unless we define complex numbers. If however we assume that there exists a number $\sqrt{(-1)}$ which we shall denote by i manipulated like an ordinary number except that it has the property $i^2 = -1$, we can write the above solution in the form $x = -3 \pm 2i$. This suggests that the notation of number may be extended in such a way that it will be consistent to use for its representation a symbol of the form $a + ib$ where a and b are real numbers and i is such that $i^2 = -1$. Any number which can be represented by the symbol $a + ib$, where a and b are real numbers, is called a complex number, a being called the *real part* and b the *imaginary part* of the number.

This introduction of complex numbers as simply an extension of the idea of numbers in order to enable us to solve all quadratic equations, should not suggest to the reader that this is the only use of complex numbers. Although they were originally introduced for this purpose, it has become apparent in the course of time that the formal manipulation of complex numbers was the key to much that had seemed obscure and awkward in the theory of real numbers and real variables, and more recently the theory of the complex variable has become a very potent tool in many branches of physics. It is therefore essential that the foundations of complex number theory should be clearly understood if any progress is to be made in many branches of mathematical physics.

§ 2. The algebra of complex numbers

The postulate that complex numbers obey the ordinary rules of addition, subtraction and multiplication, together with the rule $i^2 = -1$, is expressed formally by the following axioms: if $a+ib$ and $c+id$ are any two complex numbers

$$(a + ib) + (c + id) = a + c + i(b + d), \tag{7.2}$$

$$(a + ib) - (c + id) = a - c + i(b - d), \tag{7.3}$$

and

$$(a + ib)(c + id) = ac - bd + i(bc + ad). \tag{7.4}$$

Example 1

$$(3 + 4i) + (6 + 7i) = 9 + 11i.$$

Example 2

$$(2 - 3i)(5 + 4i) = 22 - 7i.$$

It is often convenient to use single letters α, β, γ to represent complex numbers. Using the axioms (7.2)–(7.4) it is easy for the reader to verify that if $\alpha = a+ib$, $\beta = c+id$, $\gamma = e+if$ then they obey the following algebraic laws

$$\alpha + \beta = \beta + \alpha \quad \text{(commutative law of addition)},$$

$$\alpha\beta = \beta\alpha \quad \text{(commutative law of multiplication)},$$

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \text{(associative law of addition)},$$

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma \quad \text{(associative law of multiplication)},$$

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \text{(distributive law)}.$$

We can now prove that two complex numbers $\alpha=a+ib$ and $\beta=c+id$ are equal, if and only if $a=c$, $b=d$.

We know that if a and b are non-zero *real* numbers, then neither a^2 nor b^2 can be negative, so that the equation

$$a^2+b^2=0 \quad \text{or} \quad a^2=-b^2,$$

can only be true if $a=0$, $b=0$. Suppose now that $a+ib=0$, then $a=-ib$, or by squaring both sides $a^2=-b^2$, and therefore $a=0$, $b=0$. Thus if a, b are *real* numbers,

$$a+ib=0$$

if and only if $a=0$, $b=0$. Similarly

$$a-ib=0,$$

if and only if $a=0$, $b=0$.

Thus if a, b, c, d are real numbers and

$$a+ib=c+id,$$

then

$$(a-c)+i(b-d)=0,$$

and this means that

$$a-c=0 \quad \text{or} \quad a=c,$$

and

$$b-d=0 \quad \text{or} \quad b=d.$$

From the result $(a+ib)(a-ib)=a^2+b^2$ we can immediately derive the value of the reciprocal of a complex number in the form

$$\frac{1}{a+ib}=\frac{a-ib}{(a+ib)(a-ib)}=\frac{a-ib}{a^2+b^2}=\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}. \tag{7.5}$$

Note that this result fails if and only if $a+ib=0$.

Example 3

$$\frac{1}{3+4i}=\frac{3-4i}{3^2+4^2}=\frac{3-4i}{25}=\frac{3}{25}-i\frac{4}{25}.$$

Equation (7.5) therefore establishes the rule for division by a non-zero complex number.

Example 4

$$\frac{-1+7i}{3+4i}=\frac{(-1+7i)(3-4i)}{25}=\frac{-3+28+(21+4)i}{25}=1+i.$$

§ 3. The Argand diagram

Let Ox, Oy be a pair of rectangular axes in a plane (fig. 7.1). Then in the same way as the real numbers can be represented by points on a line such as Ox, we can extend the representation to complex numbers and represent the number $a+ib$ by the point in this plane whose coordinates referred to (Ox, Oy) as axes are (a, b). In general if P is any point in this plane whose coordinates are (x, y) then the point P represents the complex number $x+iy$. If $y=0$ the point P is on the x-axis and the number is real, so that the x-axis is referred to as the *real axis*. If $x=0$, the point P is on the y-axis and the complex number is purely imaginary, so that the y-axis is called the *imaginary axis*. We shall frequently use the single letter z to denote the complex number $x+iy$ and write $z=x+iy$. The number $x-iy$ is then called the complex number conjugate to z and denoted by z^* (or sometimes $\bar{z}$). The real and imaginary parts (x and y) of z are often written Re z and Im z. Thus if $z=x+iy$, so that $z^*=x-iy$, then

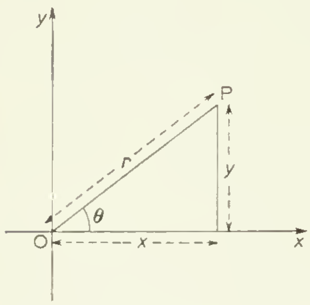

Fig. 7.1

$$z + z^* = 2x = 2 \operatorname{Re} z, \tag{7.6}$$

$$z - z^* = 2iy = 2i \operatorname{Im} z, \tag{7.7}$$

whilst

$$zz^* = (x + iy)(x - iy) = x^2 + y^2. \tag{7.8}$$

It is left as an exercise for the reader to show that if α, β are any two complex numbers, then

$$(\alpha\beta)^* = \alpha^*\beta^*, \qquad (\alpha + \beta)^* = \alpha^* + \beta^*.$$

The diagram in fig. 7.1 on which we represent complex numbers as described above, is called the *Argand diagram*.

If instead of using the rectangular coordinates (x, y) of P, we use the polar coordinates (r, θ) as shown in fig. 7.1; then

$$x = r \cos \theta, \qquad y = r \sin \theta, \tag{7.9}$$

so that

$$r = \sqrt{(x^2 + y^2)}. \tag{7.10}$$

Thus we can write the complex number z in the form

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

The distance $OP=r$ of the point P from the origin O is called the *modulus* of the complex number z and written as $|z|$, so that, using (7.8),

$$|z| = r = \sqrt{(x^2+y^2)} = \sqrt{(zz^*)}. \tag{7.11}$$

If $y=0$, this coincides with the definition of the modulus $|x|$ of a real number given in Ch. 1 § 1.1.

The polar coordinate θ is called the *amplitude* (am) or *argument* (arg) of the complex number. Since eqs. (7.9) and (7.10) do not determine a unique value for $\theta=\arg z$, we say that its *principal value* is such that

$$-\pi < \arg z \leqslant \pi. \tag{7.12}$$

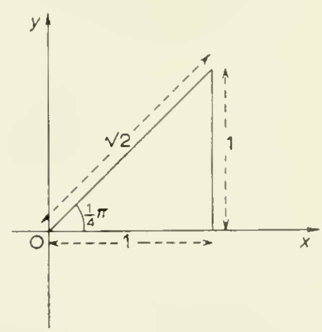

Fig. 7.2

Example 5

If $z=1+i$, then writing $z=r(\cos\theta+i\sin\theta)$, we have

$$r\cos\theta = 1, \qquad r\sin\theta = 1.$$

Thus $|z|=r=\sqrt{2}$ and $\cos\theta=1/\sqrt{2}$, $\sin\theta=1/\sqrt{2}$. So the principal value of $\theta=\arg z$ is $\frac{1}{4}\pi$, this being the value of θ which satisfies the inequality (7.12). This result is quite obvious on the Argand diagram in fig. 7.2. Other simple results are tabulated below.

TABLE 7.1

z	$r = \|z\|$	$\theta = \arg z$ (principal value)
i	1	$\frac{1}{2}\pi$
$1 - i$	$\sqrt{2}$	$-\frac{1}{4}\pi$
$-1 + i\sqrt{3}$	2	$\frac{2}{3}\pi$
$-i$	1	$-\frac{1}{2}\pi$
$\sin\alpha + i\cos\alpha$	1	$\frac{1}{2}\pi - \alpha + 2k\pi$ where k is chosen to give an angle in the range (7.12)

§ 3.1. Algebraic results interpreted on the Argand diagram

Some of the algebraic results of § 2 are now easily verified by the geometrical representation of a complex number on the Argand diagram. For example, if $z=x+iy=0$, so that $x=0$, $y=0$, then the point P representing this number z is at the origin of coordinates O. Similarly if two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are equal then from § 2 we have $x_1=x_2$, $y_1=y_2$ and so they are represented by the same point on the Argand diagram.

Further in fig. 7.3 let P be the point (x_1, y_1) and Q the point (x_2, y_2), then using $z_1=x_1+iy_1$, and $z_2=x_2+iy_2$ we have

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= x_1 + x_2 + i(y_1 + y_2)$$

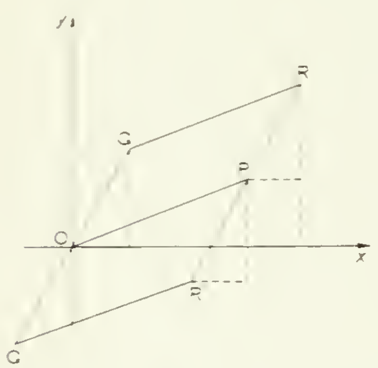

Fig. 7.3

and this complex number is represented by the point R on the diagram where OPRQ is a parallelogram.

Similarly

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$

is represented by the point R′ where OPR′Q′ is a parallelogram, OQ′ being equal in length and opposite in sense to OQ.

Note that in fig. 7.3 we have OR′=QP so that $|z_1-z_2|=$OR′=QP and therefore $|z_1-z_2|$ is the length of the line joining the point z_2 to the point z_1 on the Argand diagram.

It is now clear that in general

$$\text{OR} \leqslant \text{OP} + \text{OQ}, \tag{7.13}$$

the equality sign holding only if OP and OQ are in the same straight line. But OR is the modulus of z_1+z_2 and thus the result (7.13) can be written as

$$|z_1 + z_2| \leqslant |z_1| + |z_2|, \tag{7.14}$$

the equality sign holding only if z_1 and z_2 have the same argument. It is left as an exercise for the reader to prove this result algebraically.

Similarly OR′ ⩽ OP+OQ′, giving

$$|z_1 - z_2| \leqslant |z_1| + |z_2|.$$

We can therefore deduce that

$$|z_1 \pm z_2 \pm z_3| \leqslant |z_1 \pm z_2| + |z_3| \leqslant |z_1| + |z_2| + |z_3| .$$

We can extend this result inductively to n complex numbers $z_1, z_2, \ldots, z_n$; we get

$$|z_1 \pm z_2 \pm \ldots \pm z_n| \leqslant |z_1| + |z_2| + |z_3| + \ldots + |z_n| . \qquad (7.15)$$

§ 3.2. Multiplication and division of complex numbers

Multiplication and division of complex numbers on the Argand diagram are more easily carried out in terms of the polar coordinates (r, θ). If

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$

$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}, \quad (7.16) \end{aligned}$$

and this can be written as

$$z_1z_2 = r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}. \qquad (7.17)$$

Also

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2),$$

since $\cos^2\theta_2 + \sin^2\theta_2 = 1$. Again this can be written as

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}. \qquad (7.18)$$

It is easy to see by simple geometrical considerations that because of the results (7.17) and (7.18), the points z_1z_2 and z_1/z_2 are represented on the Argand diagram by the points S in fig. 7.4 and T in fig. 7.5 respectively. In fig. 7.4 the triangles OAP, OQS are similar, whilst in fig. 7.5 the triangles OQA, OPT are similar.

Also from eqs. (7.16) and considering fig. 7.4 we see that

$$|z_1z_2| = \mathrm{OS} = r_1r_2 = |z_1|\,|z_2| , \qquad (7.19)$$

and

$$\arg z_1z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2. \qquad (7.20)$$

These results may be expressed in words thus: the modulus of the product $z_1 z_2$ is equal to the product of the moduli, while the argument of the product $z_1 z_2$ is the *sum* of the arguments of z_1 and z_2.

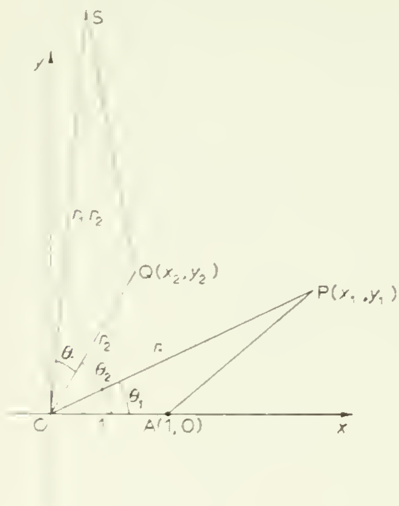

Fig. 7.4

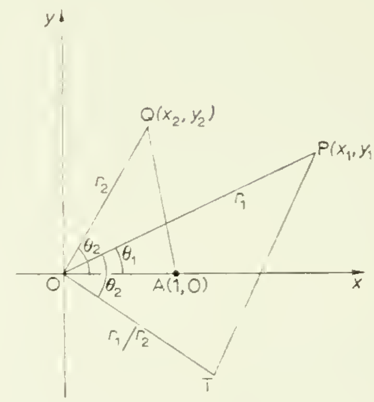

Fig. 7.5

Eq. (7.18) and consideration of fig. 7.5 gives

$$\left|\frac{z_1}{z_2}\right| = \text{OT} = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \tag{7.21}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2, \tag{7.22}$$

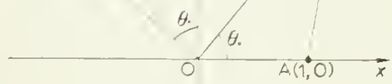

Fig. 7.6

the modulus of the quotient z_1/z_2 is the quotient of the moduli of z_1 and z_2, whilst the argument of the quotient z_1/z_2 is the *difference* of the arguments of z_1 and z_2.

In the special case when

$$z_2 = \text{OQ} = i = \cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi$$

then

$$z_1 z_2 = i z_1 = r_1\{\cos(\theta_1 + \tfrac{1}{2}\pi) + i \sin(\theta_1 + \tfrac{1}{2}\pi)\}, \tag{7.23}$$

and from this result and consideration of fig. 7.6 we see that multiplication by i simply turns OP through a right angle to OS, OS being found by the usual construction for multiplication as in fig. 7.4.

Example 6

Find the modulus and argument of

$$\frac{28}{(2-i\sqrt{3})(5-i\sqrt{3})}.$$

We have

$$(2-i\sqrt{3})(5-i\sqrt{3}) = 10-3-i7\sqrt{3} = 7(1-i\sqrt{3}).$$

Thus

$$\frac{28}{(2-i\sqrt{3})(5-i\sqrt{3})} = \frac{4}{1-i\sqrt{3}} = \frac{4(1+i\sqrt{3})}{1+3} = 1+i\sqrt{3}.$$

Thus

$$r\cos\theta = 1, \qquad r\sin\theta = \sqrt{3},$$

giving

$$r = \sqrt{(1+3)} = 2, \qquad \cos\theta = \tfrac{1}{2}, \qquad \sin\theta = \tfrac{1}{2}\sqrt{3},$$

so that $\theta = \frac{1}{3}\pi$. It is easy to verify eq. (7.19) for the modulus of the denominator as 14 and then eq. (7.21) does give $r = \frac{28}{14} = 2$.

Example 7

Show that for any two complex numbers z_1 and z_2

$$|z_1+z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}(z_1 z_2^*) + |z_2|^2. \tag{7.24}$$

Suppose $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$ where x_1, y_1, x_2, y_2 are real. Then

$$z_1+z_2 = (x_1+x_2)+i(y_1+y_2),$$

and so by eq. (7.11)

$$|z_1+z_2|^2 = (x_1+x_2)^2+(y_1+y_2)^2 = x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2,$$

which can be written as

$$|z_1+z_2|^2 = (x_1^2+y_1^2)+(x_2^2+y_2^2)+2(x_1x_2+y_1y_2) = |z_1|^2+|z_2|^2+2(x_1x_2+y_1y_2).$$

Now

$$z_1z_2^* = (x_1+iy_1)(x_2-iy_2) = x_1x_2+y_1y_2+i(y_1x_2-x_1y_2),$$

and therefore the real part of this is $\mathrm{Re}(z_1z_2^*) = x_1x_2+y_1y_2$; the result follows.

If x and y are variables then $z = x+iy$ is also a variable and we call it a complex variable. When a relationship exists between the variables x and y the point P on the Argand diagram with coordinates (x, y) traces out a line or curve called its *locus*.

Frequently the relationship between x and y which defines the locus of P can be given in terms of the complex variable z rather than x and y. For example, the equation

$$|z| = a, \tag{7.25}$$

means in polar coordinates $r=a$, or in terms of the rectangular coordinates (x, y)

$$\sqrt{(x^2 + y^2)} = a,$$

or

$$x^2 + y^2 = a^2.$$

This is the equation of a circle centre the origin of radius a, and the eq. (7.25) equally well defines this locus. Similarly since $|z-\alpha|$ where α is complex, represents the length of the line joining z to α, then

$$|z - \alpha| = a,$$

represents a circle centre α and radius a. Other loci may be expressed in similar ways.

Example 8

The equation

$$|z - \alpha| = |z - \beta|,$$

where α, β may be complex, is the locus of a point P which is such that its distance from the point α equals its distance from the point β on the Argand diagram. Thus it is the straight line which bisects perpendicularly the line joining α and β.

EXERCISE 7.1

Express the complex numbers in Nos. 1–6 in the form $a+ib$ where a and b are real.

1. $(1 - 2i)^2$. **2.** $\dfrac{4 - i}{(3 + i)^2}$. **3.** $\dfrac{4 + 3i}{1 + 2i}$.

4. $\dfrac{(3 - 2i)(1 - i)}{2 + i}$. **5.** $\dfrac{(1 + i)(2 + 3i)}{3 - i}$. **6.** $\dfrac{(1 + i)(-3 + 2i)^2}{(1 - 2i)^2}$.

Find the moduli and arguments of the complex numbers in Nos. 7–13, α, β being real.

7. $-2 - 2i$. **8.** $-\sqrt{3} + 3i$. **9.** $\sqrt{3} - i$.

10. $1 + \sqrt{2} + i$. **11.** -2. **12.** $1 + \cos\alpha + i \sin\alpha$.

13. $\cos\alpha - i \sin\beta + i(\sin\alpha + i \cos\beta)$.

14. Calculate the following products using the rules (7.19) and (7.20) and verify the results by direct multiplication:

$$(-\sqrt{3} + 3i)(\sqrt{3} + i), \qquad (-2 + 2i)(-3 - 3i).$$

15. Calculate the following quotients using the rules (7.21) and (7.22) and verify the results by direct calculation:

$$\frac{1+\sqrt{3}i}{1+i}, \qquad \frac{-3}{-\sqrt{3}+i}.$$

16. Find the moduli and arguments of the numbers

$$\frac{(1-i)^3}{(1+i)^6}, \qquad \frac{1+\cos\alpha+i\sin\alpha}{1+i}, \qquad \frac{1}{1+(\cos\alpha+i\sin\alpha)^2}.$$

17. Give a graphical construction for the product $(1+i)(-3+2i)$ and verify the result by calculation.

18. If $|z|=r$ and $\arg z=\theta$, find $|z^{-1}|$ and $\arg(z^{-1})$ in terms of r and θ.

19. Find the locus of z when

$$\text{(i)} \quad |z-1|=2, \qquad \text{(ii)} \quad |z-1|=|z-3|.$$

20. If a and b are fixed complex numbers and if z varies so that $|z|=r$, use the results (7.14) and (7.19) to show that the greatest value of $|az+b|$ is $|a|r+|b|$.

Find the greatest and least values of $|(1+i)z+(1-i)|$, for $1 \leqslant |z| \leqslant 2$ and find the values of z which give these values of the modulus.

21. A, B, C and D are the vertices of a parallelogram, described anti-clockwise, in the Argand diagram. A is the origin, and the mid points of BC and CD are z_1 and z_2 respectively. If C represents z, show that $3z=2(z_1+z_2)$.

Find the condition which z_1 and z_2 must satisfy for ABCD to be a square.

22. If z_1 and z_2 are complex numbers, prove that

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2,$$

and give a geometrical interpretation of this equality.

23. If the complex numbers a, b, c are represented on the Argand diagram by the points A, B, C, prove that $(b-c)/(a-c)$ has modulus BC/AC and has argument equal to the angle ACB.

If a, b are fixed complex numbers and z a variable complex number, show that the system of equations

$$\arg\left(\frac{z-b}{z-a}\right)=\theta,$$

where θ is a real parameter, represents a system of circles passing through the points A and B. Find the values of θ associated with the circle which has AB as diameter.

24. If z satisfies the equation

$$\left|\frac{z+a}{z+b}\right|=p,$$

where a, b, z are complex and p is a real positive constant $\neq 1$, show that z also. satisfies the equation

$$|z + c|^2 = r^2,$$

where c is complex and r a real constant, both to be determined in terms of a, b and p.

Give a geometrical interpretation of this result.

25. Find the equation of the circle in the Argand diagram with centre α (α real) and radius r in the form

$$zz^* + az + bz^* + c = 0,$$

where a, b and c can be determined in terms of α and r. Show that as z describes the above circle, in general z^{-1} also describes a circle and find its centre. What is the locus of z in the exceptional case?

26. Use the result given in eq. (7.24) to show that the locus of points in the Argand diagram given by

$$|z - p| = 2|z|,$$

is a circle with centre $-\frac{1}{3}p$ and radius $\frac{2}{3}|p|$.

27. Use the result given in eq. (7.24) to show that the locus of points in the Argand diagram such that

$$|z|^2 + 2\,\mathrm{Re}(iz) = 3$$

is a circle, and give its centre and radius.

Also show that the circle represented by it cuts the circle $|z|^2 = 3$ on the real axis, and interpret the result geometrically.

28. P_1 and P_2 are the points z_1, z_2 respectively on the Argand diagram. A variable point $P(z)$ moves so that

$$z - z_1 = \lambda i(z - z_2),$$

where λ is a variable real parameter. Find the locus of P.

§ 4. De Moivre's theorem and its applications

If n is a positive or negative integer and θ is a real number, De Moivre's theorem is

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{7.26}$$

We have already seen in eqs. (7.15) and (7.16) that by direct multiplication

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \tag{7.27}$$

so that if $\theta_1 = \theta_2 = \theta$ then

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta. \tag{7.28}$$

This means that the result (7.26) does hold for the positive integer $n=2$. Suppose then that it holds for any positive integer $n=\nu$, so that

$$(\cos\theta + i\sin\theta)^{\nu} = \cos\nu\theta + i\sin\nu\theta; \tag{7.29}$$

then by multiplying both sides of this equation by $\cos\theta+i\sin\theta$ we have

$$(\cos\theta + i\sin\theta)^{\nu+1} = (\cos\nu\theta + i\sin\nu\theta)(\cos\theta + i\sin\theta),$$

and using eq. (7.27) this becomes

$$(\cos\theta + i\sin\theta)^{\nu+1} = \cos(\nu+1)\theta + i\sin(\nu+1)\theta.$$

This establishes the result (7.26) by the method of induction (Ch. 1 § 5) for any positive integer n.

To show that the theorem is true when n is a negative integer, suppose $n=-m$ where m is a positive integer. Then

$$(\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^{m}},$$

and using eq. (7.26) for the positive integer m, this becomes

$$\frac{1}{\cos m\theta + i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta},$$

which is $\cos m\theta+i\sin(-m\theta)$, establishing the result.

§ 4.1. The exponential form of De Moivre's theorem

In Ch. 6 Example 14 the power series for the exponential function $\exp x$ was found as

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} + \ldots .$$

If we put $x=i\theta$ in this expansion we get

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \ldots + \frac{(-1)^n\theta^{2n}}{(2n)!} + \frac{i(-1)^n\theta^{2n+1}}{(2n+1)!} + \ldots$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!};$$

comparing these two infinite series with eqs. (6.41) and (6.40), putting

$x=\theta$ in these equations, we see that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{7.30}$$

Similarly

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{7.31}$$

For the present we will accept these results simply as a more convenient way of writing $\cos\theta+i\sin\theta$ and $\cos\theta-i\sin\theta$ without giving any justification for the use of a complex variable in the exponential function. Something will be said about functions of complex variables in § 5 of this chapter, but the theory of such functions will be dealt with more fully in Ch. 17.

Using the eqs. (7.30) and (7.31) we find that De Moivre's theorem can be written as

$$(e^{i\theta})^n = e^{in\theta}, \tag{7.32}$$

when n is a positive or negative integer. This generalises the result in eq. (4.21) for the real exponential function to the case of imaginary numbers, which is reasonable.

The results (7.30) and (7.31) may also be written as

$$\cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}), \tag{7.33}$$

which gives a connection with the hyperbolic cosine, and hyperbolic sine defined in eqs. (4.39) and (4.40).

§ 4.2. Expansions of $\cos n\theta$, $\sin n\theta$ and $\tan n\theta$ when n is a positive integer

When n is a positive integer we can use De Moivre's theorem in its original form (7.26) to expand $\cos n\theta$, $\sin n\theta$ in powers of $\cos\theta$, $\sin\theta$ in the following way. Using the binomial theorem on the left hand side of eq. (7.26) we get

$$\cos n\theta + i\sin n\theta = \cos^n\theta + \binom{n}{1}\cos^{n-1}\theta(i\sin\theta) + \dots$$

$$+ \binom{n}{r}\cos^{n-r}\theta(i\sin\theta)^r + \dots + (i\sin\theta)^n. \tag{7.34}$$

Equating real and imaginary parts on each side of this equation, we have

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \ldots + u_n. \tag{7.35}$$

$$\sin n\theta = n \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \ldots + v_n. \tag{7.36}$$

where

$$u_n = (-1)^{\frac{1}{2}n} \sin^n \theta \quad \text{when } n \text{ is even, or}$$

$$u_n = n(-1)^{\frac{1}{2}(n-1)} \cos \theta \sin^{n-1} \theta \quad \text{when } n \text{ is odd;}$$

whilst

$$v_n = n(-1)^{\frac{1}{2}(n-2)} \cos \theta \sin^{n-1} \theta \quad \text{when } n \text{ is even, or}$$

$$v_n = (-1)^{\frac{1}{2}(n-1)} \sin^n \theta \quad \text{when } n \text{ is odd.}$$

By using $\sin^2 \theta = 1 - \cos^2 \theta$ we see immediately that when n is either even or odd, $\cos n\theta$ and $(\sin n\theta)/\sin \theta$ can be expressed as polynomials in powers of $\cos \theta$. Further when n is even, and using $\cos^2 \theta = 1 - \sin^2 \theta$, $\cos n\theta$ may be expressed as a polynomial in $\sin \theta$, whilst when n is odd, $\sin n\theta$ can be expressed as a polynomial in $\sin \theta$. Also by division of each side of the eq. (7.36) by the corresponding side of eq. (7.35) and then cancelling the factor $\cos^n \theta$ on the right hand side we derive the result

$$\tan n\theta = \frac{n \tan \theta - \binom{n}{3} \tan^3 \theta + \ldots + A_n}{1 - \binom{n}{2} \tan^2 \theta + \ldots + B_n},$$

where A_n, B_n are functions of $\tan \theta$ whose particular forms depend on whether n is even or odd.

Example 9

With $n = 7$ in the eq. (7.34) we get

$$\cos 7\theta + i \sin 7\theta = (\cos \theta + i \sin \theta)^7 = \cos^7 \theta + 7 \cos^6 \theta (i \sin \theta) + 21 \cos^5 \theta (i \sin \theta)^2$$
$$+ 35 \cos^4 \theta (i \sin \theta)^3 + 35 \cos^3 \theta (i \sin \theta)^4 + 21 \cos^2 \theta (i \sin \theta)^5 + 7 \cos \theta (i \sin \theta)^6 + (i \sin \theta)^7.$$

Equating real and imaginary parts, this gives

$$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta, \tag{7.37}$$

and

$$\sin 7\theta = 7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta. \tag{7.38}$$

Using $\sin^2 \theta = 1 - \cos^2 \theta$ in eq. (7.37) and $\cos^2 \theta = 1 - \sin^2 \theta$ in eq. (7.38) we may write these as

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta, \tag{7.39}$$

and

$$\sin 7\theta = 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta. \tag{7.40}$$

Dividing each side of eq. (7.38) by the corresponding side of eq. (7.37) and cancelling out the factor $\cos^7 \theta$ on the right hand side, we get

$$\tan 7\theta = \frac{7 \tan \theta - 35 \tan^3 \theta + 21 \tan^5 \theta - \tan^7 \theta}{1 - 21 \tan^2 \theta + 35 \tan^4 \theta - \tan^6 \theta}.$$

§ 4.3. Expressions for $\cos^n \theta$, $\sin^n \theta$ in terms of $\cos n\theta$, $\sin n\theta$ where n is a positive integer

Using the eqs. (7.33) we can write

$$2^n \cos^n \theta = (e^{i\theta} + e^{-i\theta})^n; \tag{7.41}$$

expanding the right hand side by the binomial theorem and putting corresponding positive and negative exponentials together, we get

$$2^n \cos^n \theta = (e^{ni\theta} + e^{-ni\theta}) + \binom{n}{1}\{e^{(n-2)i\theta} + e^{-(n-2)i\theta}\}$$
$$+ \ldots + \binom{n}{r}\{e^{(n-2r)i\theta} + e^{-(n-2r)i\theta}\} + \ldots + t_n,$$

where

$$t_n = \binom{n}{\frac{1}{2}n},$$

when n is even, or

$$t_n = \binom{n}{\frac{1}{2}(n-1)}(e^{i\theta} + e^{-i\theta}),$$

when n is odd. Thus when n is even

$$2^n \cos^n \theta = 2\left[\cos n\theta + \binom{n}{1}\cos(n-2)\theta + \ldots + \binom{n}{r}\cos(n-2r)\theta + \ldots + \binom{n}{\frac{1}{2}n}\right],$$

whilst when n is odd

$$2^n \cos^n \theta = 2\left[\cos n\theta + \binom{n}{1}\cos(n-2)\theta + \ldots + \binom{n}{r}\cos(n-2r)\theta + \right.$$
$$\left. + \ldots + \binom{n}{\frac{1}{2}(n-1)}\cos\theta\right].$$

Similarly by starting from

$$(2i \sin \theta)^n = (e^{i\theta} - e^{-i\theta})^n,$$

we can determine $\sin^n \theta$ as a series of terms in $\sin(n-2r)\theta$, $r = 0, 1, \dots, \frac{1}{2}n$ or $\frac{1}{2}(n-1)$. Results of this nature can be combined to determine similar forms of expression for $\sin^n \theta \cos^m \theta$ where n, m are positive integers.

Example 10

Prove that

$$2^7 \cos^8 \theta = \cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35.$$

As in eq. (7.41) we write

$$2^8 \cos^8 \theta = (e^{i\theta} + e^{-i\theta})^8,$$

and expanding the right hand side, we get

$$2^8 \cos^8 \theta = (e^{8i\theta} + e^{-8i\theta}) + 8(e^{6i\theta} + e^{-6i\theta}) + 28(e^{4i\theta} + e^{-4i\theta}) + 56(e^{i\theta} + e^{-i\theta}) + 70,$$

which we can write as

$$2^8 \cos^8 \theta = 2 \cos 8\theta + 16 \cos 6\theta + 56 \cos 4\theta + 112 \cos 2\theta + 70;$$

this gives the required result.

Example 11

Prove that

$$16 \sin^3 \theta \cos^2 \theta = 2 \sin \theta + \sin 3\theta - \sin 5\theta.$$

We use eqs. (7.33) to give

$$-2^5 i \sin^3 \theta \cos^2 \theta = (e^{i\theta} - e^{-i\theta})^3 (e^{i\theta} + e^{-i\theta})^2. \tag{7.42}$$

Here the right hand side is first written as

$$\{(e^{i\theta} - e^{-i\theta})(e^{i\theta} + e^{-i\theta})\}^2 (e^{i\theta} - e^{-i\theta}) = (e^{2i\theta} - e^{-2i\theta})^2 (e^{i\theta} - e^{-i\theta}),$$

and then expanding and multiplying out, we get

$$(e^{5i\theta} - e^{-5i\theta}) - (e^{3i\theta} - e^{-3i\theta}) - 2(e^{i\theta} - e^{-i\theta}).$$

Thus dividing eq. (7.42) through by $-2i$, we have

$$2^4 \sin^3 \theta \cos^2 \theta = 2 \sin \theta + \sin 3\theta - \sin 5\theta.$$

Obviously, results of this kind could be used to evaluate integrals of the type discussed in Ch. 5 § 2.

EXERCISE 7.2

1. Prove that

$$\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1,$$

and

$$2^5 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 20.$$

2. Use De Moivre's theorem to prove that

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta;$$

find also the expressions for $\sin 5\theta$ in terms of $\sin \theta$ and for $\tan 5\theta$ in terms of $\tan \theta$.

3. Show that

$$512 \sin^6 \theta \cos^4 \theta = 6 - 2 \cos 2\theta - 8 \cos 4\theta + 3 \cos 6\theta + 2 \cos 8\theta - \cos 10\theta,$$

$$128 \sin^5 \theta \cos^3 \theta = \sin 8\theta - 2 \sin 6\theta - 2 \sin 4\theta + 6 \sin 2\theta.$$

4. Use the result for $\cos 5\theta$ given in Question 2 to show that the roots of the equation

$$32x^5 - 40x^3 + 10x - 1 = 0$$

are

$$\cos \tfrac{1}{15}\pi, \quad \cos \tfrac{5}{15}\pi, \quad \cos \tfrac{7}{15}\pi, \quad \cos \tfrac{11}{15}\pi, \quad \cos \tfrac{13}{15}\pi.$$

Deduce

(i) $\cos^2 \frac{1}{15}\pi + \cos^2 \frac{7}{15}\pi + \cos^2 \frac{11}{15}\pi + \cos^2 \frac{13}{15}\pi = \frac{9}{4}$,

(ii) $\sec^2 \frac{1}{15}\pi + \sec^2 \frac{7}{15}\pi + \sec^2 \frac{11}{15}\pi + \sec^2 \frac{13}{15}\pi = 96$.

5. By using the exponential values of $\cos \theta$ and $\sin \theta$, or otherwise, prove that

$$\sum_{r=1}^{n} \sin \frac{2r\pi}{n} = \sum_{r=1}^{n} \cos \frac{2r\pi}{n} = 0.$$

If n is an integer greater than unity, extend the result to show that

$$\sum_{r=1}^{n} \sin \left(\alpha + \frac{2rp\pi}{n}\right) = 0,$$

where p is any positive integer, not a multiple of n.

6. Find the modulus of $e^{i\theta} - e^{i\varphi}$.

7. Simplify

(i) $\dfrac{\cos 2\alpha + i \sin 2\alpha}{\cos \alpha + i \sin \alpha}$, (ii) $\dfrac{(\cos \theta - i \sin \theta)^2}{(\cos \theta + i \sin \theta)^3}$.

8. Show that

$$(1 - 2x \cos \theta + x^2) = (1 - xe^{i\theta})(1 - xe^{-i\theta}).$$

Hence deduce that if $|x| < 1$

$$\log(1 - 2x \cos \theta + x^2) = -2 \sum_{k=1}^{\infty} (x^k \cos k\theta)/k.$$

§ 4.4. Roots of complex numbers

Finally De Moivre's theorem can be used to find the roots of a complex number. In the first place let us examine the roots of the equation

$$z^n = 1, \tag{7.43}$$

where z is a complex number. If we write $z=r(\cos\theta+i\sin\theta)=re^{i\theta}$, we have immediately from De Moivre's theorem

$$r^n(\cos n\theta + i\sin n\theta) = 1. \tag{7.44}$$

Now on the right hand side of this equation we can write

$$1 = e^{2k\pi i} = \cos 2k\pi + i\sin 2k\pi$$

where k is any integer. Thus if r and θ are such that

$$r^n = 1, \tag{7.45}$$

and

$$\cos n\theta = \cos 2k\pi, \qquad \sin n\theta = \sin 2k\pi, \tag{7.46}$$

then the eq. (7.43) is satisfied. Since r must be real, eq. (7.45) is satisfied provided $r=1$, whilst eqs. (7.46) are satisfied provided $\theta=2k\pi/n$. If k takes the values $0, 1, 2,\ldots, n-1$, we get n distinct values for

$$z = \cos\theta + i\sin\theta = \cos 2k\pi/n + i\sin 2k\pi/n,$$

but if k takes the values $n, n+1,\ldots$ then $\theta=2\pi,\ 2\pi+2\pi/n,\ldots$ and will simply repeat the values of z obtained from $k=0, 1,\ldots$. Thus with $r=1$, $\theta=2k\pi/n$, $k=0, 1,\ldots n-1$, there are n distinct values of z satisfying eq. (7.43); that is, n distinct values of $\sqrt[n]{1}$. If we plot these n distinct values on the Argand diagram we see that they coincide with the vertices of a regular polygon of n sides inscribed in the unit circle, with one vertex of the polygon at $z=1$ ($r=1$, $k=0$). Fig. 7.7 shows this result for $n=6$, the six roots being given by

$$z_{k+1} = e^{2k\pi i/6}$$
$$= \cos(2k\pi/6) + i\sin(2k\pi/6),$$

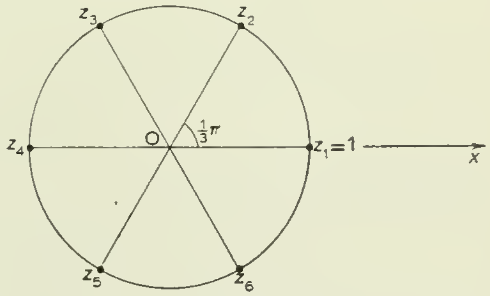

Fig. 7.7

with $k=0, 1, 2, 3, 4, 5$. Since the radius of the circle is unity, and the angles between successive radii to the points z_{k+1} are all equal, then $z_3=z_2^2$, $z_4=z_3z_2=z_2^3$ and so on. Thus if we use ω for z_1, the roots can be written as $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$. Using the result $\omega^6=1$, it is easy for the reader to verify that all these roots satisfy eq. (7.43) for $n=6$, and also from eq. (7.43) we have for their sum

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0.$$

The method used for the above particular example can be used to find the roots of any real or complex number. We will illustrate this by finding the cube root of $1-i$. This is really the solution of the equation

$$z^3 = 1 - i,$$

and as before writing $z = re^{i\theta}$ and using

$$1 - i = \sqrt{2}(\cos 7\pi/4 + i \sin 7\pi/4) = \sqrt{2}e^{i7\pi/4},$$

we see that the equation is satisfied if

$$r^3 e^{3i\theta} = \sqrt{2}e^{i7\pi/4},$$

so that there are three distinct values given by

$$r = 2^{\frac{1}{6}}; \qquad 3\theta = \tfrac{7}{4}\pi + 2k\pi \qquad (k = 0, 1, 2);$$

the values of r must be positive, so that $2^{\frac{1}{6}}$ is the positive number which when raised to the power 6, gives 2. Thus the three roots are

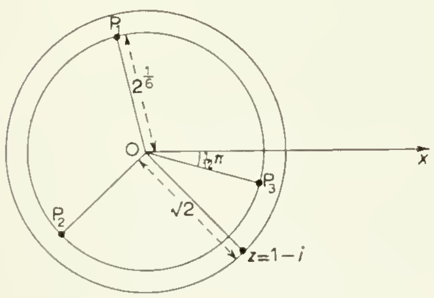

Fig. 7.8

$$z_1 = 2^{\frac{1}{6}}e^{7i\pi/12} = 2^{\frac{1}{6}}(\cos \tfrac{7}{12}\pi + i \sin \tfrac{7}{12}\pi),$$

$$z_2 = 2^{\frac{1}{6}}e^{5i\pi/4} = 2^{\frac{1}{6}}(\cos \tfrac{5}{4}\pi + i \sin \tfrac{5}{4}\pi),$$

$$z_3 = 2^{\frac{1}{6}}e^{23i\pi/12} = 2^{\frac{1}{6}}(\cos \tfrac{23}{12}\pi + i \sin \tfrac{23}{12}\pi),$$

and are represented on the Argand diagram in fig. 7.8 by the points P_1, P_2, P_3 where

$$OP_1 = OP_2 = OP_3 = 2^{\frac{1}{6}}, \quad \text{and} \quad \widehat{P_1OP_2} = \widehat{P_2OP_3} = \widehat{P_3OP_1} = \tfrac{2}{3}\pi.$$

Example 12

Write down all the roots of the equation

$$(x + 1)^6 + 64x^6 = 0.$$

We can write this equation in the form

$$\left(\frac{x+1}{x}\right)^6 = -64,$$

The six roots of $-64 = 64e^{i\pi} = 64(\cos \pi + i \sin \pi)$, are

$$2 \exp\left(i\frac{\pi}{6} + \frac{i2k\pi}{6}\right), \qquad k = 0, 1, 2, 3, 4, 5.$$

These are

$$\begin{aligned}
2e^{i\pi/6} &= 2(\cos \tfrac{1}{6}\pi + i \sin \tfrac{1}{6}\pi) = \sqrt{3} + i, \\
2e^{i\pi/2} &= 2(\cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi) = 2i, \\
2e^{i5\pi/6} &= 2(\cos \tfrac{5}{6}\pi + i \sin \tfrac{5}{6}\pi) = -\sqrt{3} + i, \\
2e^{i7\pi/6} &= 2(\cos \tfrac{7}{6}\pi + i \sin \tfrac{7}{6}\pi) = -\sqrt{3} - i, \\
2e^{i3\pi/2} &= 2(\cos \tfrac{3}{2}\pi + i \sin \tfrac{3}{2}\pi) = -2i, \\
2e^{i11\pi/6} &= 2(\cos \tfrac{11}{6}\pi + i \sin \tfrac{11}{6}\pi) = \sqrt{3} - i.
\end{aligned}$$

The value of x corresponding to the first root is given by

$$x + 1 = x(\sqrt{3} + i),$$

or

$$x = -1/(1 - \sqrt{3} - i) = (\sqrt{3} - 1 + i)/(5 - 2\sqrt{3}).$$

The value of x corresponding to the second root is

$$x + 1 = 2xi,$$

giving $x = -\frac{1}{5}(1 + 2i)$. The remaining values follow in a similar way.

Having evaluated roots of complex numbers in this way, let us again consider De Moivre's theorem for any positive or negative integer p in the form

$$(\cos \theta + i \sin \theta)^p = \cos p\theta + i \sin p\theta.$$

If q is a positive integer we can take the qth root of each side of this equation, and we see that *one value* of the qth root of the right hand side is

$$\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta,$$

so that this is *one value* of

$$(\cos \theta + i \sin \theta)^{p/q}.$$

Thus De Moivre's theorem for a positive or negative fraction can be used to give one value of a fractional power of a complex number.

EXERCISE 7.3

1. Find all the cube roots of i. Hence or otherwise express $x^6 + 1$ as a product of three real quadratic factors.

2. A regular n-gon is inscribed in a circle with radius R and centre $c = a + ib$. One vertex is at the point $z_0 = a + i(b + R)$. Find the remaining vertices

3. Show that the square roots of $\frac{1}{2}(1+\sqrt{3}i)$ are $\pm\frac{1}{2}(\sqrt{3}+i)$. Hence or otherwise find the solutions of the equation $z^4=-1+\sqrt{3}i$.

4. Show that the cube roots of $(1+i)/\sqrt{2}$ are

$\frac{1}{2}(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}})+\frac{1}{2}(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})i,\quad -\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}},\quad -\frac{1}{2}(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})-\frac{1}{2}(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})i.$

Hence solve the equation $x^6-64i=0$.

5. Find all the values of $(\sqrt{3}+i)^{\frac{1}{3}}$.

6. Show that there are exactly n roots of the equation $z^n=a$, where a is a non-zero complex number and n a positive integer. Prove that

$$x^5-y^5=(x-y)(x^2+2xy\cos\tfrac{1}{5}\pi+y^2)(x^2-2xy\cos\tfrac{2}{5}\pi+y^2).$$

7. Find all the roots of the equation $x^4-2x^2+4=0$ (giving answers in surd form), and indicate the positions of the roots on the Argand diagram.

8. (i) Find all the fifth roots of unity and show their positions on the Argand diagram. If one of the complex roots is ω, show that

$$z^5-1=(z-1)\prod_{r=1}^{4}(z-\omega^r).$$

(ii) Find all the solutions of the equation

$$(z-2)^5=z^5,$$

and show that the corresponding points on the Argand diagram lie on a line parallel to the imaginary axis.

§ 5. Function of a complex variable

If x and y are real variables, the variable $z=x+iy$ is a complex variable. The Argand diagram which is the xy-plane is then referred to as the z-plane. If in some region of this plane for each point $z=x+iy$, one or more complex numbers $w=u+iv$ where u, v are real can be determined by some rule, we say that the variable w is given as a function of z and write

$$w=u+iv=f(z).$$

Example 13

$$w=z^2=(x+iy)^2=x^2-y^2+2ixy,$$

so that

$$u=x^2-y^2,\qquad v=2xy,$$

and the value of w is determined for every value of (x, y) in the z-plane.

Note that u and v are real functions of the real variables (x, y), so that we can write

$$w = u + iv = f(z) = f_1(x, y) + if_2(x, y).$$

The function $f_1(x, y)$ is called the real part of the function $f(z)$ and the function $f_2(x, y)$ is called the imaginary part. A complex function $f(z)$ is said to be *single valued* in a region of the z-plane if for each z in the region there corresponds only one value of $w=f(z)$. If more than one value of w corresponds to each value of z, the function $f(z)$ is said to be *multiple valued.*

The functions $f(z) \equiv z$, $f(z) \equiv z^2$ are single valued functions. The function $f(z) \equiv z^{\frac{1}{2}}$ is double valued; for if $w = z^{\frac{1}{2}}$ where $z = r(\cos\theta + i\sin\theta)$ we have seen in § 4.4 that there are two distinct values of w given by

$$w_1 = r^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta),$$

$$w_2 = r^{\frac{1}{2}}\{\cos(\pi + \tfrac{1}{2}\theta) + i\sin(\pi + \tfrac{1}{2}\theta)\}.$$

All that we are concerned about in this paragraph and § 5.1 is that the reader should be able to determine the real and imaginary parts $f_1(x, y)$, $f_2(x, y)$ of a given function $f(z)$. The analytical theory of complex functions is given in Ch. 17.

Example 14

Express $(z-1)^{-1}$ in the form $f_1(x, y) + if_2(x, y)$.

Writing $z = x + iy$ we have

$$\frac{1}{z-1} = \frac{1}{(x-1)+iy} = \frac{(x-1)-iy}{(x-1)^2+y^2},$$

which can be written as

$$\frac{x-1}{(x-1)^2+y^2} + i\frac{(-y)}{(x-1)^2+y^2}.$$

§ 5.1. Some elementary complex functions

In §§ 1–4 of this chapter we have defined the operations of addition, multiplication, division and root extraction for complex numbers. These suffice to determine for any z the value of all rational algebraic functions of z of the form $P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials in z or may involve fractional powers of z. However they do not provide a direct means of defining the complex functions corresponding to the

real functions such as $\sin x$, $\exp x$, $\log x$, $\tan^{-1} x$, $\cosh x$, etc. It would be very inconvenient if the definition of one of these functions, say $\sin z$, did not conform to the usual definition of $\sin x$ when z assumes the real value x. We begin then by defining $\exp z$ as a function which becomes $\exp x$ when z has the real value x, and is also such that the familiar law of indices $\exp z_1 \exp z_2 = \exp(z_1+z_2)$ holds. A definition of $\exp z$ that fulfils these two criteria is

$$e^z \equiv e^{x+iy} = e^x(\cos y + i \sin y):$$

for when $y=0$, $z=x$ then $\exp z = \exp x$; also, using this definition,

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2), \end{aligned}$$

and by eq. (7.27) this becomes

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1+x_2}\{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\} \\ &= e^{x_1+x_2+i(y_1+y_2)} = e^{z_1+z_2}. \end{aligned}$$

We note also that when $x=0$ we have

$$e^{iy} = \cos y + i \sin y, \tag{7.47}$$

which is the result we tentatively assumed in eq. (7.30) conforming with the series expansions of the exponential function, the sine and cosine functions in Ch. 4.

Replacing y by $-y$ in eq. (7.47) we get

$$e^{-iy} = \cos y - i \sin y, \tag{7.48}$$

and adding and subtracting eq.s (7.47) and (7.48) we get the formulae

$$\begin{aligned} \cos y &= \tfrac{1}{2}(e^{iy} + e^{-iy}), \\ \sin y &= \tfrac{1}{2i}(e^{iy} - e^{-iy}). \end{aligned}$$

These results suggest immediately that we define the complex trigonometric functions of z as follows

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \tag{7.49}$$

$$\sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}), \tag{7.50}$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = (\cos z)^{-1}, \qquad \operatorname{cosec} z = (\sin z)^{-1}.$$

Using these definitions it is easy to check that all the familiar formulae of analytical trigonometry remain valid when real variables x are replaced by complex variables z. For example,

$$\sin^2 z + \cos^2 z = 1, \tag{7.51}$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{7.52}$$

and so on.

Similarly we can define the complex hyperbolic functions, remembering their real forms, as follows:

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \tag{7.53}$$

$$\sinh z = \tfrac{1}{2}(e^z - e^{-z}), \tag{7.54}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = (\cosh z)^{-1}, \qquad \operatorname{cosech} z = (\sinh z)^{-1}.$$

Comparing the results in eqs. (7.53), (7.54) with those in eqs. (7.49), (7.50), we see that

$$\cos iz = \cosh z, \qquad \sin iz = i \sinh z, \tag{7.55}$$

or conversely

$$\cosh iz = \cos z, \qquad \sinh iz = i \sin z. \tag{7.56}$$

It is now easy to see why the formulae relating the hyperbolic functions given in Ch. 4 § 6 have a strong resemblance, apart from signs, to the corresponding trigonometric formulae, since the above results apply equally when z is real and when z is purely imaginary.

The inverse trigonometric and hyperbolic functions are defined in the same way as for real variables, namely that if $w = \sin z$ then $z = \sin^{-1} w$. As for inverse functions of real variables, these functions are multiple valued. The complex logarithmic function is the inverse of the exponential function and is also multiple valued. The many valuedness of $\log z$ and allied functions will be discussed in Ch. 17.

It will be of value to the reader to know how to determine the real and imaginary parts of complex functions and also how they may be used in elementary problems. We will therefore conclude the chapter with some examples.

Example 15

Express $\sin(i+\frac{1}{4}\pi)$ in the form $a+ib$ where a and b are real.

Using eq. (7.52) we have

$$\sin(i+\tfrac{1}{4}\pi) = \sin i \cos \tfrac{1}{4}\pi + \cos i \sin \tfrac{1}{4}\pi,$$

and using eqs. (7.55) with $z=1$ this becomes

$$\sin(i+\tfrac{1}{4}\pi) = \frac{1}{\sqrt{2}} i \sinh 1 + \frac{1}{\sqrt{2}} \cosh 1.$$

Example 16

Prove that the general value of $\sin^{-1} 3$ is

$$\tfrac{1}{2}(4n+1)\pi \pm i \log(3+2\sqrt{2}).$$

There is no real angle whose sine is 3, so the value of $\sin^{-1} 3$ must be complex. Suppose

$$\sin^{-1} 3 = a + ib,$$

where a, b are real. Then

$$\sin(a+ib) = 3$$

and by the method of Example 15, we get

$$\sin a \cosh b + i \cos a \sinh b = 3. \tag{7.57}$$

Thus equating real and imaginary parts of eq. (7.57) real values a, b must satisfy the two equations

$$\sin a \cosh b = 3, \tag{7.58}$$

$$\cos a \sinh b = 0. \tag{7.59}$$

Since $\sin^{-1} 3$ is not real, b cannot be zero, so in eq. (7.59) we must have $\cos a = 0$. Also if b is real $\cosh b$ is positive, so $\sin a$ must be positive in eq. (7.58). These two conditions on a mean that $a = \frac{1}{2}(4n+1)\pi$ where n is any integer. The two eqs. (7.58) and (7.59) will then be satisfied if $\cosh b = 3$, or

$$b = \cosh^{-1} 3 = \pm \log(3+2\sqrt{2}),$$

using eq. (4.59) of Ch. 4.

The reader should verify that the same result can be found by writing

$$3 = \sin(a+ib) = \frac{1}{2i}\{e^{i(a+ib)} - e^{-i(a+ib)}\},$$

and solving this equation for $e^{i(a+ib)} = e^{ia-b}$.

Example 17

Express $\log(2+i)$ in the form $a+ib$ where a and b are real.

For the logarithmic function $\log z$ we write z in its polar coordinate form $z = re^{i\theta}$ and then

$$\log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta,$$

which is the required form. In this example

$$2 + i = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

where $r\cos\theta = 2$, $r\sin\theta = 1$, giving

$$r = \sqrt{5}, \qquad \cos\theta = 2/\sqrt{5}, \qquad \sin\theta = 1/\sqrt{5}. \tag{7.60}$$

Thus

$$\log(2+i) = \log(\sqrt{5}\,e^{i\theta}) = \log\sqrt{5} + i\theta = \tfrac{1}{2}\log 5 + i\theta,$$

where θ is given by eqs. (7.60) and is therefore multiple valued.

Example 18

Express i^i in the form $a+ib$. Here we write

$$i^i = a + ib,$$

and take logs of both sides. We have

$$i\log i = \log(a+ib) = \log \alpha e^{i\beta},$$

where $\alpha = \sqrt{(a^2+b^2)}$, $\cos\beta = a/\alpha$, $\sin\beta = b/\alpha$. Also

$$\log i = \log e^{\frac{1}{2}i\pi} = \tfrac{1}{2}i\pi,$$

so that

$$i\cdot\tfrac{1}{2}i\pi = \log\alpha + i\beta,$$

giving $\log\alpha = -\frac{1}{2}\pi$, $\beta = 0$. Therefore $b=0$, $a=\alpha=e^{-\frac{1}{2}\pi}$. This result is now obvious if in the original form i^i we write $i=e^{\frac{1}{2}i\pi}$, so that $i^i=(e^{\frac{1}{2}i\pi})^i=e^{-\frac{1}{2}\pi}$. However the above method must be used for a more general complex power such as $(\sqrt{3}+i)^i$.

Example 19

If $x+iy=\tan(\alpha+i\beta)$ where x, y, α, β are real, prove that

$$x = (\sin 2\alpha)/(\cos 2\alpha + \cosh 2\beta),$$
$$y = (\sinh 2\beta)/(\cos 2\alpha + \cosh 2\beta).$$

We write

$$\tan(\alpha+i\beta) = \frac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)} = \frac{\sin\alpha\cos i\beta + \cos\alpha\sin i\beta}{\cos\alpha\cos i\beta - \sin\alpha\sin i\beta},$$

or using eqs. (7.55) this becomes

$$\frac{\sin\alpha\cosh\beta + i\cos\alpha\sinh\beta}{\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta}.$$

Multiplying numerator and denominator by $\cos\alpha\cosh\beta + i\sin\alpha\sinh\beta$ and picking out the real and imaginary parts we get

$$x = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta},$$

$$y = \frac{\sinh\beta\cosh\beta}{\cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta}.$$

It is easy for the reader to verify that these results are equivalent to those given.

Example 20

Use complex variable methods to evaluate the integrals

$$\int e^{ax}\cos bx\,dx, \qquad \int e^{ax}\sin bx\,dx,$$

where a, b, x are real.

Using eq. (7.47) we see that the integrands in these two integrals are the real and imaginary parts of

$$e^{ax}\,e^{ibx} = e^{(a+ib)x}.$$

We therefore evaluate

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a+ib} = \frac{(a-ib)e^{ax}(\cos bx + i\sin bx)}{a^2+b^2}.$$

Thus equating real and imaginary parts

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx),$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$$

Compare these results with those given in Ch. 5, eqs. (5.9) and (5.10).

EXERCISE 7.4

Express the complex functions given in Nos. 1–8 in the form $a+ib$ where a and b are real, giving *all* values when the functions are multiple valued.

1. $\sinh(3+i)$. **2.** $\cosh^{-1}\frac{1}{2}$. **3.** $\cot\frac{1}{2}\pi i$.

4. $i^{(1+i)}$. **5.** $\log ie$. **6.** $\log(3-4i)$.

7. $(\sqrt{3}+i)^i$. **8.** $\tanh^{-1} 2$.

9. Express $\sin(x+iy)$ in the form $u+iv$ where x, y, u, v are real and show that

$$u^2+v^2 = \tfrac{1}{2}(\cosh 2y - \cos 2x).$$

10. Show that, if x and y are real

$$|\cosh(x+iy)| = \tfrac{1}{2}(\cosh 2x + \cos 2y),$$

and that if x, y, u, v are real and

$$\cosh(x+iy) = \tan(u+iv),$$

then

$$\cosh 2x + \cos 2y = 2(\cosh 2v - \cos 2u)/(\cosh 2v + \cos 2u).$$

FUNCTIONS OF MORE THAN ONE VARIABLE

§ 1. Introduction

In the preceding chapters we have been mainly concerned with functions of a single variable, although mention was made, at the end of Ch. 1, of functions of more than one variable when partial derivatives were defined. Also a distinction was made between implicit and explicit functions.

In this chapter we wish to develop the theory of functions of more than one variable because they are, of course, of quite considerable importance, since many physical and chemical quantities are dependent on the value of more than one variable. We shall state and prove some theorems for functions of several variables which are analogous to theorems already established for functions of a single variable. It is obvious that to do so we require to give precision to certain concepts with which we are already familiar for functions of one variable. We shall, in general, confine ourselves to explicit functions of two independent variables, but the definitions and methods may be readily extended to more than two variables.

§ 1.1. Continuous functions

A function $f(x, y)$ of two independent variables x and y is said to be continuous at the values $x=a$, $y=b$ of the variables, if, given any small positive number ε, we can find a positive number μ dependent on ε, which is such that

$$|f(x, y) - f(a, b)| < \varepsilon, \tag{8.1}$$

when $|x-a| \leqslant \mu(\varepsilon)$ and $|y-b| \leqslant \mu(\varepsilon)$. This result may be illustrated in a diagram as in fig. 8.1. It means that however small ε may be chosen we can find a square, as shaded in fig. 8.1 about the point (a, b) which is such that the value of the function $f(x, y)$ at any point within that square differs from the value of the function at (a, b) by less than ε.

The alternative ways of expressing this result are

$$f(x, y) = f(a, b) + \varepsilon,$$

where $\varepsilon \to 0$ as $x \to a$, $y \to b$; or

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = f(a, b),$$

the definition of this limit implying the statement given above.

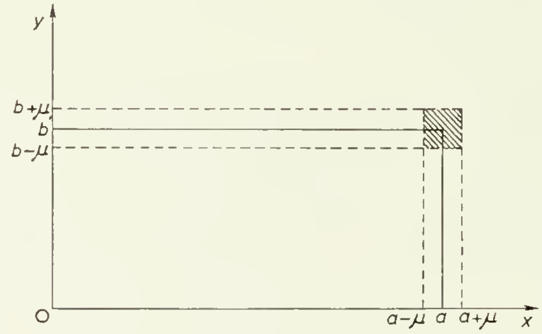

Fig. 8.1

As for functions of a single variable, remembering the properties of limits (Ch. 1 § 2.1), we see that if $f(x, y)$ and $g(x, y)$ are two continuous functions at $x=a$, $y=b$ then the following functions are also continuous at these values

(i) $f(x, y) + g(x, y)$,
(ii) $f(x, y)\,g(x, y)$ and
(iii) $f(x, y)/g(x, y)$, provided $g(a, b) \neq 0$.

Most of the ordinary functions of analysis are continuous except possibly for certain particular values of the variables x and y.

§ 1.2. Partial derivatives: geometrical illustration

In Ch. 1 § 7 we defined the partial derivatives of $z=f(x, y)$ with respect to x and y respectively as

$$\frac{\partial z}{\partial x} = \lim_{\delta x\to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}, \tag{8.2}$$

and

$$\frac{\partial z}{\partial y} = \lim_{\delta y\to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}; \tag{8.3}$$

we note that $\partial z/\partial x$ is the ordinary derivative of z with respect to x, y being treated as a constant, and similarly for $\partial z/\partial y$ in which x is treated as a constant. We will now illustrate the meaning of partial differentiation geometrically. The equation

$$z = f(x, y)$$

represents a surface in the three-dimensional space or 3-space with rectangular coordinates x, y, z. To every point A in the xy-plane, there corresponds a value of z given by this equation. The point P having an

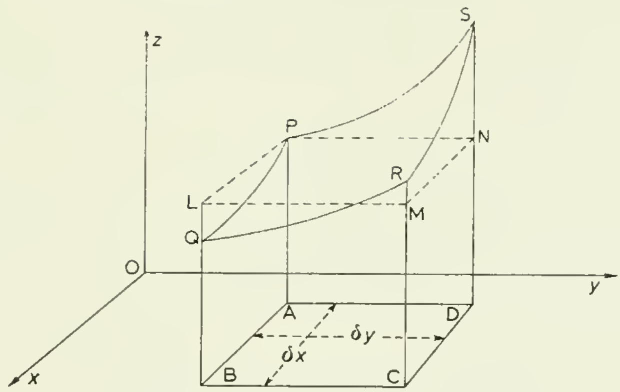

Fig. 8.2

ordinate AP parallel to Oz and equal (on a chosen scale) to this value of z, can then be plotted in the three-dimensional space Oxyz (fig. 8.2). As the values of (x, y) are varied, the locus of P will be a surface in this 3-space.

Let PQRS be an elementary portion of this surface corresponding to the points A(x, y), B$(x+\delta x, y)$, C$(x+\delta x, y+\delta y)$, D$(x, y+\delta y)$ forming a rectangle of edges δx, δy parallel to the axes Ox, Oy in the xy-plane. Let PLMN be a plane through P parallel to the xy-plane, cutting the ordinates BQ, CR, DS at L, M, N respectively, so that

$$\begin{aligned} \mathrm{LQ} &= f(x+\delta x, y) - f(x, y), \\ \mathrm{NS} &= f(x, y+\delta y) - f(x, y), \\ \mathrm{MR} &= f(x+\delta x, y+\delta y) - f(x, y). \end{aligned}$$

Note that LQ is negative in fig. 8.2.

Hence the partial derivative $\partial z/\partial x$ as defined in equation (8.2) is given by

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{\mathrm{LQ}}{\mathrm{PL}} = \lim_{\mathrm{L}\to\mathrm{P}} \tan \widehat{\mathrm{LPQ}},$$

and by definition of gradient of a curve, this is the gradient of the curve PQ at P. Thus $\partial z/\partial x$ is the gradient of the sectional curve of the surface PQRS in the direction parallel to the axis of x. Similarly

$$\frac{\partial z}{\partial y} = \lim_{N\to P} \frac{NS}{PN}$$

is the gradient of curve PS at P, which is the gradient of the sectional curve of the surface at P in the direction parallel to the axis of y.

It further appears from the figure that

$$\frac{\partial}{\partial x} f(x, y+\delta y) = \lim_{\delta x\to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \lim_{M\to N} \frac{MR - NS}{MN}.$$

which is the gradient of the curve SR at S.

Now in this result when $\delta y \to 0$ also, N moves up to P and M moves up to L, so that the plane NSRM moves up to the position PQL; then the gradient of the curve SR at S ultimately coincides with the gradient of the curve PQ at P. This means that

$$\lim_{\delta y\to 0} \frac{\partial}{\partial x} f(x, y+\delta y) = \frac{\partial f(x, y)}{\partial x}. \tag{8.4}$$

These results apply, of course, only if the surface PQRS is a continuous surface, that is if it has no breaks, and if there is no sudden change in the value of $f_x(x, y)$ itself. The result (8.4) is also obvious from the definition of continuity of a function provided that the partial derivative $f_x(x, y)$ is a continuous function. Similarly

$$\lim_{\delta x\to 0} \frac{\partial}{\partial y} f(x+\delta x, y) = \frac{\partial}{\partial y} f(x, y), \tag{8.5}$$

provided $f_y(x, y)$ is a continuous function.

§ 1.3. Total differential

It will be possible to extend the results of this paragraph to functions of more than two variables, but again we shall start with two independent variables x, y and take $z = f(x, y)$ as the dependent variable.

Suppose that δz is the small change in z corresponding to the small independent changes δx in x and δy in y. Then

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y),$$

and by simply subtracting and adding the same term $f(x, y+\delta y)$, we can write this in the form

$$\delta z = \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\},$$

or

$$\delta z = \frac{\{f(x+\delta x, y+\delta y)-f(x, y+\delta y)\}\delta x}{\delta x} + \frac{\{f(x, y+\delta y)-f(x, y)\}\delta y}{\delta y}. \tag{8.6}$$

But by definition of a partial derivative, eq. (8.2)

$$\lim_{\delta x \to 0} \frac{\{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\}}{\delta x} = \frac{\partial}{\partial x} f(x, y + \delta y),$$

and as $\delta y \to 0$ also, this becomes, by eq. (8.4),

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}.$$

Similarly from eq. (8.3)

$$\lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}.$$

So from the definitions of a limit,

$$\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \left(\frac{\partial z}{\partial x} + \varepsilon_1\right),$$

and

$$\frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \left(\frac{\partial z}{\partial y} + \varepsilon_2\right),$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\delta x, \delta y \to 0$. Thus altogether, substituting in eq. (8.6)

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + (\varepsilon_1 \delta x + \varepsilon_2 \delta y). \tag{8.7}$$

Now, in the same way as for a function of one variable, we define the *differential* of z to be the first two terms on the right hand side of this eq. (8.7), that is

$$\mathrm{d}z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y. \tag{8.8}$$

Again, if we take $z \equiv x$, we have $\partial z/\partial x = 1$ and $\partial z/\partial y = 0$, so that

$$dz \equiv dx = 1\delta x,$$

whilst if we take $z \equiv y$, we have $\partial z/\partial x = 0$ and $\partial z/\partial y = 1$, so that

$$dz \equiv dy = 1\delta y,$$

and we see that the differentials of the *independent* variables are the same as the increments in those variables. Thus we can rewrite eq. (8.8) for the differential of the dependent variable z as

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{8.9}$$

If x varies while y remains constant, so that $dy = 0$, we see that $(\partial z/\partial x)\,dx$ is the differential of z corresponding to a variation in x alone, whilst similarly $(\partial z/\partial y)\,dy$ is the differential of z corresponding to a variation of y alone. The expression (8.9) for dz is therefore often referred to as the *total differential* of z for the two variables x and y.

We note that in a similar way to the two-dimensional result, $(\partial z/\partial x)\,dx$ is the change in z along the tangent to the sectional curve PQ at P (fig. 8.2), whilst $(\partial z/\partial y)\,dy$ is the change in z along the tangent to the sectional curve PS at P. These two tangents at P define a plane, the tangent plane to the surface at P, and the total differential dz is the change in z in the direction (dx, dy) on this plane.

The result (8.9) may be readily extended, by exactly the same method, to a function of more than two variables. If $z = f(x_1, x_2, \ldots x_n)$, where $x_1, x_2, \ldots x_n$ are n independent variables, then we define the total differential of z as

$$dz = \frac{\partial z}{\partial x_1}\,dx_1 + \frac{\partial z}{\partial x_2}\,dx_2 + \ldots + \frac{\partial z}{\partial x_n}\,dx_n. \tag{8.10}$$

§ 1.4. Differentiation of functions of functions

Suppose $z = f(x, y)$, but x, y instead of being independent variables are both given as functions of a single independent variable t say. Then corresponding to a given increment δt in t, there will be increments δx, δy in x and y respectively and an increment δz in z. Thus from eq. (8.7) divided throughout by δt

$$\frac{\delta z}{\delta t} = \frac{\partial z}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial z}{\partial y}\frac{\delta y}{\delta t} + \varepsilon_1\frac{\delta x}{\delta t} + \varepsilon_2\frac{\delta y}{\delta t},$$

where ε_1, $\varepsilon_2 \to 0$ as $\delta t \to 0$ assuming that then δx, $\delta y \to 0$. Proceeding to the limit as $\delta t \to 0$, if $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$ exist and are finite, we find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{8.11}$$

But by definition of the differentials of z, x, y all given as functions of t, we have

$$\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t, \qquad \mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t, \qquad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t,$$

$\mathrm{d}t$ being the differential of the independent variable t in each case. Thus eq. (8.11) becomes

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y.$$

Comparing with (8.8) we see that this is the correct expression for the total differential of z, whether x, y are independent variables or both dependent on a single variable t. The expression (8.11) enables us to determine the derivative of z with respect to t when x and y are given as functions of t.

Example 1

Find $\mathrm{d}z/\mathrm{d}t$ when

$$z = y \sin(2x + 3y), \tag{8.12}$$

$$x = \sqrt{(1 + t^2)}, \qquad y = 2t - 1. \tag{8.13}$$

We have

$$\frac{\partial z}{\partial x} = 2y\cos(2x + 3y),$$

$$\frac{\partial z}{\partial y} = \sin(2x + 3y) + 3y\cos(2x + 3y),$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{\sqrt{(1 + t^2)}} = \frac{y + 1}{2x}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2.$$

Thus

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{y(y + 1)}{x}\cos(2x + 3y) + 2\sin(2x + 3y) + 6y\cos(2x + 3y),$$

where x, y are given in terms of t by eqs. (8.13). It is obvious that the result could be determined by simple substitution of eqs. (8.13) in eq. (8.12) and then differentiating the resulting expression with respect to t. However, the expression for z as a function of t would be very clumsy and the process of differentiation would in fact be unchanged, though set out differently.

The above application does not therefore fully demonstrate the practical usefulness of the total differential and the result in eq. (8.11). This usefulness is illustrated more fully when x and y instead of being given as functions of a single variable t are themselves functions of two other independent variables u, v. Suppose then

$$z = f(x, y), \qquad x = \varphi(u, v), \qquad y = \psi(u, v) \tag{8.14}$$

so that z is expressible as a function of the two variables u, v:

$$z = f(x, y) = f\{\varphi(u, v), \psi(u, v)\} \equiv g(u, v) \tag{8.15}$$

where the function $g(u, v)$ is a different function of u, v from $f(x, y)$ of x, y. We have inserted this result here because notation plays a considerable role in the use of the total differential of eq. (8.9). In that equation z is given as a function of x, y by the relation

$$z = f(x, y),$$

and $\partial z/\partial x$ represents the partial derivative of the function $f(x, y)$ with respect to x as defined by eq. (8.2) and could have been written as either f_x or $\partial f/\partial x$. Similarly for $\partial z/\partial y$. Thus in eq. (8.9)

$$\frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}, \qquad \frac{\partial z}{\partial y} = f_y = \frac{\partial f}{\partial y}. \tag{8.16}$$

However by means of the eqs. (8.14) we have been able to give z as a function of u, v by the relation $z=g(u, v)$ of eq. (8.15), and using this relation we define $\partial z/\partial u$ and $\partial z/\partial v$ to be

$$\frac{\partial z}{\partial u} = g_u = \frac{\partial g}{\partial u}, \qquad \frac{\partial z}{\partial v} = g_v = \frac{\partial g}{\partial v}. \tag{8.17}$$

This means that whenever z is differentiated partially with respect to x or y the function $f(x, y)$ is involved, whilst whenever z is differentiated partially with respect to u or v the function $g(u, v)$ is involved. Provided this fact is kept in mind no difficulty should be experienced.

If now v is treated as a constant in the eqs. (8.14) and (8.15) then z is given as a function of x, y where x and y are each given as functions of u. We can therefore use the result in eq. (8.11) with t replaced by u, but since z, x, y are really given as functions of two variables u, v we must use partial derivatives instead of the straight derivatives of that equation,

expressing the fact that v is kept constant. Thus

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial u}; \tag{8.18}$$

similarly if u is treated as a constant,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial v}. \tag{8.19}$$

But z being given as a function of the two independent variables u, v by the relation (8.15) we have for the total differential

$$\mathrm{d}z = \frac{\partial z}{\partial u}\,\mathrm{d}u + \frac{\partial z}{\partial v}\,\mathrm{d}v;$$

using eqs. (8.18) and (8.19), this becomes

$$\mathrm{d}z = \frac{\partial z}{\partial x}\left\{\frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v\right\} + \frac{\partial z}{\partial y}\left\{\frac{\partial y}{\partial u}\,\mathrm{d}u + \frac{\partial y}{\partial v}\,\mathrm{d}v\right\},$$

which can be written as

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y. \tag{8.20}$$

We see again that this expression for the total differential $\mathrm{d}z$ is valid even when x, y are not independent but are given as functions of two other variables u, v. In practice then this formula (8.20) for the total differential is very useful. It can be used either (i) when $\mathrm{d}x$ and $\mathrm{d}y$ are independent increments or (ii) when they are increments dependent upon a single increment $\mathrm{d}t$, giving formula (8.11), or (iii) when they depend upon two other increments $\mathrm{d}u$ and $\mathrm{d}v$, giving the formulae (8.18) and (8.19) for the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$. Further if x and y are given as functions of more than two independent variables $u, v, w, \ldots$ we can express $\mathrm{d}x$ and $\mathrm{d}y$ in terms of $\mathrm{d}u, \mathrm{d}v, \mathrm{d}w, \ldots$; then the coefficients of $\mathrm{d}u, \mathrm{d}v, \mathrm{d}w, \ldots$, are the partial derivatives $\partial z/\partial u, \partial z/\partial v, \partial z/\partial w, \ldots$ and are given by equations such as (8.18) and (8.19).

Similar equations will also apply when z is given as a function of any number of variables $x_1, x_2, \ldots x_n$. Using eq. (8.10)

$$\mathrm{d}z = \frac{\partial z}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial z}{\partial x_2}\,\mathrm{d}x_2 + \ldots + \frac{\partial z}{\partial x_n}\,\mathrm{d}x_n,$$

suppose that $x_1, x_2, \ldots x_n$ are each given as functions of variables $u, v, w \ldots$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial u} + \ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial u},$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial v} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial v} + \ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial v},$$

and so on. These results are often referred to as the *chain rule.*

Example 2

If $z=f(x, y)$ where $2x=\mathrm{e}^u+\mathrm{e}^v$, $2y=\mathrm{e}^u-\mathrm{e}^v$, show that

(i) $\displaystyle \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$,

(ii) $\displaystyle \mathrm{e}^{u+v}\frac{\partial z}{\partial x} = \mathrm{e}^v\frac{\partial z}{\partial u} + \mathrm{e}^u\frac{\partial z}{\partial v}$.

By equations (8.18) and (8.19) we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial x}\mathrm{e}^u + \frac{\partial z}{\partial y}\mathrm{e}^u\right) = \frac{1}{2}\mathrm{e}^u\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right), \tag{8.21}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{1}{2}\mathrm{e}^v\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right), \tag{8.22}$$

Thus

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{1}{2}\frac{\partial z}{\partial x}(\mathrm{e}^u + \mathrm{e}^v) + \frac{1}{2}\frac{\partial z}{\partial y}(\mathrm{e}^u - \mathrm{e}^v) = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}.$$

Also by elimination of $\partial z/\partial y$ from eqs. (8.21) and (8.22) we have

$$\mathrm{e}^v\frac{\partial z}{\partial u} + \mathrm{e}^u\frac{\partial z}{\partial v} = \mathrm{e}^{u+v}\frac{\partial z}{\partial x}.$$

§ 1.5. Further uses of differentials

If u and v are given as functions of x, y by the relations $u=\varphi(x, y)$, $v=\psi(x, y)$ it may be difficult to solve these two equations for x and y and express them as functions of u and v. If so, there is no obvious rule for calculating $\partial x/\partial u$, $\partial x/\partial v$ ... remembering that, in general,

$$\partial x/\partial u \neq (\partial u/\partial x)^{-1}.$$

However we can use the total differential results in this case. Since u and

v are given as functions of x and y, then by eq. (8.20)

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy,$$

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy,$$

remembering that here $\partial u/\partial x = \varphi_x = \partial\varphi/\partial x$ etc. Solving these two equations for the differentials dx and dy in terms of du, dv, we get

$$\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right)dx = \frac{\partial v}{\partial y}\,du - \frac{\partial u}{\partial y}\,dv, \tag{8.23}$$

and

$$\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right)dy = -\frac{\partial v}{\partial x}\,du + \frac{\partial u}{\partial x}\,dv. \tag{8.24}$$

Now if v is treated as a constant, so that $dv=0$, then dividing eq. (8.23) through by du we get

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Bigg/\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right),$$

whilst eq. (8.24) gives

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Bigg/\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right).$$

Similarly by putting $du=0$, we get the values of $\partial x/\partial v$, $\partial y/\partial v$.

Example 3

Given $u=x^3+y$, $v=x+y^3$, to find the values of $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$.

We have

$$\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 3y^2,$$

so that

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 3x^2\,dx + dy, \tag{8.25}$$

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = dx + 3y^2\,dy. \tag{8.26}$$

We note that eqs. (8.25) and (8.26) could have been written down immediately by simply taking the differential of each of the terms in $u=x^3+y$ and in $v=x+y^3$.

Solving eqs. (8.25) and (8.26) for $\mathrm{d}x$ and $\mathrm{d}y$ we get

$$(9x^2y^2 - 1)\,\mathrm{d}x = 3y^2\,\mathrm{d}u - \mathrm{d}v, \tag{8.27}$$

and

$$(9x^2y^2 - 1)\,\mathrm{d}y = -\mathrm{d}u + 3x^2\mathrm{d}v. \tag{8.28}$$

Putting $\mathrm{d}v=0$, eqs. (8.27) and (8.28) give

$$\frac{\partial x}{\partial u} = \frac{3y^2}{9x^2y^2 - 1}, \qquad \frac{\partial y}{\partial u} = -\frac{1}{9x^2y^2 - 1},$$

whilst putting $\mathrm{d}u=0$ they give

$$\frac{\partial x}{\partial v} = -\frac{1}{9x^2y^2 - 1}, \qquad \frac{\partial y}{\partial v} = \frac{3x^2}{9x^2y^2 - 1}.$$

The method is essentially the same when u, v are defined only implicitly in terms of x, y.

Example 4

Given $u+v=x^2+y^2-h^2$ and $uv=a^2x^2+b^2y^2-k^4$ (a, b, k are constants), show that

$$\frac{\partial u}{\partial x} = \frac{-2x(a^2 - u)}{u - v}, \qquad \frac{\partial v}{\partial y} = \frac{2y(b^2 - v)}{u - v}.$$

Show also that

$$\frac{\partial x}{\partial u} = -\frac{b^2 - v}{2(a^2 - b^2)x},$$

and obtain a value for $\partial y/\partial v$.

In the equations

$$u + v = x^2 + y^2 - h^2, \qquad uv = a^2x^2 + b^2y^2 - k^4,$$

take the total differential of both sides, to give

$$\mathrm{d}u + \mathrm{d}v = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y, \tag{8.29}$$

$$v\,\mathrm{d}u + u\,\mathrm{d}v = 2a^2x\,\mathrm{d}x + 2b^2y\,\mathrm{d}y. \tag{8.30}$$

To find $\partial u/\partial x$, we require the change in u corresponding to a change in x, when y remains constant. Thus putting $\mathrm{d}y=0$ in the above equations we get

$$\mathrm{d}u + \mathrm{d}v = 2x\,\mathrm{d}x,$$

$$v\,\mathrm{d}u + u\,\mathrm{d}v = 2a^2x\,\mathrm{d}x.$$

Eliminating $\mathrm{d}v$,

$$(u - v)\,\mathrm{d}u = 2x(u - a^2)\,\mathrm{d}x,$$

or

$$\frac{\partial u}{\partial x} = \frac{-2x(a^2 - u)}{u - v}.$$

Similarly putting $\mathrm{d}x=0$ and eliminating $\mathrm{d}u$, we get

$$\frac{\partial v}{\partial y} = \frac{2y(b^2 - v)}{u - v}.$$

To find $\partial x/\partial u$, we require the change in x corresponding to a change in u when v remains constant. Thus putting $\mathrm{d}v=0$ in eqs. (8.29) (8.30) we get

$$\mathrm{d}u = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y,$$

$$v\,\mathrm{d}u = 2a^2x\,\mathrm{d}x + 2b^2y\,\mathrm{d}y.$$

Eliminating $\mathrm{d}y$,

$$(b^2 - v)\,\mathrm{d}u = 2x(b^2 - a^2)\,\mathrm{d}x,$$

giving

$$\frac{\partial x}{\partial u} = -\frac{b^2 - v}{2x(a^2 - b^2)}.$$

Similarly putting $\mathrm{d}u=0$ and eliminating $\mathrm{d}x$ we get

$$\frac{\partial y}{\partial v} = \frac{a^2 - u}{2y(a^2 - b^2)}.$$

EXERCISE 8.1

1. If $u^2+uv=\log xy$, $v^2+uv=x^2+y^2$ find

$$\partial u/\partial x, \qquad \partial u/\partial y, \qquad \partial v/\partial x, \qquad \partial v/\partial y,$$

and show that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{y^2 - x^2}{xy(u + v)^2} = \frac{y^2 - x^2}{xy(x^2 + y^2 + \log xy)}.$$

2. If u, v are given as functions of x, y by the equations

$$x = u^3 + v, \qquad y = u + v^3,$$

prove that

$$\frac{\partial u}{\partial x} = \frac{3v^2}{9u^2v^2 - 1}, \qquad \frac{\partial v}{\partial x} = -\frac{1}{9u^2v^2 - 1}.$$

Hence find the value of $\partial z/\partial x$ when $z=\exp(u^2+v^2)$.

3. If $x^2+y^2=2\log uv$ and $x\log u + y\log v = u^2 + v^2$,

show that

$$\frac{\partial v}{\partial x} = \frac{v\{\log u - x(2u^2 - x)\}}{2v^2 - 2u^2 - y + x},$$

and find the corresponding expressions for $\partial v/\partial y$, $\partial u/\partial x$, $\partial u/\partial y$.

4. If $u=(x+y)/xy$, $v=1/(x+y)$ and z is given as a function of the independent variables x, y, prove that

$$\frac{\partial z}{\partial u}=\frac{x^2y^2}{x^2-y^2}\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right),\qquad \frac{\partial z}{\partial v}=\frac{x+y}{x-y}\left(y^2\frac{\partial z}{\partial y}-x^2\frac{\partial z}{\partial x}\right).$$

5. If u, v are given as functions of x, y by the relations

$$u^3+v^3+x^3-3y=0,\qquad u^2+v^2+y^2+2x=0,$$

find $\partial u/\partial x$.

§ 1.6. Particular applications: implicit functions

Suppose that we apply the eq. (8.11) to the particular case

$$z=f(x,y),\qquad x=t,\qquad y=\psi(t).$$

Here

$$z=f\{t,\psi(t)\}\equiv g(t)$$

and is therefore given as a function of a single variable t, so that using eq. (8.11) we get

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t},$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t},$$

where we remember that here $\partial z/\partial x=f_x$ and $\partial z/\partial y=f_y$. Since $x=t$, the variable t in this result may be replaced by the variable x, so that we have in fact the relations

$$z=f(x,y),\qquad y=\psi(x), \tag{8.31}$$

and

$$z=f\{x,\psi(x)\}\equiv g(x).$$

When we do use the variable x and the relations in this way, it is wiser to use the partial derivatives in the form f_x and f_y because of the possibility of ambiguity between $\mathrm{d}z/\mathrm{d}x$ and $\partial z/\partial x$. Thus when z and y are given as functions of x by the relations (8.31), then

$$\frac{\mathrm{d}z}{\mathrm{d}x}=f_x+f_y\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{8.32}$$

The relations (8.31) leading to the equation (8.32) do not often occur in

specific problems since it is simpler to eliminate the y, determine the function $g(x)$, and find $\mathrm{d}z/\mathrm{d}x$ immediately from $g(x)$. However the reason for deriving eq. (8.32) will now be demonstrated.

Suppose x, y are two variables related implicitly by the equation $f(x, y)=0$. This means that in the eqs. (8.31), (8.32), $z=0$ and $\mathrm{d}z/\mathrm{d}x=0$, so that from eq. (8.32)

$$f_x + f_y \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{8.33}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f_x}{f_y}. \tag{8.34}$$

This result in the form of eq. (8.33) is seen to be exactly equivalent to the method for finding $\mathrm{d}y/\mathrm{d}x$ from implicit functions as given in Ch. 1 § 4.6, but we now see that $\mathrm{d}y/\mathrm{d}x$ can be expressed directly in terms of f_x and f_y by eq. (8.34).

Example 5

To find $\mathrm{d}y/\mathrm{d}x$ when $f(x, y) \equiv x^3+y^3-3axy=0$.

We have

$$\frac{\partial f}{\partial x} = 3(x^2 - ay), \qquad \frac{\partial f}{\partial y} = 3(y^2 - ax),$$

giving from eq. (8.34)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(x^2 - ay)}{(y^2 - ax)}. \tag{8.35}$$

We note that simply writing down the total differential of $f(x, y)$ as the total differential of each of its terms, namely

$$3x^2\,\mathrm{d}x + 3y^2\,\mathrm{d}y - 3a(x\,\mathrm{d}y + y\,\mathrm{d}x) = 0, \tag{8.36}$$

and now dividing through by $\mathrm{d}x$, gives the same result and is more simple an operation.

We note here also that the second derivative $\mathrm{d}^2y/\mathrm{d}x^2$ may be found from the result (8.35) .Treating the right hand side of the equation as a quotient in which y is regarded as a function of x, we have by differentiation with respect to x

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\{(y^2 - ax)(2x - a\,\mathrm{d}y/\mathrm{d}x) - (x^2 - ay)(2y\,\mathrm{d}y/\mathrm{d}x - a)\}}{(y^2 - ax)^2}.$$

and substitution of $\mathrm{d}y/\mathrm{d}x$ from eq. (8.35) in the result on the right hand side gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{6ax^2y^2 - 2a^3xy - 2xy^4 - 2x^4y}{(y^2 - ax)^3}. \tag{8.37}$$

Alternatively eq. (8.35) may be written in the form

$$(y^2 - ax)\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - ay) = 0,$$

which is the form derived from eq. (8.36) after dividing through by $\mathrm{d}x$. Differentiating each term of the equation with respect to x, regarding y and $\mathrm{d}y/\mathrm{d}x$ as functions of x, we get

$$(y^2 - ax)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(2y\frac{\mathrm{d}y}{\mathrm{d}x} - a\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(2x - a\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0, \tag{8.38}$$

and again substitution of $\mathrm{d}y/\mathrm{d}x$ from eq. (8.35) gives the result above for $\mathrm{d}^2y/\mathrm{d}x^2$. By differentiation of eqs. (8.37) or (8.38) we can similarly find $\mathrm{d}^3y/\mathrm{d}x^3$ and so on for higher derivatives.

We can deal with implicit relations between more than two variables in a similar way. If $f(x, y, z) = 0$ is an implicit relation defining z as a function of the independent variables x and y, then if we treat y as a constant, we can find $\partial z/\partial x$ by the method of Ch. 1 § 4.6. Similarly if x is treated as a constant we can find $\partial z/\partial y$.

Example 6

To find $\partial z/\partial x$, $\partial z/\partial y$ when

$$z^3 + 2(x + y)z + x^2 = 0. \tag{8.39}$$

In this eq. (8.39) regard y as constant and differentiate each term partially with respect to x, regarding z as a function of x and y. We get

$$3z^2\frac{\partial z}{\partial x} + 2z + 2(x + y)\frac{\partial z}{\partial x} + 2x = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{2(x + z)}{2(x + y) + 3z^2}.$$

Similarly differentiating eq. (8.39) with respect to y, regarding x as constant, we get

$$3z^2\frac{\partial z}{\partial y} + 2z + 2(x + y)\frac{\partial z}{\partial y} = 0,$$

giving

$$\frac{\partial z}{\partial y} = -\frac{2z}{2(x + y) + 3z^2}.$$

It is left as an exercise to the reader to check that this is equivalent to taking the total differential of each term of eq. (8.39) and then putting $\mathrm{d}y = 0$ and $\mathrm{d}x = 0$ in turn. The method may be extended to a single implicit relation between any number of variables.

A similar method may be used when we have two implicit relations between three variables

$$f(x, y, z) = 0 \quad \text{and} \quad \varphi(x, y, z) = 0;$$

we regard these two relations as defining y and z as functions of the one independent variable x, since we may regard them as soluble for y and z in terms of x. We can find $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}z/\mathrm{d}x$ as in the following example.

Example 7

$$ax^2 + by^2 + cz^2 - 1 = 0, \qquad x^2 + y^2 - 4az = 0.$$

Differentiate each of the above equations throughout with respect to x, we get, cancelling a factor 2

$$ax + by\frac{\mathrm{d}y}{\mathrm{d}x} + cz\frac{\mathrm{d}z}{\mathrm{d}x} = 0,$$

and

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} - 2a\frac{\mathrm{d}z}{\mathrm{d}x} = 0.$$

Thus, by eliminating $\mathrm{d}z/\mathrm{d}x$ we get

$$(2a^2 + c)x + (2ab + c)y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(2a^2 + c)x}{(2ab + c)y},$$

and by eliminating $\mathrm{d}y/\mathrm{d}x$ we get

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{(a - b)x}{2ab + cz}.$$

It is obvious from the above that either of the three variables may be regarded as the independent variable, and that here $\mathrm{d}z/\mathrm{d}x = (\mathrm{d}x/\mathrm{d}z)^{-1}$, in contrast to the results for partial derivatives. Also the higher derivatives $\mathrm{d}^2y/\mathrm{d}x^2$, $\mathrm{d}^2z/\mathrm{d}x^2$ may be found as in Example 5 of this chapter. We have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{(2a^2 + c)\{(2ab + c)y^2 + (2a^2 + c)x^2\}}{(2ab + c)^2y^3}.$$

EXERCISE 8.2

1. If y is defined implicitly as a function of x by the equation $x^3 + y^3 - 3axy = 0$, find the value of $\mathrm{d}y/\mathrm{d}x$ and prove that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2a^3xy}{(y^2 - ax)^3}.$$

2. If

$$\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3 = 1,$$

find $\partial z/\partial x$ and $\partial x/\partial y$.

3. If y and z are defined implicitly as functions of x by the relations $x+y+z=a$, $x^2+y^2+z^2=b^2$, prove that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = \frac{a^2-3b^2}{(y-z)^3}.$$

4. If x and y are defined implicitly as functions of z by the relations

$$x^3+y^3+z^3=a^3, \qquad xy+yz+zx=b^2,$$

prove that

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{y^2(x+y)-z^2(x+z)}{x^2(x+z)-y^2(y+z)}.$$

5. If y and z are defined as functions of x by the relations

$$x+y+z=0, \qquad x^3+y^3-z^3=10,$$

determine $\mathrm{d}y/\mathrm{d}x$, $\mathrm{d}z/\mathrm{d}x$, $\mathrm{d}^2y/\mathrm{d}x^2$, $\mathrm{d}^2z/\mathrm{d}x^2$ at the point $x=1$, $y=1$, $z=-2$.

6. Two sets of three independent variables u, v, w and x, y, z are related by the three equations

$$\begin{aligned} x+y+z &= u^2+v^2+w^2, \\ x^2+y^2+z^2 &= u^3+v^3+w^3, \\ x^3+y^3+z^3 &= u+v+w. \end{aligned}$$

Prove that

$$\frac{\partial u}{\partial x} = \frac{18x^2vw-3v-3w+4x}{6(u-v)(u-w)}.$$

§ 2. Second and higher order partial derivatives

We have already seen in Ch. 1 § 7.1 that if $z=f(x, y)$ then the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ will themselves be functions of both x and y in general, and may be differentiated again partially with respect to x and y. The notation used was

$$\frac{\partial z}{\partial x} = f_x; \qquad \frac{\partial z}{\partial y} = f_y;$$

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}; \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy};$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}; \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx}.$$

The notation may be extended to higher order partial derivatives, such as

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = f_{xxx}; \qquad \frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = f_{xyy},$$

and so on.

It was noted in Examples 44 and 45 of Ch. 1 that $f_{xy}=f_{yx}$ and we now propose to prove that this result and obvious extensions of it hold under given conditions on the function $f(x, y)$ and its partial derivatives. The theorem we shall prove is:

If f_x, f_y, f_{xy}, f_{yx} are all continuous functions of x, y in the neighbourhood of any point (a, b) then

$$f_{xy} = f_{yx},$$

at that point.

We will note here that the above conditions are not the only known sufficient conditions under which the above result is true, but they are the simplest conditions under which it can be proved.

To prove the theorem, consider first the function of x

$$G(x) = f(x, b+k) - f(x, b).$$

Apply the mean value theorem (Ch. 6 § 1.3) to $G(x)$ in the interval $a<x<a+h$, to give

$$G(a+h) = G(a) + hG'(a+\theta_1 h), \qquad 0<\theta_1<1,$$

or

$$G(a+h) - G(a) = h\{f_x(a+\theta_1 h, b+k) - f_x(a+\theta_1 h, b)\},$$

where $f_x(a+\theta_1 h, b+k)$ is the value of f_x with $x=a+\theta_1 h$, $y=b+k$, and similarly for $f_x(a+\theta_1 h, b)$. But since f_x is a continuous function of x, y in the neighbourhood of (a, b) and f_{yx} exists, then we can apply the same mean value theorem to the right hand side of this equation in f_x, regarded as a function of y in the interval $b<y<b+k$. That is

$$f_x(a+\theta_1 h, b+k) - f_x(a+\theta_1 h, b) = kf_{yx}(a+\theta_1 h, b+\theta_2 k); \qquad 0<\theta_2<1.$$

But by hypothesis, f_{yx} is continuous at $x=a$, $y=b$ and therefore

$$f_{yx}(a+\theta_1 h, b+\theta_2 k) = f_{yx}(a, b) + \varepsilon_1,$$

where $\varepsilon_1 \to 0$ as $h, k \to 0$. Thus altogether

$$G(a+h) - G(a) = hk\{f_{yx}(a, b) + \varepsilon_1\}. \tag{8.40}$$

Similarly if we write

$$H(y) = f(a + h, y) - f(a, y),$$

then

$$H(b + k) - H(b) = hk\{f_{xy}(a, b) + \varepsilon_2\}, \tag{8.41}$$

where $\varepsilon_2 \to 0$ as $h, k \to 0$. But

$$\begin{aligned} G(a + h) - G(a) &= f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b) \\ &= f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b) \\ &= H(b + k) - H(b). \end{aligned}$$

Hence in the limit as $h, k \to 0$ this means that

$$\lim_{h,k\to 0} \frac{G(a + h) - G(a)}{hk} = \lim_{h,k\to 0} \frac{H(b + k) - H(b)}{hk},$$

or, from eqs. (8.40) and (8.41), that

$$f_{yx}(a, b) = f_{xy}(a, b).$$

Thus if $z = f(x, y)$ then the above result can be written as

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y},$$

holding at all points where f, f_x, f_y are continuous functions of x, y.

By repeated application, the theorem may be extended to show that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial}{\partial x}\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right), \end{aligned}$$

and so on for any number of partial derivatives provided $f(x, y)$ and all its partial derivatives are continuous functions. This latter condition is more than is absolutely *necessary* for the results to hold.

Example 8

If $z = (xy - y^2)/(x^2 + xy)$, find $\partial z/\partial x$ and show that it can be written in the form

$$\frac{\partial z}{\partial x} = \frac{y}{x^2} - \frac{2y}{(x + y)^2}.$$

Find also $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x \partial y$ and show that

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} = 0.$$

Differentiating

$$z = \frac{xy - y^2}{x^2 + xy}$$

partially with respect to x, we get

$$\frac{\partial z}{\partial x} = \frac{(x^2 + xy)(y) - (xy - y^2)(2x + y)}{(x^2 + xy)^2},$$

or simplifying,

$$\frac{\partial z}{\partial x} = \frac{y}{x^2} - \frac{2y}{(x + y)^2}.$$

Thus differentiating this again partially with respect to x, we have

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2y}{x^3} + \frac{4y}{(x + y)^3},$$

and differentiating it partially with respect to y, we have

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x^2} - \frac{2}{(x + y)^2} + \frac{4y}{(x + y)^3}.$$

Thus

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} = -\frac{2y}{x^2} + \frac{4xy}{(x + y)^3} + \frac{y}{x^2} - \frac{2y}{(x + y)^2} + \frac{4y^2}{(x + y)^3} + \frac{y}{x^2} - \frac{2y}{(x + y)^2} = 0.$$

In the above example the relation defining the dependent variable z in terms of the independent variables x, y was an explicit relation and the method for finding second and higher order partial derivatives was quite straightforward. When the relationship between the variables is implicit the methods are not quite so straightforward but they can be adapted from those given in § 1.6 of this chapter for ordinary derivatives. We give here a few examples to illustrate these methods.

Example 9

If $xy + xz + yz = 1$, find $\partial z/\partial x$ and hence $\partial^n z/\partial x^n$.

Here we regard z as a function of x and y defined by

$$xy + xz + yz = 1. \tag{8.42}$$

Differentiating this equation through partially with respect to x, regarding y as

constant, we get

$$y + z + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{y + z}{x + y}. \tag{8.43}$$

To find $\partial^2 z/\partial x^2$, we differentiate the right hand side partially with respect to x, treating y as constant and z as a function of x and y determined by (8.42). Thus

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\{(x + y)\,\partial z/\partial x - (y + z)\cdot 1\}}{(x + y)^2},$$

and by substitution of $\partial z/\partial x$ from eq. (8.43) we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(y + z)}{(x + y)^2}. \tag{8.44}$$

Again differentiating partially with respect to x,

$$\frac{\partial^3 z}{\partial x^3} = 2\,\frac{\{(x + y)^2\,\partial z/\partial x - 2(y + z)(x + y)\cdot 1\}}{(x + y)^4}$$

$$= -\frac{2\cdot 3(y + z)}{(x + y)^3}.$$

These results suggest that

$$\frac{\partial^n z}{\partial x^n} = \frac{(-1)^n\, n!(y + z)}{(x + y)^n},$$

and we shall establish this formula inductively.

If it is true for a particular value of n, then by a further differentiation we have

$$\frac{\partial^{n+1} z}{\partial x^{n+1}} = (-1)^n\, n!\,\frac{\{(x + y)^n\,\partial z/\partial x - (y + z)n(x + y)^{n-1}\}}{(x + y)^{2n}}$$

$$= \frac{(-1)^{n+1}(n + 1)!(y + z)}{(x + y)^{n+1}},$$

and this result holds also for $n+1$. The inductive proof is therefore established.

Example 10

If z is given as a function of x and y by the relation

$$z^2 = x^2(y^2 + z), \tag{8.45}$$

show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z,$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 2z.$$

Differentiating eq. (8.45) partially with respect to x, we get

$$2z\frac{\partial z}{\partial x} = 2x(y^2 + z) + x^2\frac{\partial z}{\partial x},$$

giving

$$\frac{\partial z}{\partial x} = \frac{2x(y^2 + z)}{2z - x^2}.$$

Similarly differentiating eq. (8.45) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = \frac{2x^2y}{2z - x^2}.$$

Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2(y^2 + z)}{2z - x^2} + \frac{2x^2y^2}{2z - x^2},$$

or using eq. (8.45),

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2z^2}{2z - x^2} + \frac{2(z^2 - x^2z)}{2z - x^2} = \frac{4z^2 - 2x^2z}{2z - x^2} = 2z.$$

To derive the second result in this case we take this result

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z, \tag{8.46}$$

and differentiate it partially with respect to x and y respectively, to give, after collection of like terms, the two relations

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial z}{\partial x}, \tag{8.47}$$

and

$$x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}. \tag{8.48}$$

Multiplying eq. (8.47) by x and eq. (8.48) by y and adding gives, using eq. (8.46)

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

EXERCISE 8.3

1. If $z = e^{-cy}\cos(cx + \alpha)$, prove that

$$\frac{\partial^2 z}{\partial x^2} = c\frac{\partial z}{\partial y}.$$

2. If $u = e^{ax}\sin by$, show that

$$b^2\frac{\partial^2 u}{\partial x^2} + a^2\frac{\partial^2 u}{\partial y^2} = 0.$$

3. If $z = x^n \mathrm{e}^{-y/x}$ find $\partial z/\partial x$ and $\partial z/\partial y$; also find the value of n for which the equation

$$y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x}$$

is satisfied.

4. If $z = \mathrm{e}^{x^2+xy+y^2}\cos(x - y)$, show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3(x + y)z,$$

and that

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 9(x + y)^2 z + 6z.$$

5. If $u = \log(ax + by + cz)$, find

$$\frac{\partial^{r+s+t} u}{\partial x^r \partial y^s \partial z^t}.$$

6. If $w = (x^3 + y^3 + z^3)^\alpha$, show that

$$w_{xx} = 6\alpha x(x^3 + y^3 + z^3)^{\alpha-1} + 9\alpha(\alpha - 1)x^4(x^3 + y^3 + z^3)^{\alpha-2}.$$

Determine α so that

$$\frac{1}{x}w_{xx} + \frac{1}{y}w_{yy} + \frac{1}{z}w_{zz} = 0.$$

7. Two functions $u(x, y)$ and $v(x, y)$ satisfy the equations

$$u_x = v_y, \qquad u_y = -v_x.$$

Deduce that $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$, assuming that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

8. Verify that $u = x^{-\frac{1}{2}}\exp(-y^2/4kx)$ satisfies the equation $u_x = ku_{yy}$.

9. If $u = x^2 + y^2 + z^2 + s^2 + t^2$ and $U = u^\alpha$, show that

$$U_{xx} = 4\alpha(\alpha - 1)u^{\alpha-2}x^2 + 2\alpha u^{\alpha-1}.$$

Hence prove that

$$U_{xx} + U_{yy} + U_{zz} + U_{ss} + U_{tt} \equiv 0,$$

if, and only if, $\alpha = 0$ or $\alpha = -\frac{3}{2}$.

10. If $z = f(y - ax) + \varphi(ay + x)$ where a is constant, show that

$$a\frac{\partial^2 z}{\partial x^2} + (a^2 - 1)\frac{\partial^2 z}{\partial x \partial y} - a\frac{\partial^2 z}{\partial y^2} = 0.$$

11. Prove that

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x \partial z}\right) \frac{1}{\sqrt{(4xz - y^2)}} = 0.$$

12. If
$$x \cos u + y \sin u = 1,$$
$$x \sin u - y \cos u = v,$$

define v and u as functions of x and y, show that

$$v \frac{\partial v}{\partial x} \cos u + v \frac{\partial v}{\partial y} \sin u = 1,$$

and that

$$v^2 \frac{\partial^2 v}{\partial x \partial y} + v(1 - v^2) \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = \cos 2u.$$

§ 2.1. Second derivatives. Change of variable

Equations containing partial derivatives arise very frequently in many branches of mathematical physics and most of these equations involve partial derivatives of the second order. Three simple examples are:

(i) the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{8.49}$$

(ii) the heat transfer equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \tag{8.50}$$

(iii) Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \tag{8.51}$$

These equations all arise in physical problems in which we wish to find solutions of them satisfying certain specific conditions, either initially when the time $t=0$ or for certain values of the space variables x, y and z (boundary conditions). Frequently such boundary conditions are not readily expressed in terms of the rectangular coordinates x, y, z but may be more easily expressed in terms of other variables. For example, to take a simple two-dimensional case, consider Laplace's equation in two dimensions

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{8.52}$$

Suppose we require a solution of this equation which takes a specific value on the circle $x^2+y^2=a^2$. This circle is most easily expressed in terms of polar coordinates r, θ by the equation $r=a$, and if ψ has to take specific values on $r=a$ it is better to know solutions of eq. (8.52) as functions of r, θ rather than x, y. In other words we would prefer the left hand side of eq. (8.52) to be expressed in terms of r, θ where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

This is precisely the sort of question we will consider in this paragraph; it is really an extension of the chain rule of § 1.4 to second derivatives. We consider a given relation $z=f(x, y)$, and by means of a change of variable

$$x = \varphi(u, v), \qquad y = \chi(u, v),$$

we suppose that z is expressible as a function of the variables u, v:

$$z = f(x, y) = f\{\varphi(u, v), \chi(u, v)\} = g(u, v).$$

Our object is to find the second partial derivatives of z with respect to x and y, namely

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= f_{xx} = \frac{\partial^2 f}{\partial x^2}, \\ \frac{\partial^2 z}{\partial x \partial y} &= f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \\ \frac{\partial^2 z}{\partial y^2} &= f_{yy} = \frac{\partial^2 f}{\partial y^2}, \end{aligned} \tag{8.53}$$

in terms of the second partial derivatives of z with respect to u and v, namely

$$\frac{\partial^2 z}{\partial u^2} = g_{uu} = \frac{\partial^2 g}{\partial u^2}, \quad \frac{\partial^2 z}{\partial u \partial v} = g_{uv} = \frac{\partial^2 g}{\partial u \partial v}, \quad \frac{\partial^2 z}{\partial v^2} = g_{vv} = \frac{\partial^2 g}{\partial v^2}. \tag{8.54}$$

In most cases the actual function $f(x, y)$ is not given, so the transformation of variables cannot be carried out directly, but as in the example of Laplace's eq. (8.52), it is a case of transforming a number of partial derivatives from one set of variables to another.

Notation again plays an important role in connection with these transformations and the use of the chain rule eqs. (8.18), (8.19). There is a wide variety of usage in mathematical and physical writings and

students must be prepared for such variety. In the previous paragraphs we have used the rule in the forms

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \tag{8.55}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}, \tag{8.56}$$

where we have used the dependent variable z in our partial derivatives rather than the functional symbols f and g. In fact we noted in § 1.4, that in these formulae the partial derivatives stand for the following derivatives:

$$\frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}, \qquad \frac{\partial z}{\partial y} = f_y = \frac{\partial f}{\partial y}, \tag{8.57}$$

whilst

$$\frac{\partial z}{\partial u} = g_u = \frac{\partial g}{\partial u}, \qquad \frac{\partial z}{\partial v} = g_v = \frac{\partial g}{\partial v}. \tag{8.58}$$

That is, whenever z is differentiated partially with respect to x or y the function $f(x, y)$ is involved, whilst whenever z is differentiated partially with respect to u or v the function $g(u, v)$ is involved. Further this means that the partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ are themselves given as functions of x and y by eqs. (8.57), and the partial derivatives $\partial z/\partial u$, $\partial z/\partial v$ are given as functions of u and v by the eqs. (8.58). Again if these meanings are kept in mind no difficulty should be experienced.

Let us consider the special problem of transforming the expression

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2},$$

into an expression in terms of r, θ where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We suppose that

$$z = f(x, y) = f(r\cos\theta, r\sin\theta) \equiv g(r, \theta),$$

so that eqs. (8.57), (8.58) hold, and also eqs. (8.53), (8.54). We use

equations of the type (8.55), (8.56) namely

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\,\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\,\frac{\partial \theta}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\,\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\,\frac{\partial \theta}{\partial y},$$

where

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x). \tag{8.59}$$

So evaluating $\partial r/\partial x$, $\partial r/\partial y$, $\partial \theta/\partial x$, $\partial \theta/\partial y$ in terms of r and θ, we get

$$\frac{\partial z}{\partial x} = \cos\theta\,\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial \theta}, \tag{8.60}$$

$$\frac{\partial z}{\partial y} = \sin\theta\,\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial z}{\partial \theta}. \tag{8.61}$$

Considering $\partial^2 z/\partial x^2$ first, we note that the left hand side of eq. (8.60) is a function of x, y, whilst the right hand side is a function of r, θ. Let us then use a variable ζ defined by eq. (8.60) in the form

$$\zeta = \frac{\partial z}{\partial x} = h(r, \theta),$$

where $h(r, \theta)$ stands for the right hand side of eq. (8.60). Then

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial \zeta}{\partial x},$$

and using eq. (8.60) again with z replaced by ζ, we have

$$\frac{\partial \zeta}{\partial x} = \cos\theta\,\frac{\partial \zeta}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial \zeta}{\partial \theta}.$$

But

$$\zeta = h(r, \theta) \equiv \cos\theta\,\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial \theta},$$

and thus

$$\frac{\partial \zeta}{\partial x} = \frac{\partial^2 z}{\partial x^2} = \cos\theta\,\frac{\partial}{\partial r}\left\{\cos\theta\,\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial \theta}\right\}$$
$$- \frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\left\{\cos\theta\,\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial \theta}\right\}. \tag{8.62}$$

An expression such as this, on the right hand side of eq. (8.62), is often written in the form

$$\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\,\frac{\partial}{\partial\theta}\right)\left(\cos\theta\,\frac{\partial z}{\partial r}-\frac{1}{r}\sin\theta\,\frac{\partial z}{\partial\theta}\right), \tag{8.63}$$

or even

$$\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\,\frac{\partial}{\partial\theta}\right)^2 z,$$

where this notation simply means the right hand side of eq. (8.62) or (8.63). We must emphasize also that since the coefficients $\cos\theta$ and $r^{-1}\sin\theta$ in the second bracket of (8.63) are functions of r, θ these must also be differentiated partially with respect to r and θ by the first bracket. Thus we get, using eq. (8.62)

$$\frac{\partial^2 z}{\partial x^2}=\cos\theta\left\{\cos\theta\,\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\sin\theta\,\frac{\partial z}{\partial\theta}-\frac{1}{r}\sin\theta\,\frac{\partial^2 z}{\partial r\,\partial\theta}\right\}$$

$$-\frac{1}{r}\sin\theta\left\{-\sin\theta\,\frac{\partial z}{\partial r}+\cos\theta\,\frac{\partial^2 z}{\partial\theta\,\partial r}-\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial\theta}-\frac{1}{r}\sin\theta\,\frac{\partial^2 z}{\partial\theta^2}\right\};$$

collecting like terms, and assuming that $\partial^2 z/\partial r\partial\theta=\partial^2 z/\partial\theta\partial r$, this becomes

$$\frac{\partial^2 z}{\partial x^2}=\cos^2\theta\,\frac{\partial^2 z}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 z}{\partial r\,\partial\theta}+\frac{\sin^2\theta}{r^2}\,\frac{\partial^2 z}{\partial\theta^2}$$

$$+\frac{\sin^2\theta}{r}\,\frac{\partial z}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial z}{\partial\theta}. \tag{8.64}$$

Similarly using eq. (8.61) we have

$$\frac{\partial^2 z}{\partial y^2}=\left(\sin\theta\,\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\,\frac{\partial}{\partial\theta}\right)^2 z, \tag{8.65}$$

which gives

$$\frac{\partial^2 z}{\partial y^2}=\sin^2\theta\,\frac{\partial^2 z}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 z}{\partial r\,\partial\theta}+\frac{\cos^2\theta}{r^2}\,\frac{\partial^2 z}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\,\frac{\partial z}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\,\frac{\partial z}{\partial\theta}.$$

Thus altogether,

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}, \tag{8.66}$$

which may be written in the form

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$

We note in the above results that each time we operate with $\partial/\partial x$, we replace it by the operation $\cos\theta\,\partial/\partial r-r^{-1}\sin\theta\,\partial/\partial\theta$, remembering that $\partial/\partial r$ and $\partial/\partial\theta$ in this operation act not only on the variable z given as a function of r, θ by the notation $z=g(r,\theta)$, but also on the coefficients in whatever follows. This is also true for $\partial/\partial y$ when it is replaced by $\sin\theta\,\partial/\partial r+r^{-1}\cos\theta\,\partial/\partial\theta$. These formulae for changing variables may be extended to higher order derivatives; for example,

$$\frac{\partial^3 z}{\partial x^3}=\left(\cos\theta\frac{\partial}{\partial r}-r^{-1}\sin\theta\frac{\partial}{\partial\theta}\right)^3 z,$$

which is equivalent to $\cos\theta\,\partial/\partial r-r^{-1}\sin\theta\,\partial/\partial\theta$ operating on the expression on the right hand side of eq. (8.64).

The result in eq. (8.66) may be deduced by the reverse process of changing variables from r, θ to x, y; we will give some details of this manipulation, since we can show how it is sometimes possible to simplify this type of analysis. Using the chain rule in its original form of eqs. (8.55) and (8.56) with u, v replaced by r, θ we have

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r},$$

which with $x=r\cos\theta$, $y=r\sin\theta$, gives

$$\frac{\partial z}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}, \tag{8.67}$$

Notice that in eq. (8.67) we have left the coefficients $\cos\theta$, $\sin\theta$ as functions of r, θ instead of expressing them in the clumsy forms

$$\cos\theta=\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\quad\text{and}\quad\sin\theta=\frac{y}{(x^2+y^2)^{\frac{1}{2}}} \tag{8.68}$$

in order to get the whole of the right hand side as a function of x, y. Thus here we have from eq. (8.67)

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left\{\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right\},$$

and remembering that for partial derivatives with respect to r, θ remains constant, we get

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta\,\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right). \tag{8.69}$$

To find $\dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial x}\right)$, $\dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial y}\right)$ we replace z by $\partial z/\partial x$, $\partial z/\partial y$ respectively in eq. (8.67), to give

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \cos\theta\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \sin\theta\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \cos\theta\,\frac{\partial^2 z}{\partial x^2} + \sin\theta\,\frac{\partial^2 z}{\partial y\partial x},$$

and

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \cos\theta\,\frac{\partial^2 z}{\partial x\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}.$$

Thus substituting in eq. (8.69)

$$\frac{\partial^2 z}{\partial r^2} = \cos^2\theta\,\frac{\partial^2 f}{\partial x^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 f}{\partial x\partial y} + \sin^2\theta\,\frac{\partial^2 f}{\partial y^2}.$$

We note that this analysis is easier than it would have been using eqs. (8.68) and expressing the right hand side of eq. (8.67) as a function of x and y only.

Similarly we could find $\partial^2 z/\partial\theta^2$, and it is left as an exercise for the reader to show from these results that

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\,\frac{\partial z}{\partial r} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

In general when the transformations from x, y to u, v are given as

$$x = \varphi(u, v), \qquad y = \chi(u, v),$$

it may be impossible or difficult to express u and v as functions of x and y. Then the second method given for changing variables is appropriated as in Example 11 following.

These methods may also be extended to deal with changes of variable in functions and equations in more than two variables as in Example 13 following.

Example 11

If $f(x, y)$ is transformed to $g(u, v)$ by the substitutions

$$3x = u^3 - 3uv^2, \qquad 3y = 3u^2v - v^3,$$

prove that

$$(u^2 + v^2)\left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) = 9(x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

Suppose that

$$z = f(x, y) = f\{\tfrac{1}{3}(u^3 - 3uv^2), \tfrac{1}{3}(3u^2v - v^3)\} \equiv g(u, v).$$

Then eqs. (8.55) and (8.56) give, in the notation of this question

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \tag{8.70}$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{8.71}$$

We therefore require the following results

$$\frac{\partial x}{\partial u} = u^2 - v^2, \qquad \frac{\partial x}{\partial v} = -2uv,$$

$$\frac{\partial y}{\partial u} = 2uv, \qquad \frac{\partial y}{\partial v} = u^2 - v^2,$$

and substitution in eqs. (8.70) and (8.71) gives

$$\frac{\partial g}{\partial u} = (u^2 - v^2)\frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y},$$

$$\frac{\partial g}{\partial v} = -2uv\frac{\partial f}{\partial x} + (u^2 - v^2)\frac{\partial f}{\partial y}.$$

Thus

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) = 2u\frac{\partial f}{\partial x} + (u^2 - v^2)\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)$$
$$+ 2v\frac{\partial f}{\partial y} + 2uv\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right), \tag{8.72}$$

whilst

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right) = -2u\frac{\partial f}{\partial x} - 2uv\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)$$
$$-2v\frac{\partial f}{\partial y} + (u^2 - v^2)\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right). \tag{8.73}$$

The values of $\dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial x}\right)$, $\dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial y}\right)$ are found by replacing z by $\partial f/\partial x$, $\partial f/\partial y$ respectively in eq. (8.55); and similarly $\dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial x}\right)$, $\dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial y}\right)$ are found by replacing z by $\partial f/\partial x$, $\partial f/\partial y$ respectively in eq. (8.56). These give

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = (u^2 - v^2)\frac{\partial^2 f}{\partial x^2} + 2uv\frac{\partial^2 f}{\partial x \partial y},$$

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = (u^2 - v^2)\frac{\partial^2 f}{\partial x \partial y} + 2uv\frac{\partial^2 f}{\partial y^2},$$

$$\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) = -2uv\frac{\partial^2 f}{\partial x^2} + (u^2 - v^2)\frac{\partial^2 f}{\partial x \partial y},$$

$$\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right) = -2uv\frac{\partial^2 f}{\partial x \partial y} + (u^2 - v^2)\frac{\partial^2 f}{\partial y^2}.$$

Substitution of these results in eqs. (8.72) and (8.73) and adding, yields the result

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = (u^2 + v^2)^2\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right\}.$$

Thus

$$(u^2 + v^2)\left(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) = (u^2 + v^2)^3\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right\},$$

and it is easy to see that $9(x^2+y^2)=(u^2+v^2)^3$. This example also illustrates the method of using $\partial g/\partial u$, $\partial f/\partial x, \ldots$ for the partial derivatives instead of $\partial z/\partial u$, $\partial z/\partial x, \ldots$.

Example 12

If $x=\cosh(u+v)$, $y=\sinh(u-v)$ prove that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial^2 u}{\partial x \partial y} = 0,$$

and hence that, if V is given as a function of x and y

$$4\{(x^2 - 1)(y^2 + 1)\}^{\frac{1}{2}}\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial u^2} - \frac{\partial^2 V}{\partial v^2}.$$

Here, using the methods of § 1.5, we have

$$\mathrm{d}x = (\mathrm{d}u + \mathrm{d}v)\sinh(u + v),$$
$$\mathrm{d}y = (\mathrm{d}u - \mathrm{d}v)\cosh(u - v),$$

and hence

$$2\mathrm{d}u = \frac{\mathrm{d}x}{\sinh(u + v)} + \frac{\mathrm{d}y}{\cosh(u - v)},$$

$$2\mathrm{d}v = \frac{\mathrm{d}x}{\sinh(u + v)} - \frac{\mathrm{d}y}{\cosh(u - v)}.$$

Thus

$$\frac{\partial u}{\partial x} = \frac{1}{2\sinh(u+v)} = \frac{\partial v}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\cosh(u-v)} = -\frac{\partial v}{\partial y},$$

and hence

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x \partial y},$$

so

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Also, using these results, we write

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}\left\{\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right\},$$

and then

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\left\{\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right\}\right]$$

$$= \frac{\partial^2 u}{\partial x \partial y}\left\{\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right\} + \frac{\partial u}{\partial x}\frac{\partial}{\partial y}\left\{\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right\} = \frac{\partial u}{\partial x}\frac{\partial}{\partial y}\left\{\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right\}.$$

Also

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial V}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial V}{\partial u}\right)\frac{\partial v}{\partial y} = \left\{\frac{\partial^2 V}{\partial u^2} - \frac{\partial^2 V}{\partial u \partial v}\right\}\frac{\partial u}{\partial y},$$

and similarly

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial v}\right) = \left(\frac{\partial^2 V}{\partial u \partial v} - \frac{\partial^2 V}{\partial v^2}\right)\frac{\partial u}{\partial y}.$$

Therefore

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\left\{\frac{\partial^2 V}{\partial u^2} - \frac{\partial^2 V}{\partial v^2}\right\};$$

since

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{1}{4\sinh(u+v)\cosh(u-v)} = \frac{1}{4\{(x^2-1)(y^2+1)\}^{\frac{1}{2}}},$$

the result follows at once.

Example 13

The function $f(x, y, z)$ is transformed to the function $g(u, v, w)$ by means of the relations $x^2 = vw$, $y^2 = uw$, $z = u + v$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial g}{\partial u} + v\frac{\partial g}{\partial v} + w\frac{\partial g}{\partial w}.$$

Let

$$V = f(x, y, z) = f\{(vw)^{\frac{1}{2}}, (uw)^{\frac{1}{2}}, u + v\} \equiv g(u, v, w).$$

Then we will use the chain rule in the form

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u}, \tag{8.74}$$

since we are given x, y, z as functions of u, v, w. Thus we use

$$\frac{\partial x}{\partial u} = 0, \qquad \frac{\partial x}{\partial v} = \frac{w}{2x}, \qquad \frac{\partial x}{\partial w} = \frac{v}{2x},$$

$$\frac{\partial y}{\partial u} = \frac{w}{2y}, \qquad \frac{\partial y}{\partial v} = 0, \qquad \frac{\partial y}{\partial w} = \frac{u}{2y},$$

$$\frac{\partial z}{\partial u} = 1, \qquad \frac{\partial z}{\partial v} = 1, \qquad \frac{\partial z}{\partial w} = 0,$$

and eq. (8.74) in the notation of the question gives

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}0 + \frac{\partial f}{\partial y}\frac{w}{2y} + \frac{\partial f}{\partial z},$$

or

$$u\frac{\partial g}{\partial u} = \frac{y}{2}\frac{\partial f}{\partial y} + \frac{y^2}{w}\frac{\partial f}{\partial z}.$$

Similarly

$$v\frac{\partial g}{\partial v} = \frac{x}{2}\frac{\partial f}{\partial x} + \frac{x^2}{w}\frac{\partial f}{\partial z},$$

$$w\frac{\partial g}{\partial w} = \frac{x}{2}\frac{\partial f}{\partial x} + \frac{y}{2}\frac{\partial f}{\partial y}.$$

Thus, by addition,

$$u\frac{\partial g}{\partial u} + v\frac{\partial g}{\partial v} + w\frac{\partial g}{\partial w} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + \frac{x^2 + y^2}{w}\frac{\partial f}{\partial z};$$

using $x^2+y^2=(u+v)w=zw$, we get the required result.

Sometimes simpler methods may be devised, using the ideas of complex numbers. Such methods are illustrated in the following example.

Example 14

If V is given as a function of x and y, and $x=r\cos\theta$, $y=r\sin\theta$, show that

$$\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y} = e^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\frac{\partial V}{\partial \theta}\right).$$

Deduce that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

We write

$$V = f(x, y) = f(r\cos\theta, r\sin\theta) \equiv g(r, \theta)$$

and then using eqs. (8.60) and (8.61), with z replaced by V,

$$\frac{\partial V}{\partial x} = \cos\theta\,\frac{\partial V}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial V}{\partial\theta},$$

$$\frac{\partial V}{\partial y} = \sin\theta\,\frac{\partial V}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial V}{\partial\theta}.$$

Thus

$$\frac{\partial V}{\partial x} + i\,\frac{\partial V}{\partial y} = (\cos\theta + i\sin\theta)\,\frac{\partial V}{\partial r} + \frac{i}{r}(\cos\theta + i\sin\theta)\,\frac{\partial V}{\partial\theta},$$

or using eq. (7.30),

$$\frac{\partial V}{\partial x} + i\,\frac{\partial V}{\partial y} = \mathrm{e}^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\,\frac{\partial V}{\partial\theta}\right). \tag{8.75}$$

Similarly

$$\frac{\partial V}{\partial x} - i\,\frac{\partial V}{\partial y} = \mathrm{e}^{-i\theta}\left(\frac{\partial V}{\partial r} - \frac{i}{r}\,\frac{\partial V}{\partial\theta}\right). \tag{8.76}$$

This eq. (8.76) may be written in the form

$$\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)V = \mathrm{e}^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\,\frac{\partial}{\partial\theta}\right)V.$$

In the left hand side of this equation replace V by the left hand side of eq. (8.75), whilst on the right hand side replace V by the right hand side of the same eq. (8.75). This gives

$$\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial V}{\partial x} + i\,\frac{\partial V}{\partial y}\right) = \mathrm{e}^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\,\frac{\partial}{\partial\theta}\right)\mathrm{e}^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\,\frac{\partial V}{\partial\theta}\right),$$

or

$$\frac{\partial^2 V}{\partial x^2} + i\,\frac{\partial^2 V}{\partial x\partial y} - i\,\frac{\partial^2 V}{\partial y\partial x} + \frac{\partial^2 V}{\partial y^2} = \mathrm{e}^{-i\theta}\left[\mathrm{e}^{i\theta}\left\{\frac{\partial^2 V}{\partial r^2} - \frac{i}{r^2}\,\frac{\partial V}{\partial\theta} + \frac{i}{r}\,\frac{\partial^2 V}{\partial r\partial\theta}\right\}\right.$$
$$\left. - \frac{i}{r}\left\{i\mathrm{e}^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\,\frac{\partial V}{\partial\theta}\right) + \mathrm{e}^{i\theta}\left(\frac{\partial^2 V}{\partial\theta\partial r} + \frac{i}{r}\,\frac{\partial^2 V}{\partial\theta^2}\right)\right\}\right],$$

and assuming that

$$\frac{\partial^2 V}{\partial x\partial y} = \frac{\partial^2 V}{\partial y\partial x} \quad\text{and}\quad \frac{\partial^2 V}{\partial r\partial\theta} = \frac{\partial^2 V}{\partial\theta\partial r},$$

this reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\,\frac{\partial V}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial\theta^2}.$$

EXERCISE 8.4

1. If $u=f(r)$ and $x=r\cos\theta$, $y=r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

2. If $x=\cosh u\cos v$, $y=\sinh u\sin v$, show that if V is given as a function of x and y

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\left(\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}\right)(\sinh^2 u\cos^2 v+\cosh^2 u\sin^2 v)^{-1}.$$

3. If φ is given as a function of two independent variables x, y which are related to two other independent variables u, v by $x=\mathrm{e}^u\cos v$, $y=\mathrm{e}^u\sin v$, show that

$$\frac{\partial^2\varphi}{\partial u\partial v}-\frac{\partial\varphi}{\partial v}=xy\left(\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial x^2}\right)+(x^2-y^2)\frac{\partial^2\varphi}{\partial x\partial y}.$$

4. V is given as a function of two independent variables x and y which are themselves functions of two other independent variables u, v such that $x=uv$, $2y=u^2-v^2$. Prove that

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\left(\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}\right)(u^2+v^2)^{-1}.$$

5. If z is given as a function of two independent variables x, y and $x=u^2-v^2$, $y=2uv$, show that

$$\frac{\partial^2 z}{\partial u\partial v}=2\left(y\frac{\partial^2 z}{\partial y^2}+2x\frac{\partial^2 z}{\partial x\partial y}-y\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}\right).$$

6. If z is given as a function of two independent variables x, y which are related to two other independent variables u, v by $x^2=uv$, $y^2=u/v$ prove that

$$4v^2\frac{\partial^2 z}{\partial v^2}+4v\frac{\partial z}{\partial v}-2u\frac{\partial z}{\partial u}=x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

7. Given $x=r\cosh\theta$, $y=r\sinh\theta$, prove that, if V is given as a function of x, y

$$\frac{\partial^2 V}{\partial x^2}-\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}.$$

8. If $z=f(x, y)$ where $2x=\mathrm{e}^u+\mathrm{e}^v$, $2y=\mathrm{e}^u-\mathrm{e}^v$, show that

$$\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

9. If $x=u^2v^{-1}$, $y=v^2u^{-1}$, and z is given as a function of x and y with continuous first and second partial derivatives, express $\partial^2 z/\partial u\partial v$ in terms of x, y and the partial derivatives of z with respect to x and y. Verify the result when $z=x^2y^2$.

10. If $x=u+v$, $y=uv$, prove the operator formulae

$$\frac{\partial}{\partial u}=\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial v}=\frac{\partial}{\partial x}+u\frac{\partial}{\partial y},$$

$$\frac{\partial^2}{\partial u^2}=\frac{\partial^2}{\partial x^2}+2v\frac{\partial^2}{\partial x\partial y}+v^2\frac{\partial^2}{\partial y^2},$$

and analogous formulae for $\partial^2/\partial v^2$, $\partial^2/\partial u\partial v$.

Prove that this change of variable transforms

$$(u^2-uv)\frac{\partial^2 w}{\partial u^2}+(uv-v^2)\frac{\partial^2 w}{\partial v^2}+(v^2-u^2)\frac{\partial^2 w}{\partial u\partial v}-(u+v)\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)=0,$$

into

$$\frac{\partial^2 w}{\partial x\partial y}=0.$$

11. A twice differentiable function $z(x, y)$ is transformed into a function $w(u, v)$ by means of a substitution $x=\varphi(u, v)$, $y=\psi(u, v)$ which has the property that

$$\frac{\partial x}{\partial u}=\frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u}.$$

Show that

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial v}\right)^2\right].$$

12. Show that the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+u=0,$$

is transformed into the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}+f=0,$$

by the substitution $u=f(r)$, where f is an arbitrary twice differentiable function of

$$r=(x^2+y^2+z^2)^{\frac{1}{2}}.$$

13. Show that the equation

$$xy\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x^2}\right)+(x^2-y^2)\frac{\partial^2 z}{\partial x\partial y}-y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0,$$

is reduced to

$$\frac{\partial^2 z}{\partial u \partial v} = 0,$$

by the substitution $x = e^u \cos v$, $y = e^u \sin v$.

14. If $w(u, v)$ is any twice differentiable function of the variables u and v, where

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y$$

and if $w(u, v) \equiv z(x, y)$ prove the following identity

$$\frac{\partial^2 z}{\partial x \partial y} \equiv u \frac{\partial w}{\partial v} - v \frac{\partial w}{\partial u} + uv\left(\frac{\partial^2 w}{\partial v^2} - \frac{\partial^2 w}{\partial u^2}\right) + (u^2 - v^2 - 1)\frac{\partial^2 w}{\partial u \partial v}.$$

15. A function F of two independent variables r and θ is transformed into a function G of variables u and s by means of the relations $r \cos \theta = 1/u$; $\tan \theta = s$. Prove that

(i) $r \dfrac{\partial F}{\partial r} = -u \dfrac{\partial G}{\partial u}$,

(ii) $\dfrac{\partial F}{\partial \theta} = us \dfrac{\partial G}{\partial u} + (1 + s^2)\dfrac{\partial G}{\partial s}$.

If F satisfies the equation

$$\cos \theta \frac{\partial^2 F}{\partial r \partial \theta} + r \sin \theta \frac{\partial^2 F}{\partial r^2} = 0,$$

prove that

$$\frac{\partial G}{\partial u} = (1 + s^2)^{\frac{1}{2}} \varphi(u),$$

where $\varphi(u)$ is some function of u.

16. Show that the partial differential equation

$$\frac{\partial^2 z}{\partial y^2} + 2(x - x^3)\frac{\partial z}{\partial x} + 2x^2 y \frac{\partial z}{\partial y} + x^2 y^2 z = 0,$$

remains unchanged in form when the independent variables are changed to u, v where $u = 1/x$, $v = xy$.

17. If z is given as a function of the independent variables x, y which are each expressible in terms of another pair of independent variables u, v, obtain the relations

$$\frac{\partial z}{\partial u} = p \frac{\partial x}{\partial u} + q \frac{\partial y}{\partial u},$$

$$\frac{\partial^2 z}{\partial u^2} = r\left(\frac{\partial x}{\partial u}\right)^2 + 2s \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + t\left(\frac{\partial y}{\partial u}\right)^2 + p \frac{\partial^2 x}{\partial u^2} + q \frac{\partial^2 y}{\partial u^2},$$

where

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}.$$

18. The function $f(x, y)$ of two independent variables satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Prove that the function $\varphi(x, y) \equiv f(u, v)$ where $u = x/(x^2+y^2)$, $v = y/(x^2+y^2)$ also satisfies Laplace's equation.

19. $F(x, y, z)$ is a twice differentiable function of x, y, z and

$$x = r \sin \theta \cos \varphi, \qquad y = r \sin \theta \sin \varphi, \qquad z = r \cos \theta.$$

If

$$V = F(x, y, z) \equiv G(r, \theta, \varphi),$$

show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} + $$
$$+ \frac{2}{r} \frac{\partial V}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial V}{\partial \theta}.$$

[Hint: use successive plane polar transformations given by

(i) $x = u \cos \varphi, \quad y = u \sin \varphi,$

and

(ii) $z = r \cos \theta, \quad u = r \sin \theta.$]

§ 3. Homogeneous functions: Euler's theorem

The function $f(x, y)$ of the two variables x, y is said to be a *homogeneous function* of degree n if the replacement of x, y by tx, ty respectively leads to the identity

$$f(tx, ty) \equiv t^n f(x, y). \tag{8.77}$$

n need not necessarily be an integer, and may be positive, negative or zero.

Example 15

x^2+xy+y^2 is a homogeneous function of degree 2.

Replacing x by tx, y by ty, we get

$$(tx)^2 + (tx)(ty) + (ty)^2 \equiv t^2(x^2 + xy + y^2).$$

Example 16

$x \sin(x/y)/(x^2+y^2)$ is a homogeneous function of degree -1. We have

$$\frac{tx \sin(tx/ty)}{(tx)^2 + (ty)^2} = \frac{1}{t}\left\{\frac{x \sin(x/y)}{x^2 + y^2}\right\}.$$

Example 17

$xy^{\frac{1}{2}}+\{(x^2+y^2)x\}^{\frac{1}{2}}$ is a homogeneous function of degree $\frac{3}{2}$.

We have

$$tx(ty)^{\frac{1}{2}} + \{(t^2x^2 + t^2y^2)tx\}^{\frac{1}{2}} = t^{\frac{3}{2}}[xy^{\frac{1}{2}} + \{(x^2 + y^2)x\}^{\frac{1}{2}}].$$

The definition of a homogeneous function of two variables may obviously be extended to a function of any number of variables.

The function $f(x_1, x_2, \dots x_m)$ of a number of variables $x_1, x_2, \dots, x_m$ is said to be a homogeneous function of degree n in these variables if

$$f(tx_1, tx_2, \dots, tx_m) \equiv t^n f(x_1, x_2, \dots, x_m). \tag{8.78}$$

Example 18

$(x^3-3x^2y+6xyz)/(yz+zx)$ is a homogeneous function of degree 1.

We have

$$\frac{(tx)^3 - 3(tx)^2ty + 6(tx)(ty)(tz)}{(ty)(tz) + (tz)(tx)} = t\left\{\frac{x^3 - 3x^2y + 6xyz}{yz + zx}\right\}.$$

If $f(x_1, x_2, \dots x_m)$ is a homogeneous function of degree n which is partially differentiable with respect to each of the variables, then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_m \frac{\partial f}{\partial x_m} = nf(x_1, x_2, \dots x_m).$$

In the function $f(x_1, x_2, \dots x_m)$, put $x_1=tu_1$, $x_2=tu_2, \dots x_m=tu_m$. Then

$$f(x_1, x_2, \dots x_m) = f(tu_1, tu_2, \dots tu_m).$$

But since the function f is a homogeneous function of degree n, then by eq. (8.78)

$$f(tu_1, tu_2, \dots, tu_m) \equiv t^n f(u_1, u_2, \dots, u_m).$$

Differentiate this equation partially with respect to t on both sides:

$$u_1 \frac{\partial f}{\partial x_1} + u_2 \frac{\partial f}{\partial x_2} + \dots + u_m \frac{\partial f}{\partial x_m} = nt^{n-1} f(u_1, u_2, \dots u_m),$$

Multiplying by t, we get

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_m \frac{\partial f}{\partial x_m} = nt^n f(u_1, u_2, \dots u_m),$$

or

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_m \frac{\partial f}{\partial x_m} = nf(x_1, x_2, \dots x_m). \tag{8.79}$$

This result is known as *Euler's theorem* for homogeneous functions.

Example 19

If

$$f(x, y, z) = \sin\left(\frac{xy + yz}{x^2 + y^2 + z^2}\right),$$

show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 0.$$

The function

$$\sin\left(\frac{xy + yz}{x^2 + y^2 + z^2}\right),$$

is homogeneous of degree zero, so that applying Euler's theorem with $n=0$ the result follows.

Example 20

Given that $\theta = \tan^{-1} u$, when u is a homogeneous function of degree 2 in x, y, z, prove that

$$x \frac{\partial \theta}{\partial x} + y \frac{\partial \theta}{\partial y} + z \frac{\partial \theta}{\partial z} = \sin 2\theta.$$

We have

$$\frac{\partial \theta}{\partial x} = \frac{d\theta}{du} \frac{\partial u}{\partial x} = \frac{1}{1 + u^2} \frac{\partial u}{\partial x},$$

and similarly for $\partial\theta/\partial y$, $\partial\theta/\partial z$. Thus

$$x \frac{\partial \theta}{\partial x} + y \frac{\partial \theta}{\partial y} + z \frac{\partial \theta}{\partial z} = \frac{1}{1 + u^2} \left\{ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right\} = \frac{2u}{1 + u^2},$$

using eq. (8.79). Since $u = \tan \theta$,

$$2u/(1 + u^2) = \sin 2\theta.$$

giving the result.

Example 21

If $f(x, y, z)$ is a homogeneous function of degree n in x, y, z and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0, \tag{8.80}$$

show that $F(x, y, z) = f(x, y, z)/r^{2n+1}$, where $r^2 = x^2 + y^2 + z^2$, also satisfies this equation.

We have

$$\frac{\partial F}{\partial x} = \frac{1}{r^{2n+1}} \frac{\partial f}{\partial x} - \frac{(2n+1)x}{r^{2n+3}} f,$$

and

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{r^{2n+1}} \frac{\partial^2 f}{\partial x^2} - \frac{2(2n+1)x}{r^{2n+3}} \frac{\partial f}{\partial x} - (2n+1)\left\{\frac{1}{r^{2n+3}} - \frac{(2n+3)x^2}{r^{2n+5}}\right\} f.$$

Similar equations hold for $\partial^2 F/\partial y^2$ and $\partial^2 F/\partial z^2$, with x replaced by y, z respectively. Thus

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} &= \frac{1}{r^{2n+1}}\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right\} \\ &\quad - \frac{2(2n+1)}{r^{2n+3}}\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right\} \\ &\quad - (2n+1)\left\{\frac{3}{r^{2n+3}} - \frac{(2n+3)(x^2+y^2+z^2)}{r^{2n+5}}\right\} f.\end{aligned}$$

But by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf,$$

and using eq. (8.80) with $x^2+y^2+z^2=r^2$, we get

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \frac{-2(2n+1)n}{r^{2n+3}} f - \frac{(2n+1)}{r^{2n+3}}\{3-(2n+3)\}f = 0.$$

EXERCISE 8.5

1. If $u=x^nF(x/y)$ where F denotes an arbitrary function, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$$

and hence that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

2. If $f(x, y)$ is a homogeneous function of degree n, prove that

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f.$$

Verify this result when $f(x, y) = (x^4+y^4)^{\frac{1}{2}}$.

3. Verify Euler's theorem for the function

$$f(x, y) = \frac{x-2y}{x^2+y^2}.$$

4. Verify Euler's theorem for the function

$$f(x, y, z) = \frac{x}{y} \exp\left(\frac{x}{z}\right).$$

5. If $f(x, y) = \tan^{-1}\{(x^2+y^2)/(x+y)\}$, prove that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f = \tfrac{1}{4} \sin 4f.$$

6. The function $f(x, y)$ satisfies the relation

$$f(k^r x, k^s y) = k^n f(x, y);$$

prove that

$$rx\frac{\partial f}{\partial x} + sy\frac{\partial f}{\partial y} = nf,$$

where f stands for $f(x, y)$.

7. The function $V(x, y, z)$ satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Show that if $X = \lambda x/r^2$, $Y = \lambda y/r^2$, $Z = \lambda z/r^2$, λ is a constant and $r^2 = x^2+y^2+z^2$, then $V(X, Y, Z)/r$ also satisfies the same equation.

§ 4. Taylor's theorem for a function of two variables

Remembering Taylor's theorem for a function of one variable (Ch. 6 § 3), the object in the two variable case is to expand the function $f(a+h, b+k)$ as a series in powers of h and k. Let $f(x, y)$ be a function which is such that all its partial derivatives with respect to x and y, up to and including the nth, are continuous functions of x and y in the closed intervals $(a, a+h)$, $(b, b+k)$. Write

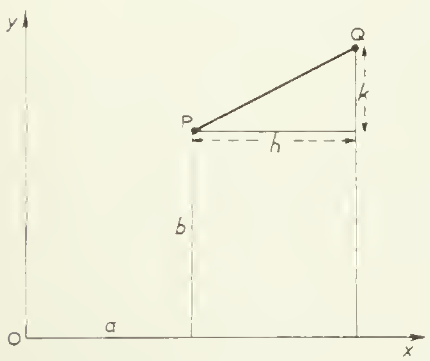

Fig. 8.3

$$x = a + ht, \qquad y = b + kt, \tag{8.81}$$

so that as t varies between the value 0 and the value 1, the point (x,y) moves along the line PQ in the xy-plane between the points P(a, b) and Q$(a+h, b+k)$, fig. 8.3. Now for all points on this line we have

$$f(x, y) = f(a + ht, b + kt) \equiv \varphi(t), \tag{8.82}$$

that is, the function $f(x, y)$ of two variables is expressed as a function of one variable t, provided $0 \leqslant t \leqslant 1$. Further we have, using an equation of the type (8.11)

$$\varphi'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y},$$

and the right hand side is a function of x and y. Thus by differentiation with respect to t again, we have

$$\begin{aligned}\varphi''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi'(t) &= \frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= h\frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + k\frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right),\end{aligned}$$

and this can be written in the form

$$\varphi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f. \tag{8.83}$$

Under the conditions given for the function $f(x, y)$ we note that written in full this becomes

$$h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2},$$

since h and k are constant.

The differentiation can be repeated any number of times to give the nth derivative of $\varphi(t)$ as

$$\varphi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f; \tag{8.84}$$

it may be proved by induction that this takes the form

$$h^n\frac{\partial^n f}{\partial x^n} + nh^{n-1}k\frac{\partial^n f}{\partial x^{n-1}\partial y} + \ldots \binom{n}{r} h^{n-r}k^r\frac{\partial^n f}{\partial x^{n-r}\partial y^r} + \ldots + k^n\frac{\partial^n f}{\partial y^n}, \tag{8.85}$$

provided all the partial derivatives in the expression are continuous, so that the order of differentiation does not matter.

We see therefore that $\varphi(t)$ and its first n derivatives are continuous functions of t in the interval (0, 1) and so by Maclaurin's theorem on a

function of one variable we have

$$\varphi(t) = \varphi(0) + t\varphi'(0) + \frac{t^2}{2!}\varphi''(0) + \dots$$
$$+ \frac{t^{n-1}}{(n-1)!}\varphi^{(n-1)}(0) + \frac{t^n}{n!}\varphi^{(n)}(\theta t), \qquad (8.86)$$

where $0<\theta<1$. But from eqs. (8.81) when $t=0$, $x=a$, $y=b$, whilst when $t=1$, $x=a+h$, $y=b+k$; so that putting $t=1$ in eq. (8.86) we write, using eqs. (8.82) and (8.84)

$$\varphi(1) = f(a+h, b+k), \qquad \varphi(0) = f(a, b)$$
$$\varphi'(0) = \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y)\right]_{\substack{x=a\\y=b}},$$
$$\varphi^{n-1}(0) = \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(x, y)\right]_{\substack{x=a\\y=b}},$$

and then eq. (8.86) becomes

$$f(a+h, b+k) = f(a, b) + \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y)\right]_{\substack{x=a\\y=b}}$$
$$+ \dots + \frac{1}{(n-1)!}\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(x, y)\right]_{\substack{x=a\\y=b}} + R_n \qquad (8.87)$$

where

$$R_n = \frac{1}{n!}\varphi^{(n)}(\theta) = \frac{1}{n!}\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(x, y)\right]_{\substack{x=a+\theta h\\y=b+\theta k}},$$

and $0<\theta<1$. If $R_n \to 0$ as $n\to\infty$ the result may be given as an infinite series.

If we put $a=b=0$, we get at once Maclaurin's form of the theorem

$$f(h, k) = f(0, 0) + \left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y)\right]_{\substack{x=0\\y=0}}$$
$$+ \dots + \frac{1}{n!}\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f(x, y)\right]_{\substack{x=0\\y=0}} + \dots.$$

If, in this form we replace h by x and k by y, we must be more careful in writing it out; in fact it must be written out in full because $\partial/\partial x$ and $\partial/\partial y$ in the brackets must not operate on the coefficients $h=x$ and $k=y$

because these coefficients were specifically assumed constant in deriving eq. (8.87). Thus the result becomes, using eq. (8.85)

$$f(x, y) = f(0, 0) + \left\{x\left(\frac{\partial f}{\partial x}\right)_0 + y\left(\frac{\partial f}{\partial y}\right)_0\right\} + \dots$$
$$+ \frac{1}{n!}\left\{x^n\left(\frac{\partial^n f}{\partial x^n}\right)_0 + \dots + \binom{n}{r} x^{n-r}y^r\left(\frac{\partial^n f}{\partial x^{n-r}\partial y^r}\right)_0 + \dots\right.$$
$$\left. + y^n\left(\frac{\partial^n f}{\partial y^n}\right)_0\right\} + \dots, \tag{8.88}$$

the zero suffices denoting the value of the function at $x=0$, $y=0$.

Example 22

Write out Taylor's theorem with $a=3$, $b=3$ for $f(x, y)=x^4+y^4-3x^2y+6$.

We have the following results: $f(3, 3) = 87$,

$$\begin{array}{llll}
\dfrac{\partial f}{\partial x} = 4x^3 - 6xy, & \left(\dfrac{\partial f}{\partial x}\right)_{\substack{x=3\\y=3}} = 54, \\
\dfrac{\partial f}{\partial y} = 4y^3 - 3x^2, & \left(\dfrac{\partial f}{\partial y}\right)_{\substack{x=3\\y=3}} = 81, \\
\dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 6y, & \left(\dfrac{\partial^2 f}{\partial x^2}\right)_{\substack{x=3\\y=3}} = 90, \\
\dfrac{\partial^2 f}{\partial x\partial y} = -6x, & \left(\dfrac{\partial^2 f}{\partial x\partial y}\right)_{\substack{x=3\\y=3}} = -18, \\
\dfrac{\partial^2 f}{\partial y^2} = 12y^2, & \left(\dfrac{\partial^2 f}{\partial y^2}\right)_{\substack{x=3\\y=3}} = 108, \\
\dfrac{\partial^3 f}{\partial x^3} = 24x, & \left(\dfrac{\partial^3 f}{\partial x^3}\right)_{\substack{x=3\\y=3}} = 72, \\
\dfrac{\partial^2 f}{\partial x^3\partial y} = -6, & \left(\dfrac{\partial^3 f}{\partial x^2\partial y}\right)_{\substack{x=3\\y=3}} = -6, \\
\dfrac{\partial^3 f}{\partial x\partial y^2} = 0, & \left(\dfrac{\partial^3 f}{\partial x\partial y^2}\right)_{\substack{x=3\\y=3}} = 0, \\
\dfrac{\partial^3 f}{\partial y^3} = 24y, & \left(\dfrac{\partial^3 f}{\partial y^3}\right)_{\substack{x=3\\y=3}} = 72.
\end{array}$$

and the only 4th partial derivatives which have a non-zero value are

$$\frac{\partial^4 f}{\partial x^4} = 24, \qquad \frac{\partial^4 f}{\partial y^4} = 24;$$

hence all higher partial derivatives vanish. Thus the series terminates and substituting in eq. (8.87) we get

$$f(3+h, 3+k) \equiv (3+h)^4 + (3+k)^4 - 3(3+h)^2(3+k) + 6 = 87 + (54h + 81k)$$
$$+ \frac{1}{2!}(90h^2 - 36hk + 108k^2) + \frac{1}{3!}(72h^3 - 18h^2k + 72k^3) + \frac{1}{4!}(24h^4 + 24k^4),$$

the reader can easily verify from the left hand side that this result is correct.

Example 23

Expand $f(x, y) = \cos x \cos y$ by Maclaurin's theorem as far as $n=2$.

We have the following results: $f(0, 0) = 1$,

$$\frac{\partial f}{\partial x} = -\sin x \cos y, \quad \left(\frac{\partial f}{\partial x}\right)_0 = 0,$$
$$\frac{\partial f}{\partial y} = -\cos x \sin y, \quad \left(\frac{\partial f}{\partial y}\right)_0 = 0,$$
$$\frac{\partial^2 f}{\partial x^2} = -\cos x \cos y, \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_0 = -1,$$
$$\frac{\partial^2 f}{\partial x \partial y} = -\sin x \sin y, \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 = 0,$$
$$\frac{\partial^2 f}{\partial y^2} = -\cos x \cos y, \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_0 = -1.$$

Thus from eq. (8.88)

$$\cos x \cos y = 1 - \frac{1}{2!}(x^2 + y^2) + \dots.$$

Note that this does agree with the series formed by multiplying together the separate series for $\cos x$ and $\cos y$.

EXERCISE 8.6

1. Write out Taylor's theorem with $a=1$, $b=2$ for the function

$$f(x, y) = x^2y + xy^2 + 1.$$

2. Write Taylor's series for $xy^3 - y^2 + y + 2$ in powers of $x-1$ and $y-2$. [Hint: put $x=1+h$, $y=2+k$ and use Taylor's series with $a=1$, $b=2$ in powers of h and k.]

3. Write Maclaurin's series for $\exp(-y^2+2xy)$ in powers of x and y, going far enough to include all terms of the fourth degree.

4. By using Maclaurin's series on a function of two variables show that

$$\log(1 + xy) = xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \dots.$$

5. Expand $\exp xy$ in a power series in x and y as far as all terms of the sixth degree.

6. Expand $\exp x \sin y$ in a power series in x and y as far as all terms of the third degree.

7. Find a linear function of x and y which is a good approximation for

$$f(x, y) = \sin^{-1}\left(\frac{x - y}{1 + xy}\right),$$

when x and y are small.

§ 5. Maximum and minimum values of a function of two variables

We define maximum and minimum values of a function $f(x, y)$ in a similar way as for a function of one variable.

A function $f(x, y)$ is said to have a maximum value at the point $x=a$, $y=b$ if

$$f(a + h, b + k) - f(a, b) < 0, \tag{8.89}$$

for all sufficiently small values of h and k, positive or negative.

A function $f(x, y)$ is said to have a minimum value at the point $x=a$, $y=b$ if

$$f(a + h, b + k) - f(a, b) > 0, \tag{8.90}$$

for all sufficiently small values of h and k, positive or negative.

Let us suppose that $f(x, y)$ is a function satisfying the conditions of Taylor's theorem given in § 4 for $n=3$. Then using eq. (8.87)

$$\begin{aligned} f(a + h, b + k) = f(a, b) &+ \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} \\ &+ \tfrac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}} + R_3. \end{aligned} \tag{8.91}$$

For convenience let us use the following notation:

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\y=b}} \equiv (f_x)_{\substack{x=a\\y=b}} \equiv f_a,$$

$$\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\y=b}} \equiv (f_y)_{\substack{x=a\\y=b}} \equiv f_b,$$

and similarly

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}} \equiv f_{aa}, \qquad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}} \equiv f_{ab}, \qquad \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}} \equiv f_{bb}.$$

Eq. (8.91) can now be written in the form

$$f(a+h,\, b+k)-f(a,\, b)=(hf_a+kf_b)+\tfrac{1}{2}(h^2f_{aa}+2hkf_{ab}+k^2f_{bb})+R_3, \tag{8.92}$$

where R_3 is a cubic in h, k.

For sufficiently small values of h, k, the sign of the right hand side of eq. (8.92) is the sign of the first term

$$hf_a + kf_b. \tag{8.93}$$

For a maximum (or minimum) value of $f(x, y)$ at (a, b), the sign of this term must be negative (or positive) for all independent values of h and k, positive or negative.

Putting $k=0$ and changing the sign of h would change the sign of the expression (8.93) unless $f_a=0$. Similarly putting $h=0$ shows that $f_b=0$.

Thus a necessary condition for a maximum or a minimum value of $f(x, y)$ at (a, b) is

$$f_a = 0, \qquad f_b = 0. \tag{8.94}$$

Points where these conditions (8.94) are satisfied are said to be *stationary points* and the function $f(x, y)$ is said to have a *stationary value.*

The points $(x, y)=(a, b)$ where $f(x, y)$ has a stationary value are therefore determined as the roots of the simultaneous equations

$$f_x = 0, \qquad f_y = 0,$$

and we note that these two equations can be expressed in terms of the differential $\mathrm{d}f$ of f in the simple form

$$\mathrm{d}f = f_x\, \mathrm{d}x + f_y\, \mathrm{d}y = 0,$$

which must be satisfied for all arbitrary values of the differentials $\mathrm{d}x$ and $\mathrm{d}y$. This form of the necessary condition for a stationary value may be extended to a function $f(x_1, x_2,\dots x_n)$ of the n independent variables $x_1, x_2,\dots x_n$ using the results of § 1.3. Thus

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\,\mathrm{d}x_2 + \dots + \frac{\partial f}{\partial x_n}\,\mathrm{d}x_n,$$

must be zero for all arbitrary values of $\mathrm{d}x_1, \mathrm{d}x_2,\dots \mathrm{d}x_n$, which means that

$$\frac{\partial f}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} = 0, \quad \dots \quad \frac{\partial f}{\partial x_n} = 0.$$

When the conditions (8.94) are satisfied for two variables we have

$$f(a+h, b+k) - f(a, b) = \tfrac{1}{2}\{h^2 f_{aa} + 2hk f_{ab} + k^2 f_{bb}\} + R_3, \tag{8.95}$$

and again, for h, k sufficiently small the sign of the right hand side depends on the sign of the expression

$$h^2 f_{aa} + 2hk f_{ab} + k^2 f_{bb}. \tag{8.96}$$

Let us now write

$$h = u \cos \varphi, \qquad k = u \sin \varphi,$$

where u is positive, and the signs of h, k will depend on the value of φ. The expression (8.96) can then be written as

$$u^2\{f_{aa} \cos^2 \varphi + 2f_{ab} \cos \varphi \sin \varphi + f_{bb} \sin^2 \varphi\}. \tag{8.97}$$

Suppose f_{aa}, f_{ab}, f_{bb} are not all zero, otherwise this term vanishes in the eq. (8.95) and we would need to consider the next term in the Taylor series. If $f_{aa}=f_{bb}=0$, $f_{ab}\neq 0$, then the sign of (8.97) is determined by the sign of $f_{ab} \cos \varphi \sin \varphi$; this changes sign if φ, and therefore k, changes sign. Then the value of the function cannot be a maximum or a minimum at (a, b). The function $f(x, y)$ is said to have a *saddle point* at (a, b).

Further if $f_{aa}\neq 0$, the expression (8.97) can be written as

$$\frac{u^2}{f_{aa}}\{(f_{aa} \cos \varphi + f_{ab} \sin \varphi)^2 + (f_{aa}f_{bb} - f_{ab}^2) \sin^2 \varphi\}, \tag{8.98}$$

or if $f_{bb}\neq 0$, it can be written as

$$\frac{u^2}{f_{bb}}\{(f_{ab} \cos \varphi + f_{bb} \sin \varphi)^2 + (f_{aa}f_{bb} - f_{ab}^2) \cos^2 \varphi\}. \tag{8.99}$$

There are now three cases to consider:

$$\text{(i)} \quad f_{aa}f_{bb} - f_{ab}^2 > 0. \tag{8.100}$$

When this condition is satisfied, we note that

$$f_{aa}f_{bb} > f_{ab}^2 > 0,$$

so that f_{aa}, f_{bb} are of the same sign. Also the expressions (8.98) and (8.99) are of invariable sign and depend on the sign of f_{aa} (or f_{bb}). The condition (8.89) is satisfied if f_{aa} is negative while the condition (8.90) is satisfied if f_{aa} is positive. Thus, provided

$$f_a = 0, \qquad f_b = 0, \qquad f_{aa}f_{bb} - f_{ab}^2 > 0,$$

$f(x, y)$ has a *maximum* value at (a, b) if f_{aa} is negative while it has a *minimum* value at (a, b) if f_{aa} is positive.

$$\text{(ii)} \quad f_{aa}f_{bb} - f_{ab}^2 < 0. \tag{8.101}$$

When this condition is satisfied the expression (8.98) is not of invariable sign. When $\varphi=0$ it is positive and when $\varphi=\tan^{-1}(-f_{aa}/f_{bb})$ it is negative. Thus we again have a stationary value of the function which is neither a maximum nor a minimum, that is, a *saddle point*. Note that $f_{aa}=f_{bb}=0$, $f_{ab}\neq 0$ is covered by the condition (8.101).

$$\text{(iii)} \quad f_{aa}f_{bb} - f_{ab}^2 = 0. \tag{8.102}$$

The expression (8.98) has the same sign as f_{aa} except possibly for the value $\varphi=\tan^{-1}(-f_{aa}/f_{bb})$. Further investigation is necessary to determine whether the function has a maximum or minimum value at (a, b). In this book no useful purpose is served by elaborating the conditions necessary for a maximum or minimum value in this case.

The above results can be interpreted geometrically; this will be shown in the following examples.

Example 24

To find the nature of the stationary points of the function $f(x, y) \equiv x^3-3x^2-4y^2$.

We have

$$f_x = 3x^2 - 6x, \qquad f_y = -8y.$$

The points $(x, y)=(a, b)$ where $f(x, y)$ has a stationary value are determined as the roots of the simultaneous equations

$$f_x = 3x^2 - 6x = 0, \qquad f_y = -8y = 0,$$

giving $x=0$, $y=0$ or $x=2$, $y=0$. Thus there are two points (0, 0) and (2, 0) where $f(x, y)$ is stationary. Also

$$f_{xx} = 6x - 6, \qquad f_{xy} = 0, \qquad f_{yy} = -8,$$

so that

$$f_{xx}f_{yy} - f_{xy}^2 = -8(6x - 6). \tag{8.103}$$

When $x=0$, the expression (8.103) is positive, and f_{xx}, f_{yy} are both negative. Thus the point (0, 0) is a maximum point.

When $x=2$, the expression (8.103) is negative, so that the point (2, 0) is a saddle point.

The geometrical interpretation of these results is as follows. If we write, as in § 1.2,

$$z = f(x, y),$$

then this equation represents a surface in 3-space. If we regard the plane xOy as a horizontal plane and z the height of any point $P(x, y, z)$ on the surface, the curves

$$f(x, y) = z$$

for particular constant values of z (c say) are appropriately called level curves or contours on the surface. Such curves may be drawn in a plane, as on a map, and the values of z on each curve will give the height of that particular curve. The curves of this example are

$$x^3 - 3x^2 - 4y^2 = z = c.$$

Without going into details the contour lines are as shown in fig. 8.4. The point (0, 0) where also $z=0$, is a maximum point; and the point (2, 0) where $z=-4$, is a saddle point. The reason for this latter name is now obvious, and it could also be described as a 'mountain pass'. The contour line through this point is given by $z=-4$.

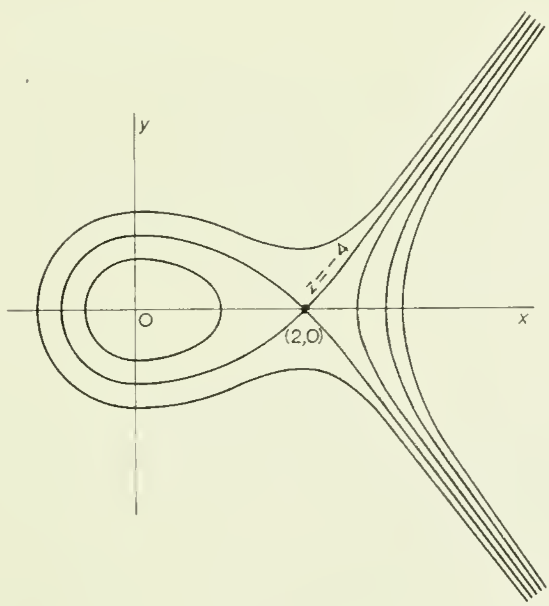

Fig. 8.4

Example 25

$$f(x, y) \equiv (x^2 + y^2)^2 - 2a^2(x^2 - y^2).$$

Stationary points of this function are given by

$$f_x = 4x(x^2 + y^2) - 4a^2x = 4x(x^2 + y^2 - a^2) = 0, \tag{8.104}$$

$$f_y = 4y(x^2 + y^2) + 4a^2y = 4y(x^2 + y^2 + a^2) = 0. \tag{8.105}$$

From eq. (8.104) either $x=0$ or $x^2+y^2=a^2$. Substituting $x=0$ in eq. (8.105), the only real value of y is $y=0$. Substituting $x^2+y^2=a^2$ in eq. (8.105) we get $y=0$ and then $x=\pm a$.

Thus the three points (0, 0), (a, 0), ($-a$, 0) are stationary points. We have also

$$f_{xx} = 12x^2 - 4a^2, \qquad f_{xy} = 8xy, \qquad f_{yy} = 12y^2 + 4a^2.$$

At (0; 0) we have $f_{xx}f_{yy}-f_{xy}^2=-16a^2<0$, so that this point is a saddle point.

At (a, 0) we have $f_{xx}f_{yy}-f_{xy}^2=32a^2>0$ and f_{xx}, f_{yy} are both positive. These results also apply to the point ($-a$, 0). Thus the points ($\pm a$, 0) are minimum points.

The contour lines are

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = c,$$

and are best expressed in polar coordinates by

$$r^4 - 2a^2r^2 \cos 2\theta = c.$$

These curves are known as Cassini's ovals and are illustrated in fig. 8.5. The particular curve $z=c=0$ through the saddle point is given by

$$r^2 = 2a^2 \cos 2\theta,$$

and is the 'lemniscate of Bernoulli'.

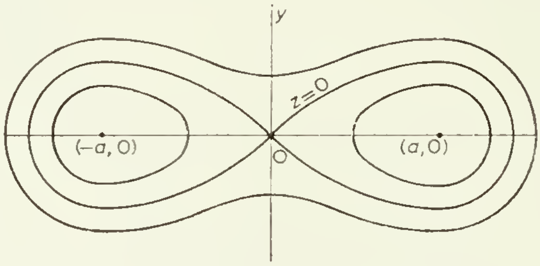

Fig. 8.5

EXERCISE 8.7

1. Show that the function $f(x, y) \equiv xy^2 - 2xy + 2x^2 - 3x$, has three stationary points, one of which is a minimum.

2. Show that the function $f(x, y) \equiv x^2y - 4x^2 - y^2$, has three stationary points, one of which is a maximum.

3. Show that the function $f(x, y) \equiv xy(12 - 4x - 3y)$, has four stationary points, one of which is a maximum.

4. Discuss the stationary values of $x^3y^2(6 - x - y)$.

5. Find the stationary points of the function $z = f(x, y)$ given implicitly by

$$2x^2 + 6y^2 + 2z^2 + 8xz - 4x - 8y + 3 = 0.$$

6. Examine the stationary values of the function $2 \sin(x + 2y) + 3 \cos(2x - y)$, for $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$.

7. Determine and discriminate the stationary values of the function

$$x^3 + y^3 - 3axy.$$

8. Find the stationary point of the function $x^{-1} + 2x \operatorname{cosec} y - x \cot y$, for $x > 0$, $0 < y < \frac{1}{2}\pi$, and determine its nature.

§ 5.1. The stationary values of implicit functions. Lagrange multipliers

We often need to find the stationary values of a function of several variables which are not all independent but may be connected by several relations. Since we have dealt with a function of two variables in § 5 let us consider firstly a function $f(x, y, z, u)$ of the four variables x, y, z, u which are not all independent, but themselves satisfy the two distinct differentiable relations

$$\Phi(x, y, z, u) = 0, \qquad \Psi(x, y, z, u) = 0. \tag{8.106}$$

For definiteness we shall think of x and y as the independent variables and think of z and u as being defined in terms of x and y by the relations (8.106). Thus $f(x, y, z, u)$ is really a function of the two independent variables x and y. To deal with the problem of finding the stationary values of $f(x, y, z, u)$ under these conditions, we could, if possible, solve the equations (8.106) for z, u in terms of x, y and substitute in the function $f(x, y, z, u)$ and so reduce the problem to that of § 5. However this procedure is not always feasible and so we require a more practicable method of dealing with this problem. Also the following method is often more convenient theoretically and we shall note later that it treats the variables x, y, z, u in a more symmetrical way.

Following the method of § 5 for a function of two variables we see that if the function $f(x, y, z, u)$ is to have a stationary value, the necessary condition can be expressed as: the differential

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y + f_z\,\mathrm{d}z + f_u\,\mathrm{d}u, \tag{8.107}$$

must be zero for arbitrary values of $\mathrm{d}x$, $\mathrm{d}y$; $\mathrm{d}z$ and $\mathrm{d}u$ are determined in terms of $\mathrm{d}x$, $\mathrm{d}y$ from the equations $\mathrm{d}\Phi=0$ and $\mathrm{d}\Psi=0$, namely

$$0 = \Phi_x\,\mathrm{d}x + \Phi_y\,\mathrm{d}y + \Phi_z\,\mathrm{d}z + \Phi_u\,\mathrm{d}u, \tag{8.108}$$

$$0 = \Psi_x\,\mathrm{d}x + \Psi_y\,\mathrm{d}y + \Psi_z\,\mathrm{d}z + \Psi_u\,\mathrm{d}u. \tag{8.109}$$

But assuming that $\Phi_z\Psi_u-\Phi_u\Psi_z\neq 0$, which the solution of these equations for $\mathrm{d}z$, $\mathrm{d}u$ requires, it is also possible to find numbers λ_1, λ_2 such that

$$f_z + \lambda_1\Phi_z + \lambda_2\Psi_z = 0, \tag{8.110}$$

$$f_u + \lambda_1\Phi_u + \lambda_2\Psi_u = 0. \tag{8.111}$$

With λ_1, λ_2 defined in this way, we can eliminate $\mathrm{d}z$, $\mathrm{d}u$ from (8.107) by adding to it λ_1 times (8.108) and λ_2 times (8.109). Then in terms of the

independent differentials dx, dy we have

$$\mathrm{d}f = (f_x + \lambda_1 \Phi_x + \lambda_2 \Psi_x)\,\mathrm{d}x + (f_y + \lambda_1 \Phi_y + \lambda_2 \Psi_y)\,\mathrm{d}y.$$

But since df=0 for arbitrary values of dx, dy then we must have

$$f_x + \lambda_1 \Phi_x + \lambda_2 \Psi_x = 0, \tag{8.112}$$

$$f_y + \lambda_1 \Phi_y + \lambda_2 \Psi_y = 0. \tag{8.113}$$

The four equations (8.110)–(8.113), together with the eqs. (8.106) determine the possible values of x, y, z, u, λ_1, λ_2 for a stationary value of $f(x, y, z, u)$. We note that the four eqs. (8.110)–(8.113) express the condition that the function

$$f(x, y, z, u) + \lambda_1 \Phi(x, y, z, u) + \lambda_2 \Psi(x, y, z, u), \tag{8.114}$$

has a stationary value, regarding it as a function of the four variables x, y, z, u as if they were all independent, and λ_1, λ_2 as constants. This indicates that the method treats the four variables in a more symmetrical way. We see also how the method may be extended to any number of variables and relations between them. We will simply state the general result as follows.

Given a function $f(x_1, x_2, x_3, \ldots, x_n)$ of n variables, connected by $h(<n)$ distinct differentiable relations

$$\Phi_1(x_1, \ldots x_n) = 0, \qquad \Phi_2(x_1, \ldots x_n) = 0, \quad \ldots \quad \Phi_h(x_1, \ldots x_n) = 0, \tag{8.115}$$

we wish to find the values of $x_1, x_2, \ldots x_n$ which make the function $f(x_1, x_2, \ldots x_n)$ stationary. We equate to zero the partial derivatives of the auxiliary function

$$f + \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \ldots + \lambda_h \Phi_h,$$

regarding $\lambda_1, \lambda_2, \ldots \lambda_h$ as constants and all the variables $x_1, x_2, \ldots x_n$ as independent. The n equations so determined, together with the h eqs. (8.115) determine the values of $x_1, x_2, \ldots x_n$ and $\lambda_1, \lambda_2, \ldots \lambda_h$. The values of $\lambda_1, \lambda_2, \ldots \lambda_h$ are not necessarily required and they are often referred to as Lagrange's undetermined multipliers, the method being that of Lagrange.

Example 26

Show that the function $f(x, y) = x^3 y^2$ has four stationary points under the condition

$$x^2 - xy = a^2, \qquad (a > 0). \tag{8.116}$$

Here it is quite easy to solve the eq. (8.116) for y in terms x and to substitute in the function x^3y^2, so that a function of a single variable x is determined. Stationary values can then be found by the methods of Ch. 6 § 4.1. We suggest that the reader verifies the following results in this way. The Lagrange method is as follows. Consider the stationary values of the function

$$F(x, y) = x^3y^2 + \lambda(x^2 - xy - a^2),$$

regarding λ as a constant and x, y as independent. The required equations are

$$\frac{\partial F}{\partial x} = 3x^2y^2 + \lambda(2x - y) = 0,$$

$$\frac{\partial F}{\partial y} = 2x^3y + \lambda(-x) = 0.$$

Eliminating λ between these two equations we get

$$3x^2y^2 + 2x^2y(2x - y) = 0,$$

or

$$x^2y(y + 4x) = 0,$$

giving $x=0$, $y=0$ or $y=-4x$. The value $x=0$ does not satisfy eq. (8.116) since $a>0$. The value $y=0$ gives $x=\pm a$. Substituting $y=-4x$ in eq. (8.116) we get $x=\pm a/\sqrt{5}$, $y=\mp 4a/\sqrt{5}$. Thus the four points $(\pm a, 0)$, $(\pm a/\sqrt{5}, \mp 4a/\sqrt{5})$ give stationary values of the function.

Example 27

If $x+y+z=1$ and $xyz=-1$, show that $x^2+y^2+z^2$ has three equal stationary values.

We consider the stationary values of the function

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda_1(x + y + z - 1) + \lambda_2(xyz + 1).$$

The equations are

$$\frac{\partial F}{\partial x} = 2x + \lambda_1 + \lambda_2 yz = 0,$$

$$\frac{\partial F}{\partial y} = 2y + \lambda_1 + \lambda_2 zx = 0,$$

$$\frac{\partial F}{\partial z} = 2z + \lambda_1 + \lambda_2 xy = 0.$$

Elimination of λ_1, λ_2 between these three equations gives

$$(x - y)(y - z)(z - x) = 0,$$

so that either $x=y$, $y=z$ or $z=x$. When $x=y$, the two equations

$$x + y + z = 1, \qquad xyz = -1,$$

give

$$2x^3 - x^2 - 1 = 0, \quad \text{or} \quad (x - 1)(2x^2 + x + 1) = 0,$$

with one real solution $x=1$. Thus $x=1$, $y=1$, $z=-1$ is a stationary point. By symmetry, or using the results $y=z$, $z=x$ in turn, we find that $(1, -1, 1)$ and $(-1, 1, 1)$ are also stationary points. These three points yield the same stationary value of $x^2+y^2+z^2$. The reader can again verify these results by expressing $x^2+y^2+z^2$ as a function of x only, using the two conditions to find y, z as functions of x.

Example 28

To find the path of least time between two points A and B in two different media as in fig. 8.6. That is, to find the path of a ray of light from A to B.

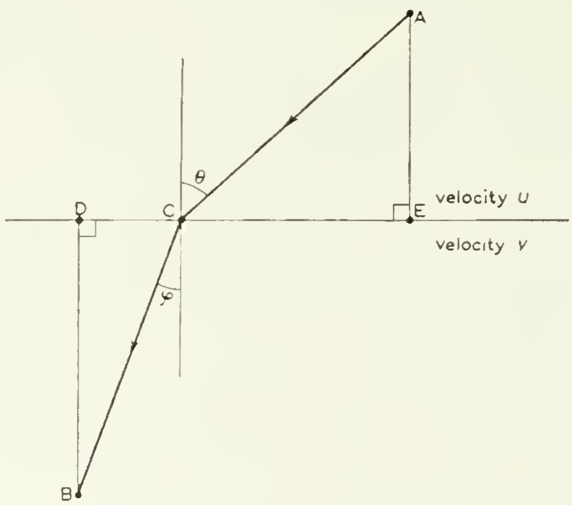

Fig. 8.6

Let the velocities in the two media be u, v. Then the time t of passage from A to B, with $\mathrm{AE}=a$ and $\mathrm{BD}=b$, is

$$t = \frac{\mathrm{AC}}{u} + \frac{\mathrm{CB}}{v} = \frac{a}{u}\sec\theta + \frac{b}{v}\sec\varphi.$$

This must be a minimum subject to the condition that DE is of constant length, that is

$$a\tan\theta + b\tan\varphi = \text{const.} \tag{8.117}$$

We write down the condition that

$$F(\theta, \varphi) = (a/u)\sec\varphi + (b/v)\sec\varphi + \lambda(a\tan\theta + b\tan\varphi - \text{const.})$$

has a stationary value. We have

$$\frac{\partial F}{\partial\theta} = (a/u)\sec\theta\tan\theta + \lambda a\sec^2\theta = 0 \tag{8.118}$$

$$\frac{\partial F}{\partial\varphi} = (b/v)\sec\varphi\tan\varphi + \lambda b\sec^2\varphi = 0 \tag{8.119}$$

and we solve eqs. (8.117)–(8.119) for λ, θ, φ. Since $\sec\theta$ or $\sec\varphi$ cannot be zero, we can divide eq. (8.118) by $a\sec^2\theta$ and eq. (8.119) by $b\sec^2\varphi$ to give

$$(\sin\theta)/u + \lambda = 0, \qquad (\sin\varphi)/v + \lambda = 0,$$

and therefore $(\sin\theta)/u = (\sin\varphi)/v$. This fixes the position of the point C in fig. 8.6. The actual values of θ and φ can be found by using eq. (8.117).

EXERCISE 8.8

1. Show that the function $f(x, y, z) = x^2+y^2+z^2$ has four stationary values on the surface $xyz = a^3$.

2. Show that the function xyz has fourteen stationary points on the surface of the unit sphere $x^2+y^2+z^2 = 1$.

3. Show that a stationary value of $x^2+y^2+z^2$ for values of x, y, z subject to the conditions

$$lx + my + nz = 0, \qquad yz + zx + xy = a^2,$$

is

$$\frac{2a^2(l^2 + m^2 + n^2)}{l^2 + m^2 + n^2 - 2mn - 2nl - 2lm}.$$

4. Show that on the circle of intersection of the sphere $x^2+y^2+z^2 = a^2$ and the plane $x+y+z = a$, the value of the expression $x^4+y^4+z^4$ oscillates between a^4 and $\frac{11}{27}a^4$, taking each of these values three times.

5. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the condition

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \tag{8.120}$$

[Note: the maximum and minimum values in this example give the squared lengths of the principal axes of the quadric surface given by eq. (8.120).]

VECTOR ALGEBRA

§ 1. Introduction and definitions

Many entities in geometry and physics, such as length, volume, mass, temperature, energy, are characterised by their magnitudes alone, so that once a scale has been set up to measure magnitude in terms of length, they are determined completely by their positions on the scale. Such entities are therefore called *scalars*.

Other entities such as displacement, velocity, acceleration, force, electric intensity, possess not only a definite numerical magnitude but also a definite direction and a definite sense in that direction. The simplest example of such an entity is the straight line displacement between two points A and B say, in space. Such a displacement having a magnitude, direction and sense is called a *vector* displacement. We employ the notation **AB** to represent this vector displacement†; the position of the points A and B determine the length and general direction of the displacement and the order of the letters corresponds to the sense of the displacement from A to B. Two such vector displacements combine together into a single displacement in a particular way referred to as the law of addition of displacements. The precise nature of this law will be stated in § 2.

Suppose now that we consider any other entity which has magnitude, direction and sense. If we choose a scale so that its magnitude can be represented by a length, then we say that we can represent such an entity by a vector displacement **AB**, if the length of **AB** represents the magnitude of the entity, using the chosen scale, the direction of **AB** and the sense from A to B are the same as the direction and sense of the given entity. Such entities belong to the special class of quantities which

† In some publications an arrow is placed over the letters to indicate the vector character of the symbol and to distinguish it from the simple geometrical segment AB. In writing, one usually puts a 'tilde' $\sim$ below vector quantities.

we call vector quantities or simply *vectors*. Although not directly apparent in this statement, an essential part of the definition of a vector is that any two vectors of like physical character, such as two vectors representing velocity, combine together into a single one of similar character, by the same law of addition as the vector displacements which are used to represent them.

§ 1.1. Equality of vectors

Two vectors of like physical character represented by the two vector displacements AB and CD are said to be equal if the mid point of the

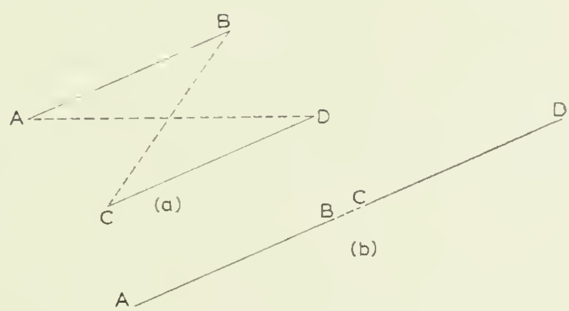

Fig. 9.1

segment AD coincides with the mid point of the segment BC so that ABDC is a parallelogram as in fig. 9.1. We then write

$$AB = CD.$$

When A and B do not coincide this definition of equality implies that

(i) the segments AB and CD are of equal length,
(ii) the lines AB and CD are in the same direction,
(iii) the sense of the displacement from A to B is the same as that from C to D.

Since ABDC is a parallelogram the fact that $AB = CD$ implies also that $AC = BD$, $CA = DB$ and $BA = DC$.

§ 1.2. Localised and free vectors

The equivalence of vectors represented by displacements in different positions implied in the definition in § 1.1 means that the actual po-

sitions of the vectors are not essential to their representation and that a vector represented for example by the displacement AB can equally well be represented by another displacement CD equivalent to AB by the definition in § 1.1. Thus although in practice we represent vectors by displacements in particular positions we must remember that in general these positions are not necessarily those of the actual vector, and that any other parallel displacement in the same sense equally represents the vector. There are however certain vectors, as for example, position vectors in geometry and forces in mechanics, which have the additional property of being associated with certain definite points in space and which cannot therefore be properly represented by displacements from any but these particular points. To distinguish the two types we call those associated with particular points *localised vectors*; the more general ones without any particular position we call *free vectors*. Generally speaking, unless anything to the contrary is stated, it will be assumed that the vectors of our discussion are all *free* vectors and do not therefore necessarily occupy the positions of the displacements used to represent them.

We shall frequently use a single letter of special type, for example $\boldsymbol{a}$, $\boldsymbol{r}$, $\boldsymbol{u}$ to symbolise a vector either in the free or localised sense. To say that a vector $\boldsymbol{r}$ is localised at the point A means that there exists a second point B so that the vector displacement AB represents $\boldsymbol{r}$ in magnitude, direction and sense; we then write $\boldsymbol{r}=AB$. On the other hand, to say that a vector $\boldsymbol{r}$ is free implies not only that there exist two points A, B such that $\boldsymbol{r}=AB$ but that these two points can be chosen in an infinite number of ways. All that is necessary is that the displacement AB should be parallel to $\boldsymbol{r}$ and in the same sense, and should represent $\boldsymbol{r}$ in magnitude on the chosen scale.

§ 1.3. Magnitude of a vector

We call the *modulus* of a vector $\boldsymbol{r}$ the positive or zero number which measures on the given scale the magnitude of the vector; we write it as

$$\text{mod}\,\boldsymbol{r}, \quad |\boldsymbol{r}| \quad \text{or simply} \quad r.$$

A vector whose modulus is zero is a *zero vector*. A zero vector can have an arbitrary direction since in going zero distance it does not matter in which direction one goes. All such vectors are therefore equivalent and are denoted by the common symbol $\mathbf{0}$.

A vector of modulus unity, frequently called a *unit vector* is often used to specify the direction and sense of a vector without specifying its magnitude.

§ 2. Addition and subtraction of vectors

Vectors by definition obey the same law of addition as rectilinear displacements. If from a point O fig. 9.2 we mark off a displacement OA representing the vector $\boldsymbol{a}$ on a chosen scale, and then from the end point A we mark off an additional displacement AC representing the vector $\boldsymbol{b}$ on the same scale, we arrive at the point C. The result of superposing these two displacements is therefore the same as would have been derived from a direct displacement from O to C along the vector OC. In this sense we say that the displacement OC is the *sum* or *resultant* of the two displacements OA and AC and we write

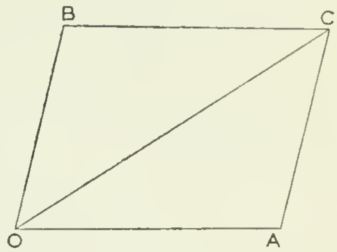

Fig. 9.2

$$OA + AC = OC. \tag{9.1}$$

Further, if OC represents the vector $\boldsymbol{c}$ on the same chosen scale we can write

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}, \tag{9.2}$$

the relation (9.2) being independent of the actual positions of the vectors since they are free vectors and are only represented by the displacements OA, AC, OC in the particular positions shown. The vector $\boldsymbol{c}$ represented by OC is the third side of a triangle of which the first two sides represent the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; the rule (9.1) for the addition of vectors is therefore known as the *triangle law of addition.* If however, we complete the parallelogram OACB where $OB = AC$, then OB can equally well represent the vector $\boldsymbol{b}$ and the rule of addition can be given in the following form: if displacements OA and OB representing vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively are drawn from the same point O, the vector $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ is represented by the diagonal OC of the parallelogram of which OA and OB are adjacent sides.

We note here that since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel in general, OA and AC will not be in the same straight line. If this is so

$$\mathrm{OC} < \mathrm{OA} + \mathrm{AC}.$$

Thus we can write generally

$$\mathrm{OC} \leqslant \mathrm{OA} + \mathrm{AC},$$

the equality sign holding only when OAC is a straight line. In terms of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ this implies the 'triangle inequality'

$$|\boldsymbol{a} + \boldsymbol{b}| \leqslant |\boldsymbol{a}| + |\boldsymbol{b}|, \tag{9.3}$$

the equality sign holding only when $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

The summation process defined by the triangle law obeys the commutative and associative laws of addition of algebra; that is,

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}, \tag{9.4}$$

and

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}). \tag{9.5}$$

Equation (9.4) is obvious from fig. 9.2 in that we can write

$$OC = OB + BC,$$

from the triangle law of addition.

Equation (9.5) follows from fig. 9.3 even if the three displacements representing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ do not lie in a plane. We have from

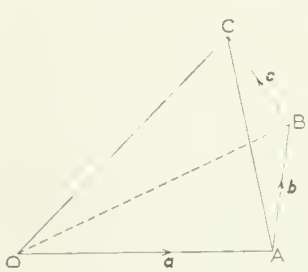

Fig. 9.3

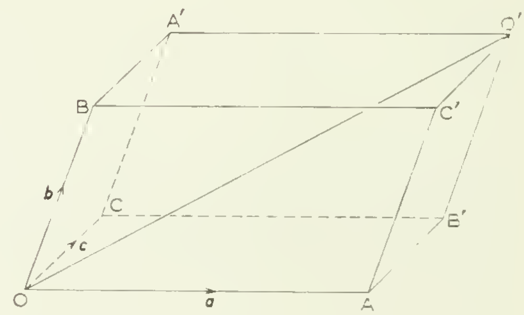

Fig. 9.4

fig. 9.3 and the triangle law of addition

$$OC = OB + BC = (OA + AB) + BC = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}.$$

But also

$$OC = OA + AC = OA + (AB + BC) = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}).$$

It is instructive also to see the result from the parallelepiped in fig. 9.4. We have

$$OO' = OC' + C'O' = (OA + AC') + C'O' = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c},$$

or

$$OO' = OA' + A'O' = (OC + CA') + A'O' = (\boldsymbol{c} + \boldsymbol{b}) + \boldsymbol{a},$$

and in fact the reader may verify from this fig. 9.4 that there are twelve different ways of arranging this summation, all of which are equal to OO'. Thus we can write the summation as $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ without indicating the order in which the vectors are added; the argument can obviously be extended to sums of any number of vectors.

We can therefore immediately derive two general results.

(i) If starting from a point O, displacement vectors $OA, AB, BC, \ldots JK$ are marked off successively to represent the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots \boldsymbol{k}$, and these displacement vectors are not necessarily all in the same plane, then the sum or resultant $\boldsymbol{r}$ of these vectors is represented by the displacement OK. We can write in terms of the displacements,

$$OK = OA + AB + BC + \ldots + JK,$$

or in terms of the vectors $\boldsymbol{r}, \boldsymbol{a}, \boldsymbol{b}, \ldots \boldsymbol{k}$, this becomes

$$\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} + \ldots + \boldsymbol{k}. \tag{9.6}$$

(ii) If any number of vectors can be represented by the sides of a closed polygon, not necessarily plane, traversed in some definite order, then their total sum is zero, or in terms of the displacements in (i)

$$OA + AB + \ldots + JK + KO = \mathbf{0}. \tag{9.7}$$

When $\boldsymbol{r}$ is the sum of several vectors $\boldsymbol{a}, \boldsymbol{b}, \ldots \boldsymbol{k}$ as in eq. (9.6) we say that $\boldsymbol{r}$ is resolved into the *components* $\boldsymbol{a}, \boldsymbol{b}, \ldots \boldsymbol{k}$. Obviously we can resolve a vector into components in any number of different ways even when the number of components is limited. Certain special and important cases of resolution of vectors will be used later.

§ 2.1. Multiplication of a vector by a scalar

If λ is a real number, we define the product of a vector $\boldsymbol{a}$ by λ, represented by $\lambda\boldsymbol{a}$, as the vector which has

(i) a magnitude equal to the magnitude of $\boldsymbol{a}$ multiplied by the magnitude $|\lambda|$ of λ,

(ii) the same direction as $\boldsymbol{a}$,

(iii) the same or opposite sense as $\boldsymbol{a}$, according as λ is positive or negative.

These conditions determine the vector $\lambda\boldsymbol{a}$ uniquely and we can easily establish from this definition the associative and distributive laws; these

state that if λ and μ are two real numbers,

$$\lambda(\mu \boldsymbol{a}) = (\lambda\mu)\boldsymbol{a} = \mu(\lambda \boldsymbol{a}), \tag{9.8}$$

$$(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}, \tag{9.9}$$

$$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}, \tag{9.10}$$

eq. (9.10) following from the elementary properties of similar triangles.

Further

$$\lambda\mathbf{0} = \mathbf{0},$$

so that if $\lambda\boldsymbol{a}=\mathbf{0}$ then either $\lambda=0$ or $\boldsymbol{a}=\mathbf{0}$.

If $\lambda \neq 0$ we can write

$$\left(\frac{1}{\lambda}\right)\boldsymbol{a} = \frac{\boldsymbol{a}}{\lambda}.$$

We can introduce the concept of vector subtraction by writing $-\boldsymbol{b}$ for $(-1)\boldsymbol{b}$, so that

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-1)\boldsymbol{b};$$

thus $\boldsymbol{a}-\boldsymbol{b}$ is the vector sum of $\boldsymbol{a}$ and the vector $(-1)\boldsymbol{b}$ which coincides with $\boldsymbol{b}$ in magnitude and direction but is in the opposite sense. Fig. 9.5 shows that $OC=\boldsymbol{a}-\boldsymbol{b}=BA$.

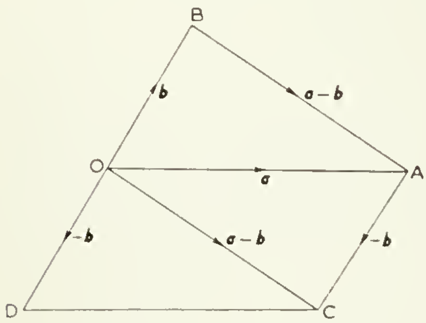

Fig. 9.5

Using $-\boldsymbol{b}$ for a vector of the same magnitude and direction as $\boldsymbol{b}$ but in the opposite sense, is in accord with the idea of representing a vector by a displacement, for if $\boldsymbol{b}=OB$ then $-\boldsymbol{b}=BO$; we have already seen that

$$OB + BO = \mathbf{0},$$

so that

$$\boldsymbol{b} - \boldsymbol{b} = \mathbf{0},$$

as we expect.

We may also notice in fig. 9.5 that

$$OB + BA = OA,$$

or

$$\boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}.$$

§ 3. Resolution of vectors into components

If, choosing a fixed point O, we can find a second point A so that OA represents a non-zero vector $\boldsymbol{a}$, then OA is in the direction of $\boldsymbol{a}$. The vector $\boldsymbol{a}$ is then said to be parallel to a given line or plane if OA is parallel to the given line or plane. If from the same point O we draw a second displacement OB to represent another non-zero vector $\boldsymbol{b}$, then the angle AOB ($\leqslant\pi$) is defined as the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We write $\widehat{\boldsymbol{ab}}$ for the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Obviously if λ and μ are two real numbers of the same sign then

$$\widehat{(\lambda\boldsymbol{a})(\mu\boldsymbol{b})} = \widehat{\boldsymbol{ab}} = \widehat{\boldsymbol{ba}}, \tag{9.11}$$

whilst if λ and μ are of opposite sign

$$\widehat{(\lambda\boldsymbol{a})(\mu\boldsymbol{b})} = \pi - \widehat{\boldsymbol{ab}}. \tag{9.12}$$

We shall now prove three results in connection with the resolution of vectors into components setting them out as three theorems and giving examples on the use of each in turn.

§ 3.1. Parallel vectors

Theorem 1. *Every vector* $\boldsymbol{r}$ *parallel to a non-zero vector* $\boldsymbol{a}$ *can be expressed uniquely in the form*

$$\boldsymbol{r} = \lambda\boldsymbol{a}, \tag{9.13}$$

where λ *is a real number.*

The vectors $a\boldsymbol{r}$ and $r\boldsymbol{a}$ have the same magnitude ar and the same direction since $\boldsymbol{r}$ and $\boldsymbol{a}$ are parallel. We can therefore write

$$a\boldsymbol{r} = r\boldsymbol{a}, \tag{9.14}$$

when $\boldsymbol{r}$ and $\boldsymbol{a}$ have the same sense, and

$$a\boldsymbol{r} = -r\boldsymbol{a}, \tag{9.15}$$

when $\boldsymbol{r}$ and $\boldsymbol{a}$ have opposite senses. Thus dividing through by the real number a we have

$$\boldsymbol{r} = \pm\frac{r}{a}\boldsymbol{a},$$

the sign being uniquely determined by the relative sense of $\boldsymbol{r}$ and $\boldsymbol{a}$. Thus $\lambda = \pm r/a$, and is equal in magnitude to the ratio of the moduli of the two vectors. If $\boldsymbol{a}$ is a unit vector so that $a = 1$, then

$$\boldsymbol{r} = \pm r\boldsymbol{a}. \tag{9.16}$$

It follows that if $\boldsymbol{r}$ is any vector, then $\boldsymbol{r}/r$ is the unit vector in the same direction and sense as $\boldsymbol{r}$.

This theorem together with the vector law of addition which we have already proved in § 2, enables us to demonstrate the use of vectors in deriving some well-known geometrical results. Although we wish to use vectors later mainly to represent physical quantities it is much easier to demonstrate the algebraic properties of vectors and their uses in geometrical problems. These problems will include some in three dimensions or solid geometry with which the reader may not be so familiar. In these problems we shall frequently use the displacement vector OA from a point O to a point A. This vector is called the *position vector* of A relative to O, and when O is given determines immediately the position of A. The position vector of any point A relative to O is, of course, a vector localised at O.

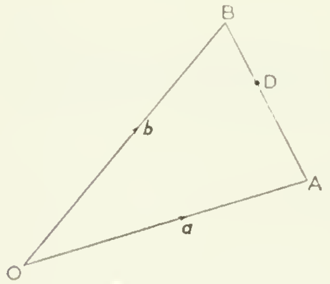

Fig. 9.6

Example 1

If $OA=\boldsymbol{a}$, $OB=\boldsymbol{b}$ and D is a point dividing AB in the ratio $p:q$, prove that

$$OD = \frac{p\boldsymbol{b}+q\boldsymbol{a}}{p+q}.$$

In fig. 9.6 by the law of addition

$$OD = OA + AD. \tag{9.17}$$

Then since AD and AB are in the same straight line and $\mathrm{AD}=p\mathrm{AB}/(p+q)$ we have by Theorem 1

$$AD = \frac{p}{p+q}\,AB.$$

Also, by the law of addition

$$AB = AO + OB = -\boldsymbol{a} + \boldsymbol{b}.$$

Therefore from eq. (9.17)

$$OD = \boldsymbol{a} + \frac{p}{p+q}(-\boldsymbol{a}+\boldsymbol{b}) = \left(1-\frac{p}{p+q}\right)\boldsymbol{a} + \frac{p\boldsymbol{b}}{p+q},$$

or

$$OD = \frac{q\boldsymbol{a}+p\boldsymbol{b}}{p+q}. \tag{9.18}$$

Example 2

OABC is a parallelogram; a point D is taken on OA so that OD : OA $=1:n$; then a point E is taken on DC so that DE : EC $=1:n$. Prove that O, E, B are collinear and $(n+1)\mathrm{OE}=\mathrm{OB}$.

In fig. 9.7, let $OA = \boldsymbol{a}$, $OC = \boldsymbol{c}$. We are given

$$\frac{\mathrm{OD}}{\mathrm{OA}} = \frac{1}{n} \quad \text{and} \quad \frac{\mathrm{DE}}{\mathrm{EC}} = \frac{1}{n}.$$

Then from Theorem 1, we have

$$OD = \frac{1}{n} OA = \frac{1}{n} \boldsymbol{a}, \tag{9.19}$$

and

$$DE = \frac{1}{n+1} DC = \frac{1}{n+1} (DO + OC),$$

or

$$DE = \frac{1}{n+1} \left(- \frac{1}{n} \boldsymbol{a} \right) + \frac{1}{n+1} \boldsymbol{c}. \tag{9.20}$$

Now

$$(n+1)OE = (n+1)(OD + DE),$$

and using eqs. (9.19) and (9.20) this becomes

$$(n+1)OE = \boldsymbol{a} + \boldsymbol{c} = OA + OC = OB.$$

Thus these two vectors $(n+1)OE$ and OB being equal, must have the same direction and magnitude. That is, OE and OB are parallel and since they both go through O they must be in the same straight line, so that O, E, B are collinear. Also since the magnitudes are equal, $(n+1)\mathrm{OE} = \mathrm{OB}$.

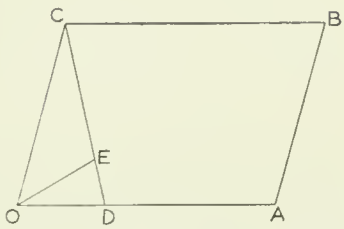

Fig. 9.7

Example 3

Prove vectorially that the four diagonals of a parallelepiped meet and bisect each other. Referring to the parallelepiped in fig. 9.4, let

$$OA = CB' = A'O' = BC' = \boldsymbol{a}, \qquad OB = AC' = CA' = B'O' = \boldsymbol{b}, \qquad OC = AB' = BA' = C'O' = \boldsymbol{c}.$$

Suppose P is the mid point of OO′, then

$$OP = \tfrac{1}{2} OO' = \tfrac{1}{2}(OA + AC' + C'O') = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}). \tag{9.21}$$

Suppose Q is the mid point of AA′, then by the result in Example 1, we have

$$OQ = \tfrac{1}{2}(OA + OA') = \tfrac{1}{2}(OA + OC + CA') = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}). \tag{9.22}$$

Thus from eqs. (9.21) and (9.22), we have $OP = OQ$, or

$$PQ = OQ - OP = 0,$$

and therefore Q coincides with P. Thus the diagonals OO′ and AA′ meet and bisect each other. Similarly it can be shown that the mid points of BB′ and CC′ also coincide with P.

EXERCISE 9.1

1. On any two non-intersecting straight lines L, L′ fixed in space there are two marked points C on L and C′ on L′. Variable points A, B on L and A′, B′ on L′ are such that

$$\frac{\mathrm{AC}}{\mathrm{CB}} = \frac{\mathrm{A'C'}}{\mathrm{C'B'}} = \lambda,$$

where λ may be positive or negative. Prove that

$$AA' + \lambda BB' = (1 + \lambda)CC'.$$

2. ABCD, ACLM, ALPN,..., is a series of coplanar parallelograms constructed so that any one of the series has one of its sides coincident with the diagonal of the parallelogram which precedes it in the series, Q is the centre of the first parallelogram, QM meets AL in R, RN meets AP in S and so on. Prove that

$$\mathrm{AR} = \tfrac{1}{3}\mathrm{AL}, \qquad \mathrm{AS} = \tfrac{1}{4}\mathrm{AP},$$

and so on.

3. OA, OB are any two lines drawn from a point O and points A′B′ are taken on these lines respectively so that OA′$=p$OA, OB′$=q$OB. Prove that the line joining O to the mid point of AB divides A′B′ in the ratio $p : q$.

4. The three vertices of a triangle ABC have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ relative to a fixed point O. Prove that the medians of the triangle meet and divide each other in the ratio 2 : 1. This point of intersection of the medians is called the centroid G of the triangle; show that

$$OG = \tfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}).$$

5. In the notation of Question 4, D, E, F are the mid points of BC, CA, AB respectively, show that

$$OD + OE + OF = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}.$$

6. A, B, C, D are the vertices of a tetrahedron and A′ is the centroid of the face BCD, B′ of the face CDA, C′ of ADB, D′ of ABC. Prove that (i) AA′, BB′, CC′, DD′ intersect and divide each other in the ratio 3 : 1; (ii) the centroids of the two tetrahedra ABCD, A′B′C′D′ coincide.

Hint: in this question the point of intersection of AA′, BB′, CC′, DD′ in (i) is called the centroid of ABCD. If O is any point in space and G the centroid then

$$OG = \tfrac{1}{4}(OA + OB + OC + OD).$$

The point G is also referred to as the *mean* position of the points A, B, C, D. In general the mean position G of n points A, B, C, D,... is such that

$$OG = \frac{1}{n}(OA + OB + OC + \ldots). \tag{9.23}$$

7. Show that the mean position of any number of points is independent of the position of the origin of vectors O.

8. The sides of a quadrilateral ABCD are divided in the ratio $p:q$ at H, K, L, M. Prove that the mean position of H, K, L, M is the same as that of the four points A, B, C, D.

9. Any n points in space are divided in $n!/r!(n-r)!$ ways into two groups, one of r points and the other of $n-r$ points. H is the mean centre of one group of r points and K the mean centre of the corresponding group of $n-r$ points. Prove that the $n!/r!(n-r)!$ lines HK are concurrent.

10. If

$$(m-n)OP + (n-l)OQ + (l-m)OR = \mathbf{0},$$

where l, m, n are real numbers, not all equal, prove that P, Q, R are collinear.

§ 3.2. Vector parallel to a plane

THEOREM 2. *Every vector* $\boldsymbol{r}$ *parallel to the plane which is parallel to two non-parallel non-zero vectors* $\boldsymbol{a}$, $\boldsymbol{b}$ *can be expressed uniquely in the form*

$$\boldsymbol{r} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}, \tag{9.24}$$

where λ, μ *are real numbers.*

Let the plane of the paper be the plane to which all three vectors $\boldsymbol{r}$, $\boldsymbol{a}$, $\boldsymbol{b}$ are parallel. In this plane we can draw a vector AB to represent $\boldsymbol{r}$. Through the point A we can draw a line in the plane parallel to the vector $\boldsymbol{a}$ and through B a line parallel to $\boldsymbol{b}$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel these two lines will meet in a point which we denote by C.

Then AC is a vector parallel to $\boldsymbol{a}$ and by Theorem 1 can be written as $\lambda\boldsymbol{a}$; similarly CB can be written as $\mu\boldsymbol{b}$. But

$$AB = AC + CB,$$

and therefore

$$\boldsymbol{r} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}.$$

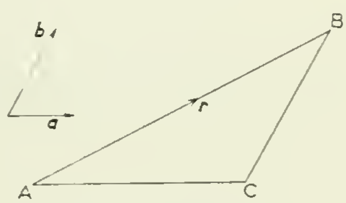

Fig. 9.8

The resolution is thus possible in at least one way. To prove that this resolution is unique let us suppose that there is a second resolution

$$\boldsymbol{r} = \lambda'\boldsymbol{a} + \mu'\boldsymbol{b}. \tag{9.25}$$

Subtracting eq. (9.25) from eq. (9.24) we then have

$$(\lambda - \lambda')\boldsymbol{a} + (\mu - \mu')\boldsymbol{b} = \mathbf{0},$$

or

$$(\lambda - \lambda')\boldsymbol{a} = (\mu' - \mu)\boldsymbol{b}. \tag{9.26}$$

Now since $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors, this result could only be true if both vectors in eq. (9.26) were zero. But $\boldsymbol{a}$ and $\boldsymbol{b}$ are also non-zero vectors and so we must have $\lambda - \lambda' = \mu - \mu' = 0$. Thus the resolution is unique.

As a corollary we may note that if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are any three non-zero vectors satisfying a linear relation of the form

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c} = \boldsymbol{0}, \tag{9.27}$$

where λ, μ, ν are non-zero real numbers, then $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are all parallel to the same plane.

Example 4

Two points $A_1(a_1\boldsymbol{u}_1)$ and $B_1(b_1\boldsymbol{u}_1)$ are taken on the line in the direction of the unit vector $\boldsymbol{u}_1$ through the origin O of vectors, and two other points $A_2(a_2\boldsymbol{u}_2)$ and $B_2(b_2\boldsymbol{u}_2)$ are taken on a second line through O in the direction of the unit vector $\boldsymbol{u}_2$. Prove that the point of intersection of A_1A_2 and B_1B_2 has position vector

$$\frac{b_1a_1(b_2 - a_2)\boldsymbol{u}_1 - b_2a_2(b_1 - a_1)\boldsymbol{u}_2}{b_2a_1 - b_1a_2},$$

relative to O.

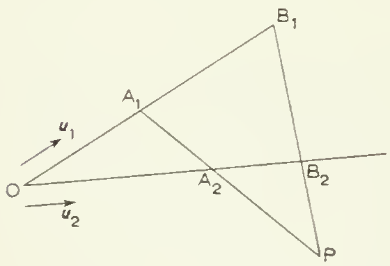

Fig. 9.9

In fig. 9.9 let P be the point of intersection of A_1A_2 and B_1B_2. We can express OP in two different forms. Since P is a point on A_1A_2 we can write

$$OP = OA_1 + A_1P = OA_1 + \lambda A_1A_2,$$

where λ is undetermined as yet. That is

$$OP = a_1\boldsymbol{u}_1 + \lambda(a_2\boldsymbol{u}_2 - a_1\boldsymbol{u}_1). \tag{9.28}$$

But P is also a point on B_1B_2, so similarly

$$OP = OB_1 + B_1P = OB_1 + \mu B_1B_2 = b_1\boldsymbol{u}_1 + \mu(b_2\boldsymbol{u}_2 - b_1\boldsymbol{u}_1), \tag{9.29}$$

where μ is undetermined. But from the eqs. (9.28) and (9.29) λ, μ must be chosen so that

$$a_1\boldsymbol{u}_1 + \lambda(a_2\boldsymbol{u}_2 - a_1\boldsymbol{u}_1) = b_1\boldsymbol{u}_1 + \mu(b_2\boldsymbol{u}_2 - b_1\boldsymbol{u}_1). \tag{9.30}$$

In this eq. (9.30) $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are unit vectors and are assumed to be non-parallel; so as in Theorem 2 this equation can only be satisfied if, after collecting terms, the coefficients of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both zero. This gives

$$a_1 - \lambda a_1 = b_1 - \mu b_1,$$

$$\lambda a_2 = \mu b_2.$$

Eliminating μ for instance, we find

$$\lambda = \frac{b_2(a_1 - b_1)}{a_1 b_2 - a_2 b_1},$$

and so substituting this value in eq. (9.28) we have

$$OP = \frac{a_1 b_1(b_2 - a_2)\boldsymbol{u}_1 - b_2 a_2(b_1 - a_1)\boldsymbol{u}_2}{a_1 b_2 - a_2 b_1}.$$

As a corollary to Theorem 2 we note that if $\boldsymbol{r}$ and $\boldsymbol{s}$ are two vectors satisfying the conditions of Theorem 2, so that

$$\boldsymbol{r} = \lambda \boldsymbol{a} + \mu \boldsymbol{b},$$

and

$$\boldsymbol{s} = l\boldsymbol{a} + m\boldsymbol{b},$$

where λ, μ, l, m are real numbers, then

$$\boldsymbol{r} + \boldsymbol{s} = (\lambda + l)\boldsymbol{a} + (\mu + m)\boldsymbol{b}. \tag{9.31}$$

EXERCISE 9.2

1. O, A, B, C, D are fixed points on a line l in the direction of the unit vector $\boldsymbol{u}$, the distances of A, B, C, D from O being a, b, c, d respectively. H and K are two points on another line from O and HC and KD meet in M, whilst HA and KB meet in L.

Prove that for all positions of HK, the line LM meets the given line l in the point at a distance from O equal to

$$\frac{cd(a - b) - ab(c - d)}{ad - bc}.$$

2. ABCD is a plane quadrilateral, E is the point of intersection of AD and BC, F that of DC and AB. Taking A as origin of vectors and $AB = \boldsymbol{b}$, $AD = \boldsymbol{d}$, $AF = k\boldsymbol{b}$, $AE = k'\boldsymbol{d}$ prove

(i) $$AC = \frac{k(1 - k')\boldsymbol{b} + k'(1 - k)\boldsymbol{d}}{1 - kk'},$$

(ii) the mid points of AC, BD, and FE are collinear and the second one divides the join of the other two in the ratio $(1 - kk') : 1$,
(iii) that if AC and BD intersect in Q and DB and EF in R, then Q and R divide DB internally and externally in the ratio $k'(1 - k) : k(1 - k')$.

3. The points D, E, F are taken on the sides of a triangle ABC so that

$$\frac{\mathrm{AD}}{\mathrm{DB}} = -\frac{l}{m}, \quad \frac{\mathrm{BE}}{\mathrm{EC}} = -\frac{m}{n}, \quad \frac{\mathrm{CF}}{\mathrm{FA}} = -\frac{n}{l}.$$

Show that D, E, F are collinear (Menelaus's theorem).

4. The points D, E, F are as given in Question 3 but without the negative sign. Show that CD, AE, BF meet in the point

$$\frac{mn\boldsymbol{a} + nl\boldsymbol{b} + lm\boldsymbol{c}}{lm + mn + nl},$$

where O is the origin of vectors and $OA = \boldsymbol{a}$ etc.

5. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are the position vectors of three points A, B, C in space, prove that the position vector of any point P on the plane ABC can be written in the form

$$\boldsymbol{r} = \frac{\alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}}{\alpha + \beta + \gamma},$$

and interpret the ratios $\alpha : \beta : \gamma$ in terms of the ratios in which AP, BP, CP cut the opposite sides of the triangle ABC.

Prove that the orthocentre of the triangle is at the point

$$\cot A \cot B \cot C\{\boldsymbol{a} \tan A + \boldsymbol{b} \tan B + \boldsymbol{c} \tan C\}.$$

§ 3.3. A particular case of Theorem 2

The resolution (9.24) of a vector into two components is particularly important when $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors; the theorem then states that it is always possible to resolve a vector $\boldsymbol{r}$ into two components, one parallel to a given vector $\boldsymbol{a}$, and the other perpendicular to the vector $\boldsymbol{a}$.

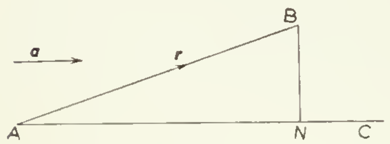

Fig. 9.10

Again in fig. 9.10 let AB represent the vector $\boldsymbol{r}$. Through A draw a line AC parallel to $\boldsymbol{a}$. These two lines define a plane to which both $\boldsymbol{a}$ and $\boldsymbol{r}$ are parallel. In this plane let BN be drawn perpendicular to AC and therefore perpendicular to $\boldsymbol{a}$. Then AN, parallel to $\boldsymbol{a}$, is the component of $\boldsymbol{r}$ in the direction of $\boldsymbol{a}$; it will be denoted by $\boldsymbol{r}_a$. The component of $\boldsymbol{r}$ perpendicular to $\boldsymbol{a}$ is NB and we denote this by $\boldsymbol{r}'_a$. Thus

$$\boldsymbol{r} = AB = AN + NB = \boldsymbol{r}_a + \boldsymbol{r}'_a;$$

it is easy to see that the signed moduli of these components are

$$r_a = \text{AN} = r \cos \widehat{\boldsymbol{ra}}, \quad \text{and} \quad r'_a = \text{NB} = r \sin \widehat{\boldsymbol{ra}},$$

the sense of $\boldsymbol{r}_a$ being opposite to the sense of $\boldsymbol{a}$ if the angle between $\boldsymbol{r}$ and $\boldsymbol{a}$ is obtuse. Similarly if $\boldsymbol{s}$ is any other vector also parallel to the plane defined by $\boldsymbol{r}$ and $\boldsymbol{a}$, we can write

$$\boldsymbol{s} = \boldsymbol{s}_a + \boldsymbol{s}'_a,$$

and then

$$\boldsymbol{r} + \boldsymbol{s} = \boldsymbol{r}_a + \boldsymbol{s}_a + \boldsymbol{r}'_a + \boldsymbol{s}'_a. \tag{9.32}$$

In this eq. (9.32) $\boldsymbol{r}_a$, $\boldsymbol{s}_a$ are both parallel to $\boldsymbol{a}$, while $\boldsymbol{r}'_a$, $\boldsymbol{s}'_a$ are both perpendicular to $\boldsymbol{a}$ in the plane of fig. 9.10, and are therefore parallel to each other. Thus $\boldsymbol{r}_a+\boldsymbol{s}_a$ is the component of $\boldsymbol{r}+\boldsymbol{s}$ parallel to $\boldsymbol{a}$, whilst $\boldsymbol{r}'_a+\boldsymbol{s}'_a$ is the component of $\boldsymbol{r}+\boldsymbol{s}$ perpendicular to $\boldsymbol{a}$, and we have immediately the results

$$(\boldsymbol{r} + \boldsymbol{s})_a = \boldsymbol{r}_a + \boldsymbol{s}_a, \tag{9.33}$$

$$(\boldsymbol{r} + \boldsymbol{s})'_a = \boldsymbol{r}'_a + \boldsymbol{s}'_a, \tag{9.34}$$

when all the vectors are parallel to the same plane. These results are, of course, particular examples of the eq. (9.31).

When *all* the vectors with which we are concerned are parallel to a single plane, then we can resolve all of them into components parallel to two perpendicular vectors in that plane. We use two *unit* vectors in the plane denoted by $\boldsymbol{i}$ and $\boldsymbol{j}$. We can then write every vector $\boldsymbol{r}$ in the form

$$\boldsymbol{r} = r_1\boldsymbol{i} + r_2\boldsymbol{j},$$

where r_1 and r_2 are real numbers, the components of $\boldsymbol{r}$ in the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$; represented by the segments AC, CB in fig. 9.11. Thus

$$r_1 = \mathrm{AC} = r \cos \widehat{\boldsymbol{ri}},$$

$$r_2 = \mathrm{CB} = r \sin \widehat{\boldsymbol{ri}} = r \cos \widehat{\boldsymbol{rj}}.$$

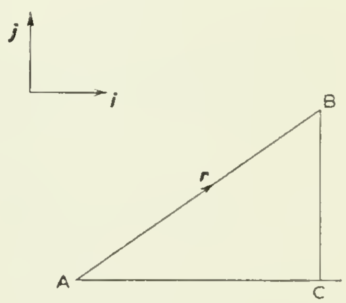

Fig. 9.11

Obviously from these results, or using Pythagoras' theorem in fig. 9.11,

$$r_1^2 + r_2^2 = r^2. \tag{9.35}$$

Further if $\boldsymbol{s}$ is any other vector parallel to the plane, then

$$\boldsymbol{s} = s_1\boldsymbol{i} + s_2\boldsymbol{j},$$

and

$$\boldsymbol{r} + \boldsymbol{s} = (r_1 + s_1)\boldsymbol{i} + (r_2 + s_2)\boldsymbol{j}.$$

Example 5

Given the following vectors:

$\boldsymbol{a}$ of magnitude 10, in direction North,
$\boldsymbol{b}$ of magnitude 20, in direction South East,

$\boldsymbol{c}$ of magnitude 8, in direction West,
$\boldsymbol{d}$ of magnitude 24, in direction 30° South of West,
$\boldsymbol{e}$ of magnitude 20, in direction North East,

evaluate the vector $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{d}+\boldsymbol{e}$.

Choosing unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ along the East and North directions respectively, we have immediately

$$\begin{aligned}
\boldsymbol{a} &= 10\boldsymbol{j},\\
\boldsymbol{b} &= 10\boldsymbol{i}\sqrt{2} - 10\boldsymbol{j}\sqrt{2},\\
\boldsymbol{c} &= -8\boldsymbol{i},\\
\boldsymbol{d} &= -24\boldsymbol{i}\cos 30° - 24\boldsymbol{j}\sin 30° = -12\boldsymbol{i}\sqrt{3} - 12\boldsymbol{j},\\
\boldsymbol{e} &= 10\boldsymbol{i}\sqrt{2} + 10\boldsymbol{j}\sqrt{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{d}+\boldsymbol{e} = {} & (10\sqrt{2} - 8 + 12\sqrt{3} + 10\sqrt{2})\boldsymbol{i}\\
& + (10 - 10\sqrt{2} + 12 + 10\sqrt{2})\boldsymbol{j} \approx 41.07\boldsymbol{i} + 22\boldsymbol{j}.
\end{aligned}$$

The magnitude of this vector is given by eq. (9.35), and its value is approximately

$$\sqrt{\{(41.07)^2 + (22)^2\}} \approx 46.59.$$

Its direction is approximately

$$\tan^{-1}\frac{22}{41.07} \approx 28°,$$

North of East.

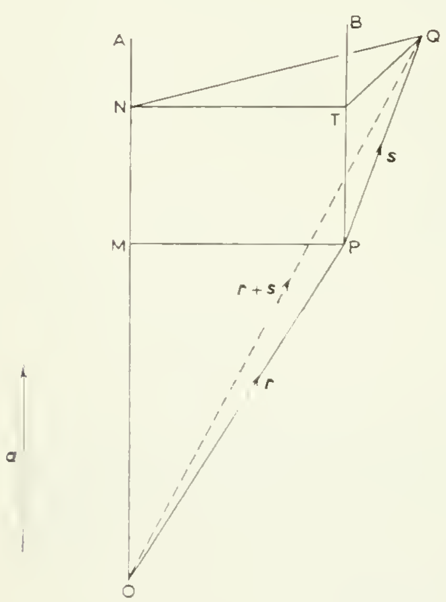

Fig. 9.12

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The results in eqs. (9.33) and (9.34) are true when $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{a}$ are not all parallel to the same plane; the diagram is then a three-dimensional one as shown in fig. 9.12. Suppose OA is parallel to the vector $\boldsymbol{a}$ and OP represents the vector $\boldsymbol{r}$, the plane of the paper being taken parallel to both $\boldsymbol{r}$ and $\boldsymbol{a}$. Draw PM perpendicular to OA, so that

$$r_a = OM, \qquad r'_a = MP. \tag{9.36}$$

Suppose $\boldsymbol{s}$ is not parallel to the plane of the paper; let PQ represent the vector $\boldsymbol{s}$. Draw PB parallel to $\boldsymbol{a}$ and from Q draw the perpendicular QT on to PB. Then

$$\boldsymbol{s} = PQ = PT + TQ = \boldsymbol{s}_a + \boldsymbol{s}'_a.$$

Draw TN perpendicular to OA, so that MPTN is a rectangle, and therefore

$$MP = NT \quad \text{and} \quad PT = MN.$$

We note immediately that $r'_a = MP = NT$ and $s'_a = TQ$ are both perpendicular to $\boldsymbol{a}$ but are not now parallel to each other. However, since PB is perpendicular to both NT and TQ, it is perpendicular to the plane NTQ. Therefore r'_a and s'_a both lie in the plane NTQ which is perpendicular to $\boldsymbol{a}$. Also NQ lies in this plane and is also perpendicular to $\boldsymbol{a}$. But $OQ = \boldsymbol{r} + \boldsymbol{s}$ and since NQ is perpendicular to $\boldsymbol{a}$ and ON parallel to $\boldsymbol{a}$, we have the following results

$$(r+s)_a = ON = OM + MN = r_a + s_a, \tag{9.37}$$

$$(r+s)'_a = NQ = NT + TQ = r'_a + s'_a. \tag{9.38}$$

EXERCISE 9.3

1. Three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are all parallel to the same plane and are defined in the following table:

	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
Magnitude	12	6.8	15.5
Direction	75°	310°	120°,

the directions being the angles measured counter-clockwise from a given direction $\boldsymbol{i}$ in the plane. Determine the following vector sums and differences in terms of components along $\boldsymbol{i}$ and along a perpendicular unit vector $\boldsymbol{j}$,

(i) $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$, (ii) $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$, (iii) $\boldsymbol{a} + \boldsymbol{b}$.

§ 3.4. Resolution into three components

THEOREM 3. *Every vector* $\mathbf{r}$ *can be resolved in a unique manner into three components parallel respectively to any three non-zero vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *which are not all parallel to the same plane.*

This means that if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ satisfy these conditions we can write any vector $\boldsymbol{r}$ as

$$\boldsymbol{r} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c}, \tag{9.39}$$

where λ, μ, ν are uniquely determined real numbers. Let OO′ represent the vector $\boldsymbol{r}$ as in fig. 9.13. Through O draw three lines OA, OB, OC parallel respectively to the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. These three lines will define planes parallel respectively to the pairs of vectors $(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{b}, \boldsymbol{c})$ and $(\boldsymbol{c}, \boldsymbol{a})$. Through O′ draw O′C′ parallel to $\boldsymbol{c}$ to meet the plane OAB in C′. Through C′ draw a line C′M parallel to $\boldsymbol{b}$ to meet OA in M. Then it is at once obvious from Theorem 1 that

$$\boldsymbol{r} = OO' = OM + MC' + C'O = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c}. \tag{9.40}$$

To show that λ, μ, ν are unique, we again assume that $\boldsymbol{r}$ could be written in the form

$$\boldsymbol{r} = \lambda'\boldsymbol{a} + \mu'\boldsymbol{b} + \nu'\boldsymbol{c}. \tag{9.41}$$

Then by subtraction we get

$$(\lambda - \lambda')\boldsymbol{a} = (\mu' - \mu)\boldsymbol{b} + (\nu' - \nu)\boldsymbol{c}. \tag{9.42}$$

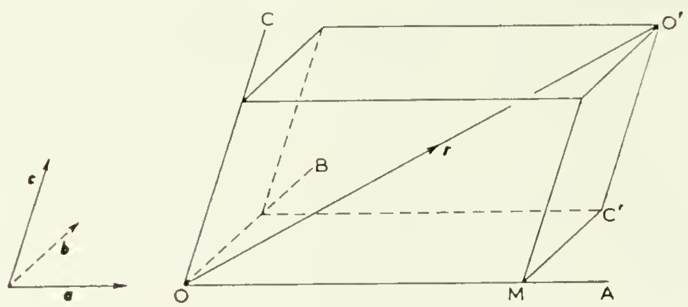

Fig. 9.13

But from Theorem 2, the right hand side of eq. (9.42) represents a vector parallel to the plane $(\boldsymbol{b}, \boldsymbol{c})$, while the left hand side is a vector parallel to $\boldsymbol{a}$. Since $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are non-zero and are not all parallel to the same plane, this result can only be true if $(\lambda-\lambda')\boldsymbol{a}=\boldsymbol{0}$, and hence $\lambda-\lambda'=0$. Similarly $\mu'-\mu=0$ and $\nu'-\nu=0$.

If now $\boldsymbol{s}$, $\boldsymbol{t}$,... are any other vectors which we can write in the forms

$$\boldsymbol{s} = \lambda_1\boldsymbol{a} + \mu_1\boldsymbol{b} + \nu_1\boldsymbol{c},$$
$$\boldsymbol{t} = \lambda_2\boldsymbol{a} + \mu_2\boldsymbol{b} + \nu_2\boldsymbol{c},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

then

$$\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t} + \ldots = (\lambda + \lambda_1 + \lambda_2 + \ldots)\boldsymbol{a} + (\mu + \mu_1 + \mu_2 + \ldots)\boldsymbol{b} + (\nu + \nu_1 + \nu_2 + \ldots)\boldsymbol{c}, \tag{9.43}$$

showing that vectors can be compounded by compounding their like components.

Again this resolution of a vector $\boldsymbol{r}$ into three components is especially important when $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are mutually perpendicular vectors. In this case we use three mutually perpendicular *unit* vectors which we denote by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and we also specify that in this order they form a *right-handed set* of directions.

This means that if $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are all drawn from the same point O as in

fig. 9.14, the sense of $\boldsymbol{k}$ is related to the direction of rotation from $\boldsymbol{i}$ to $\boldsymbol{j}$ through the smaller angle $(\frac{1}{2}\pi)$ between them, as advance is to rotation in a right hand screw. The same thing then applies to $\boldsymbol{i}, \boldsymbol{j}$ respectively in the cyclic interchange of the letters to $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{i}$ and $\boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}$. These vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are often referred to as a *triad of unit vectors*.

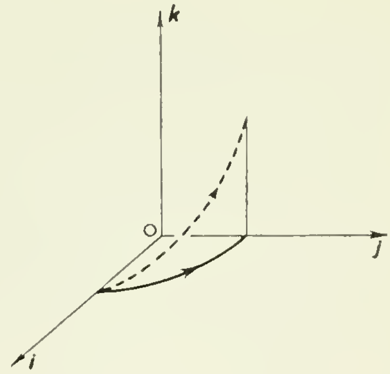

Fig. 9.14

We can then write *any* vector $\boldsymbol{r}$ in the form

$$\boldsymbol{r} = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}. \qquad (9.44)$$

When however $\boldsymbol{r}$ represents the position vector of any point P relative to a fixed origin O, as in fig. 9.15 we usually write it in the form

$$\boldsymbol{r} = \boldsymbol{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},$$

where $\boldsymbol{OP}$ is the diagonal of the rectangular parallelepiped shown. Then we have x=OL as the projection of OP in the direction $\boldsymbol{i}$, whilst similarly y, z are the projections of OP in the directions $\boldsymbol{j}, \boldsymbol{k}$ respectively. As we shall see in Ch. 10 x, y, z are the rectangular coordinates of the point P in three-dimensional analytical

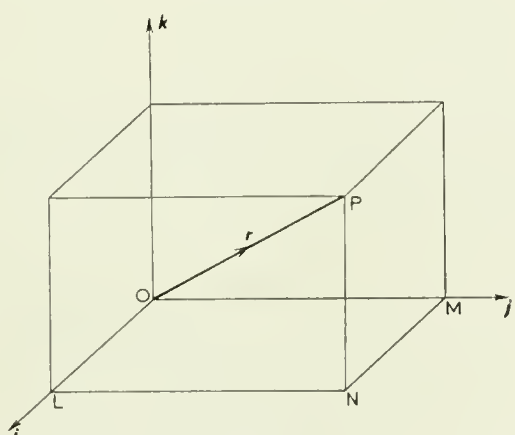

Fig. 9.15

geometry referred to axes Ox, Oy, Oz in the directions of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ respectively. It is obvious from the geometry of the diagram in fig. 9.15, that the length of the vector $\boldsymbol{r}=\boldsymbol{OP}$ is given by

$$r^2 = \mathrm{OP}^2 = \mathrm{OL}^2 + \mathrm{LP}^2 = \mathrm{OL}^2 + \mathrm{LN}^2 + \mathrm{NP}^2,$$

or

$$r^2 = x^2 + y^2 + z^2. \tag{9.45}$$

If $\boldsymbol{r}$ is a unit vector, then $r^2 = 1$, or

$$x^2 + y^2 + y^2 = 1, \tag{9.46}$$

so only two of the x, y, z are arbitrary, the third being determined, apart from sign, by eq. (9.46).

Example 6

Three vectors of lengths a, $2a$, $3a$ meet in a point and are directed along the diagonals of the three faces of a cube meeting at the point. Determine their resultant

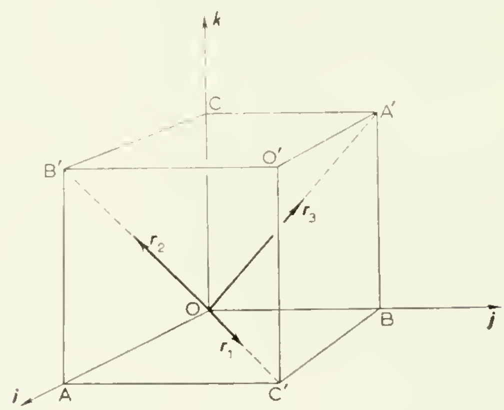

Fig. 9.16

in the form $\boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ and find its modulus. The cube is shown in fig. 9.16. Choose unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ along OA, OB, OC forming a right-handed set. Let the vector of length a be $\boldsymbol{r}_1$ along OC′, the vector of length $2a$ be $\boldsymbol{r}_2$ along OB′, and the vector of length $3a$ be $\boldsymbol{r}_3$ along OA′. Thus

$$\boldsymbol{r}_1 = a\boldsymbol{i} \cos \tfrac{1}{4}\pi + a\boldsymbol{j} \sin \tfrac{1}{4}\pi = \frac{a}{\sqrt{2}} (\boldsymbol{i} + \boldsymbol{j}),$$

$$\boldsymbol{r}_2 = 2a\boldsymbol{i} \cos \tfrac{1}{4}\pi + 2a\boldsymbol{k} \sin \tfrac{1}{4}\pi = \frac{2a}{\sqrt{2}} (\boldsymbol{i} + \boldsymbol{k}),$$

$$\boldsymbol{r}_3 = 3a\boldsymbol{j} \cos \tfrac{1}{4}\pi + 3a\boldsymbol{k} \sin \tfrac{1}{4}\pi = \frac{3a}{\sqrt{2}} (\boldsymbol{j} + \boldsymbol{k}).$$

The resultant $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 = \frac{a}{\sqrt{2}} (\boldsymbol{i} + \boldsymbol{j}) + \frac{2a}{\sqrt{2}} (\boldsymbol{i} + \boldsymbol{k}) + \frac{3a}{\sqrt{2}} (\boldsymbol{j} + \boldsymbol{k})$$

$$= \frac{3a}{\sqrt{2}} \boldsymbol{i} + \frac{4a}{\sqrt{2}} \boldsymbol{j} + \frac{5a}{\sqrt{2}} \boldsymbol{k},$$

and the modulus R of this vector is given by

$$R^2 = \tfrac{1}{2}a^2(9 + 16 + 25) = 25a^2,$$

or $R = 5a$.

EXERCISE 9.4

1. Find the sum of the vectors $2\boldsymbol{i}-5\boldsymbol{j}+\boldsymbol{k}$, $4\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k}$, $7\boldsymbol{i}-13\boldsymbol{j}+9\boldsymbol{k}$. Calculate the modulus of each vector and also the modulus of their resultant.

2. The position vectors of two points A, B are $5\boldsymbol{i}-8\boldsymbol{j}+3\boldsymbol{k}$, $-\boldsymbol{i}-2\boldsymbol{j}$ respectively. Find the vector $\boldsymbol{AB}$. A point P divides AB in the ratio 2 : 1, find $\boldsymbol{OP}$ and determine its modulus.

3. Prove that the points with position vectors $3\boldsymbol{i}+8\boldsymbol{j}-\boldsymbol{k}$, $8\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$, $-\boldsymbol{i}+3\boldsymbol{j}+8\boldsymbol{k}$, are at the vertices of an equilateral triangle. Find the position vector of the centroid of the triangle.

§ 4. Products of two vectors

Although we have proved that all vectors can be expressed as a sum of components along particular directions, and although this will enable us later to show the relationship between vectors and analytical geometry in two and three dimensions, it is the object in vector algebra and vector analysis to treat vectors without resolving them into components, as far as possible. We shall therefore begin by giving definitions and deriving properties of vectors without the use of components; later, we shall show what these same definitions and properties yield when the vectors are expressed in terms of their particular components along the triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

§ 4.1. The inner product of two vectors

The *inner product* or *scalar product* of a vector $\boldsymbol{a}$ by a vector $\boldsymbol{b}$, denoted by $\boldsymbol{a}\cdot\boldsymbol{b}$, is a *real number* defined by the equation

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab \cos \widehat{\boldsymbol{ab}}. \tag{9.47}$$

The fundamental properties of the inner product are:

(i) $\boldsymbol{a}\cdot\boldsymbol{0} = 0, \qquad \boldsymbol{0}\cdot\boldsymbol{a} = 0;$ (9.48)

and if $\boldsymbol{a}\cdot\boldsymbol{b}=0$, then either $\boldsymbol{a}=\boldsymbol{0}$ or $\boldsymbol{b}=\boldsymbol{0}$ or $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

(ii) $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$ (commutative law), (9.49)

(iii) $\lambda(\boldsymbol{a}\cdot\boldsymbol{b}) = (\lambda\boldsymbol{a})\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\lambda\boldsymbol{b}),$ (9.50)

(iv) $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}_b\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{b}_a,$ (9.51)

(v) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c},$ (distributive law). (9.52)

Properties (i) and (ii) are immediate consequences of the definition; so also is (iii) remembering eqs. (9.11) and (9.12).

Property (iv) is proved as follows: remembering (§ 3.3) that $\boldsymbol{b}_a$ is the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$, then its signed modulus b_a is given by $b_a = b\cos\widehat{\boldsymbol{ab}}$. Thus

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\widehat{\boldsymbol{ab}} = ab_a = \boldsymbol{a}\cdot\boldsymbol{b}_a. \tag{9.53}$$

Similarly for $\boldsymbol{a}_b$, the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

To prove property (v) we use eqs. (9.33) and (9.53). We have

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})_a = \boldsymbol{a}\cdot(\boldsymbol{b}_a+\boldsymbol{c}_a);$$

since $\boldsymbol{b}_a$ and $\boldsymbol{c}_a$ are both parallel to $\boldsymbol{a}$, then

$$|\boldsymbol{b}_a+\boldsymbol{c}_a| = \pm\,(b_a+c_a),$$

the $\pm$ sign depending on the relative sense of $\boldsymbol{a}$ and $\boldsymbol{b}_a+\boldsymbol{c}_a$. Thus

$$\boldsymbol{a}\cdot(\boldsymbol{b}_a+\boldsymbol{c}_a) = a(b_a+c_a) = ab_a+ac_a = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}.$$

This result can of course, be extended to any number of vectors, and because of property (ii) the order may also be reversed, so that we have results such as

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{d}, \tag{9.54}$$

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c}, \tag{9.55}$$

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\cdot(\boldsymbol{e}+\boldsymbol{f}+\boldsymbol{g}) &= \boldsymbol{a}\cdot\boldsymbol{e}+\boldsymbol{a}\cdot\boldsymbol{f}+\boldsymbol{a}\cdot\boldsymbol{g}\\ &\quad+\boldsymbol{b}\cdot\boldsymbol{e}+\boldsymbol{b}\cdot\boldsymbol{f}+\boldsymbol{b}\cdot\boldsymbol{g}+\boldsymbol{c}\cdot\boldsymbol{e}+\boldsymbol{c}\cdot\boldsymbol{f}+\boldsymbol{c}\cdot\boldsymbol{g}.\end{aligned} \tag{9.56}$$

Example 7

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are mutually perpendicular vectors of equal magnitude, then $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ is equally inclined to each of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Let $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{r}$. Then

$$\boldsymbol{r}\cdot\boldsymbol{a} = (\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\cdot\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{a}+\boldsymbol{c}\cdot\boldsymbol{a},$$

and since $\cos\widehat{\boldsymbol{aa}}=\cos 0=1$, whilst $\cos\widehat{\boldsymbol{ab}}=\cos\widehat{\boldsymbol{ac}}=\cos\frac{1}{2}\pi=0$, this gives

$$\boldsymbol{r}\cdot\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{a} = a^2.$$

But $\boldsymbol{r}\cdot\boldsymbol{a}=ra\cos\widehat{\boldsymbol{ra}}$ and so

$$\cos\widehat{\boldsymbol{ra}} = a/r;$$

similarly

$$\cos\widehat{\boldsymbol{rb}} = b/r, \quad \text{and} \quad \cos\widehat{\boldsymbol{rc}} = c/r.$$

But $a=b=c$ and therefore

$$\cos\widehat{\boldsymbol{ra}} = \cos\widehat{\boldsymbol{rb}} = \cos\widehat{\boldsymbol{rc}},$$

so that

$$\widehat{\boldsymbol{ra}} = \pm\widehat{\boldsymbol{rb}} = \pm\widehat{\boldsymbol{rc}},$$

or $\boldsymbol{r}$ is equally inclined to $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

We note now that the three mutually perpendicular vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ obey the following relations, derived directly from eq. (9.47):

$$\boldsymbol{i}\cdot\boldsymbol{i} = 1, \qquad \boldsymbol{j}\cdot\boldsymbol{j} = 1, \qquad \boldsymbol{k}\cdot\boldsymbol{k} = 1, \tag{9.57}$$

$$\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{i} = 0, \qquad \boldsymbol{i}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0, \qquad \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{j} = 0. \tag{9.58}$$

Using these relations we can determine the value of the inner product $\boldsymbol{a}\cdot\boldsymbol{b}$ when both $\boldsymbol{a}$ and $\boldsymbol{b}$ are expressed in terms of their components along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. Suppose

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \qquad \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k},$$

so that

$$\boldsymbol{a}\cdot\boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})\cdot(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}).$$

Expanding this result in the form of eq. (9.56) and using eqs. (9.57) and (9.58), we find

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{9.59}$$

Note also that in a similar way

$$\boldsymbol{a}\cdot\boldsymbol{i} = a\cos\widehat{\boldsymbol{ai}} = a_1,$$

is the projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{i}$ whilst $a_2=\boldsymbol{a}\cdot\boldsymbol{j}$, $a_3=\boldsymbol{a}\cdot\boldsymbol{k}$ are the projections of $\boldsymbol{a}$ in the directions of $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. Thus we can formally write

$$\boldsymbol{a} = (\boldsymbol{a}\cdot\boldsymbol{i})\boldsymbol{i} + (\boldsymbol{a}\cdot\boldsymbol{j})\boldsymbol{j} + (\boldsymbol{a}\cdot\boldsymbol{k})\boldsymbol{k}. \tag{9.60}$$

Further, if we write $\boldsymbol{a}\cdot\boldsymbol{a}=\boldsymbol{a}^2$ then from eq. (9.47) $\boldsymbol{a}^2=a^2$ so that $a=(\boldsymbol{a}^2)^{\frac{1}{2}}$, this again enables us to determine the modulus a of a vector $\boldsymbol{a}$ in terms of its components, since

$$\boldsymbol{a}^2 = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})\cdot(a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = a_1^2 + a_2^2 + a_3^2, \tag{9.61}$$

so that

$$a = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}. \tag{9.62}$$

We have already noted this result in eq. (9.45).

In addition,

$$(\boldsymbol{a} \pm \boldsymbol{b})^2 = (\boldsymbol{a} \pm \boldsymbol{b}) \cdot (\boldsymbol{a} \pm \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a} \pm 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b},$$

which can be written as

$$(\boldsymbol{a} \pm \boldsymbol{b})^2 = \boldsymbol{a}^2 \pm 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b}^2 \tag{9.63}$$

This result is analogous to the expansion of $(a \pm b)^2$ when a and b are numbers, but we must remember that each term on the right hand side of eq. (9.63) is a scalar product.

Example 8

Show that the three vectors

$$\begin{aligned} \boldsymbol{a} &= 3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}, \\ \boldsymbol{b} &= -7\boldsymbol{i} - 12\boldsymbol{j} - 9\boldsymbol{k}, \\ \boldsymbol{c} &= -39\boldsymbol{i} + 31\boldsymbol{j} - 11\boldsymbol{k} \end{aligned}$$

are mutually orthogonal.

Using eq. (9.59) we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 3(-7) + 2(-12) + (-5)(-9) = -21 - 24 + 45 = 0,$$

so that from property (i) eq. (9.48) $\boldsymbol{a}$ and $\boldsymbol{b}$ must be at right angles. Similarly

$$\boldsymbol{a} \cdot \boldsymbol{c} = 3(-39) + 2(31) + 5(11) = 0,$$

and

$$\boldsymbol{b} \cdot \boldsymbol{c} = +7(39) - 12(31) + 9(11) = 0,$$

so that the three vectors are mutually orthogonal.

EXERCISE 9.5

1. Find the modulus of each of the vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ and the unit vectors in the direction of each. Find also the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

2. If $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = -4\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$, verify that

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}.$$

3. If the position vectors of two points A, B relative to O are $2\boldsymbol{i} - 7\boldsymbol{j} - 13\boldsymbol{k}$, $8\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$, find the position vector of P, the mid point of AB. Find also the angles between $\boldsymbol{OP}$ and $\boldsymbol{OA}$ and between $\boldsymbol{OP}$ and $\boldsymbol{OB}$. What is the unit vector in the direction of $\boldsymbol{OP}$?

The following questions should be answered without the use of components.

4. Three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are such that $(\boldsymbol{c}-\frac{1}{2}\boldsymbol{a})\cdot\boldsymbol{a}=(\boldsymbol{c}-\frac{1}{2}\boldsymbol{b})\cdot\boldsymbol{b}$; prove that $\boldsymbol{c}-\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$ is perpendicular to $\boldsymbol{a}-\boldsymbol{b}$.

5. If A, B, C, D are any four points in space, prove that

$$DA\cdot BC + DB\cdot CA + DC\cdot AB = 0.$$

6. C, D are points that divide AB internally and externally in the ratio $p : q$. Prove that

$$OC\cdot OD = \frac{q^2a^2 - p^2b^2}{q^2 - p^2},$$

where $OA=\boldsymbol{a}$, $OB=\boldsymbol{b}$. Deduce that the vectors OC, OD are perpendicular if $bp=aq$ and interpret the result geometrically.

7. If O is the centroid of the triangle ABC and P any other point, prove that

$$\text{AP}^2 + \text{BP}^2 + \text{CP}^2 = \text{AO}^2 + \text{BO}^2 + \text{CO}^2 + 3\text{OP}^2.$$

8. If one pair of opposite edges of a tetrahedron are equal in length and both perpendicular to the line joining their mid points, prove that the same is true of the other pairs of opposite sides.

9. Using the distributive law, eq. (9.52), for the scalar product of two vectors, prove by vector methods that the perpendiculars from the three vertices of a triangle to the opposite sides are concurrent.

§ 4.2. The vector product of two vectors

The *vector product* of a vector $\boldsymbol{a}$ by a second vector $\boldsymbol{b}$, denoted by $\boldsymbol{a}\times\boldsymbol{b}$ is a *vector* defined in the following way

(i) the modulus or magnitude of the vector is $ab \sin \widehat{\boldsymbol{ab}}$, equal to the area of the parallelogram in fig. 9.17 of which $OA=\boldsymbol{a}$ and $OB=\boldsymbol{b}$ are adjacent sides.

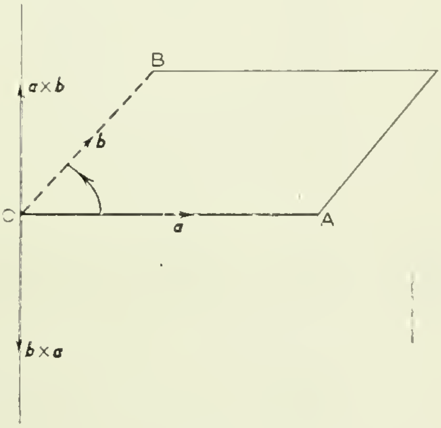

Fig. 9.17

(ii) the direction of the vector $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to the directions of both $\boldsymbol{a}$ and $\boldsymbol{b}$, that is to the plane AOB which is parallel to both vectors.

(iii) the sense of $\boldsymbol{a}\times\boldsymbol{b}$ is such that $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}\times\boldsymbol{b}$ in this order form a right-handed system (§ 3.4) as shown in fig. 9.17; $\boldsymbol{a}$ and $\boldsymbol{a}\times\boldsymbol{b}$ are drawn in the plane of the paper and $\boldsymbol{b}$ is thought of as being behind the paper.

The fundamental properties of the vector product are:

$$\text{(i)}\quad \boldsymbol{a} \times \mathbf{0} = \mathbf{0} \times \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{a} = \mathbf{0}, \tag{9.64}$$

and $\boldsymbol{a}\times\boldsymbol{b}=\mathbf{0}$ means that either $\boldsymbol{a}=\mathbf{0}$, $\boldsymbol{b}=\mathbf{0}$ or $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$.

$$\text{(ii)}\quad \lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda\boldsymbol{b}), \tag{9.65}$$

$$\text{(iii)}\quad \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}, \tag{9.66}$$

$$\text{(iv)}\quad \begin{aligned} &\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} \\ &(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}. \end{aligned} \tag{9.67}$$

Properties (i) and (ii) result directly from the definition. Property (iii) also follows at once, the difference in sign between $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{b}\times\boldsymbol{a}$ being due to (iii) in the definition of the vector product; this property implies that the commutative law of algebra does not apply to vector products, so that care must be taken in preserving the order of the letters in a vector product. In particular, we must be careful in formulating the distributive laws (iv) for vector products; in the first equation of (iv) $\boldsymbol{a}$ must be in front of both $\boldsymbol{b}$ and $\boldsymbol{c}$ on the right hand side, whilst in the second equation $\boldsymbol{a}$ and $\boldsymbol{b}$ must be in front of $\boldsymbol{c}$. Formulae (iv) are the only ones requiring proof and if the first of these is true then the second can be derived from it by using property (iii).

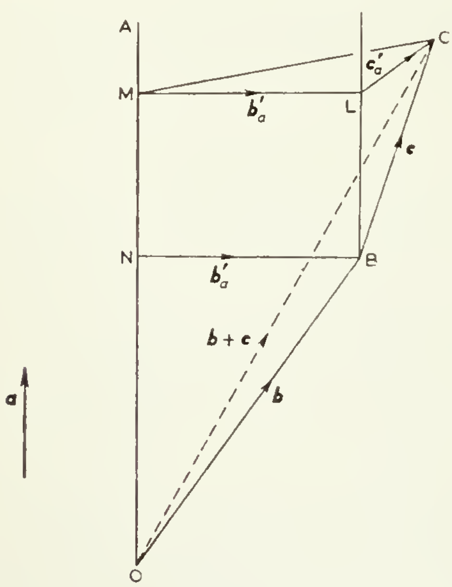

Fig. 9.18

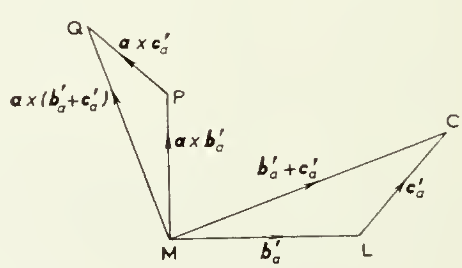

Fig. 9.19

★ To prove (iv) consider fig. 9.18 similar to fig. 9.12 in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are drawn in the plane of the paper and $\boldsymbol{c}$ behind. Then the magnitude of $\boldsymbol{b}'_a$ the component of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$ is $b'_a = \text{NB} = b \sin \widehat{\boldsymbol{ab}}$ and so

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}'_a, \tag{9.68}$$

the directions and senses of both these vectors being perpendicular to ONB and into the plane as it is drawn in fig. 9.18. Thus

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})'_a,$$

and using eq. (9.38) with $\boldsymbol{b}$ and $\boldsymbol{c}$ in place of $\boldsymbol{r}$ and $\boldsymbol{s}$, we get

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})'_a = \boldsymbol{a} \times (\boldsymbol{b}'_a + \boldsymbol{c}'_a). \tag{9.69}$$

Now consider fig. 9.19 in which the plane MLC is drawn in the plane of the paper, and $\boldsymbol{a}$ is normal to MLC and outwards. If this triangle is turned through a right angle in the counterclockwise sense and its sides multiplied in magnitude by a to give MPQ, then since $\boldsymbol{a}$ is perpendicular to all vectors in this plane and outwards, we have

$$\begin{aligned} MP &= \boldsymbol{a} \times ML = \boldsymbol{a} \times \boldsymbol{b}'_a, \\ MQ &= \boldsymbol{a} \times MC = \boldsymbol{a} \times (\boldsymbol{b}'_a + \boldsymbol{c}'_a), \\ PQ &= \boldsymbol{a} \times LC = \boldsymbol{a} \times \boldsymbol{c}'_a. \end{aligned}$$

But in the triangle MPQ

$$MQ = MP + PQ,$$

so that

$$\boldsymbol{a} \times (\boldsymbol{b}'_a + \boldsymbol{c}'_a) = \boldsymbol{a} \times \boldsymbol{b}'_a + \boldsymbol{a} \times \boldsymbol{c}'_a,$$

and eqs. (9.68) and (9.69) then give

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}.$$ ★

Example 9

If $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{0}$ prove that $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{b}\times\boldsymbol{c}=\boldsymbol{c}\times\boldsymbol{a}$ and interpret the result geometrically.

Given

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0},$$

then multiplying through vectorially by $\boldsymbol{a}$ we get

$$\boldsymbol{a} \times (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{0},$$

or

$$\boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{0}.$$

But $\boldsymbol{a}\times\boldsymbol{a}=\boldsymbol{0}$ and $\boldsymbol{a}\times\boldsymbol{c}=-\boldsymbol{c}\times\boldsymbol{a}$, therefore

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}.$$

Similarly by multiplying through vectorially by $\boldsymbol{b}$ we get

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}.$$

The geometrical interpretation is, that because $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$ the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ can be represented by the sides of a triangle ABC taken in order as shown in fig. 9.20. Then each of the vectors $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\times\boldsymbol{a}$ has magnitude equal to twice the area of the triangle ABC, is normal to ABC and in sense out of the paper as drawn in fig. 9.20.

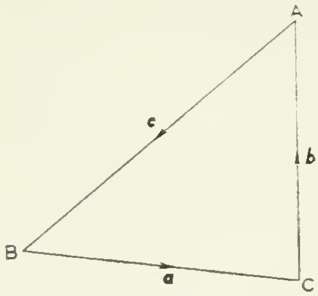

Fig. 9.20

The triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ satisfy the following relations which follow directly from the definition of the vector product

$$\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k},\quad \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i},\quad \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}, \tag{9.70}$$

$$\boldsymbol{i}\times\boldsymbol{i}=\mathbf{0},\quad \boldsymbol{j}\times\boldsymbol{j}=\mathbf{0},\quad \boldsymbol{k}\times\boldsymbol{k}=\mathbf{0}. \tag{9.71}$$

We can now determine the form of the vector products in terms of components along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. If

$$\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k},\quad\text{and}\quad \boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k},$$

then

$$\boldsymbol{a}\times\boldsymbol{b}=(a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k})\times(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}),$$

and using the distributive law this becomes

$$\begin{aligned}&a_1b_1\ \boldsymbol{i}\times\boldsymbol{i}+a_1b_2\ \boldsymbol{i}\times\boldsymbol{j}+a_1b_3\ \boldsymbol{i}\times\boldsymbol{k}\\ &+a_2b_1\ \boldsymbol{j}\times\boldsymbol{i}+a_2b_2\ \boldsymbol{j}\times\boldsymbol{j}+a_2b_3\ \boldsymbol{j}\times\boldsymbol{k}\\ &+a_3b_1\ \boldsymbol{k}\times\boldsymbol{i}+a_3b_2\ \boldsymbol{k}\times\boldsymbol{j}+a_3b_3\ \boldsymbol{k}\times\boldsymbol{k},\end{aligned}$$

so that with eqs. (9.70) and (9.71), we get

$$\boldsymbol{a}\times\boldsymbol{b}=(a_2b_3-a_3b_2)\boldsymbol{i}+(a_3b_1-a_1b_3)\boldsymbol{j}+(a_1b_2-a_2b_1)\boldsymbol{k}. \tag{9.72}$$

Example 10

Find a unit vector perpendicular to each of the vectors

$$\boldsymbol{a}=2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k},\qquad \boldsymbol{b}=3\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}.$$

By definition the vector $\boldsymbol{a}\times\boldsymbol{b}$ is a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. We have from eq. (9.72)

$$\boldsymbol{a}\times\boldsymbol{b}=\{(-1)(-1-1(4)\}\boldsymbol{i}+\{1(3)-2(-1)\}\boldsymbol{j}+\{2(4)-(-1)3\}\boldsymbol{k}=-3\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k}.$$

We note that the scalar product of this vector with either $\boldsymbol{a}$ or $\boldsymbol{b}$ does give zero, so that it is perpendicular to each of them.

The magnitude of the vector $-3\boldsymbol{i}+5\boldsymbol{j}+11\boldsymbol{k}$ is

$$(9+25+121)^{\frac{1}{2}}=(155)^{\frac{1}{2}},$$

and therefore a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$(-3\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k})/(155)^{\frac{1}{2}}$$

EXERCISE 9.6

1. Given two points A, B having position vectors $4\boldsymbol{i}-8\boldsymbol{j}+\boldsymbol{k}$, $7\boldsymbol{i}-5\boldsymbol{k}$ relative to an origin O, find the position vector of the point P dividing AB in the ratio 2 : 1. Find a vector perpendicular to $\boldsymbol{OP}$ and $\boldsymbol{AB}$ and show that this is in the same direction as a vector perpendicular to $\boldsymbol{OA}$ and $\boldsymbol{OB}$.

2. Prove that the area of the triangle whose vertices have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is half the modulus of the vector

$$\boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}.$$

3. A line makes angles α, β, γ, δ with the four diagonals of a cube. Prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \tfrac{4}{3}.$$

4. A unit vector $\boldsymbol{v}$ makes angles α, β with two directions defined by unit vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ where $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \cos\theta$. Show that $\boldsymbol{v}$ is given by

$$\boldsymbol{v} = \lambda \boldsymbol{u}_1 + \mu \boldsymbol{u}_2 + \nu \boldsymbol{u}_1 \times \boldsymbol{u}_2,$$

where λ, μ, ν are given by

$$\begin{aligned} \lambda \sin^2\theta &= \cos\alpha - \cos\beta\cos\theta, \\ \mu \sin^2\theta &= \cos\beta - \cos\alpha\cos\theta, \\ \nu^2 \sin^4\theta &= \sin^2\theta - \cos^2\alpha - \cos^2\beta + 2\cos\alpha\cos\beta\cos\theta. \end{aligned}$$

§ 5. Division by a vector

Since a vector can be multiplied by another vector to give either a scalar or a vector, we would expect to be able to divide by vectors in two different ways. There is however no convenient and useful definition of division by a vector. To see why this is so let us consider the possible definitions of 'vector division' as the inverse of scalar and vector multiplication.

(i) If λ is a scalar and we want to divide λ by a vector $\boldsymbol{a}$, forming a quotient $\lambda/\boldsymbol{a}$, then we could define this quotient to be a vector $\boldsymbol{b}$ satisfying

$$\lambda = \boldsymbol{a} \cdot \boldsymbol{b}, \tag{9.73}$$

this being a sensible equation in vector algebra. However if $\boldsymbol{c}$ is any vector perpendicular to $\boldsymbol{a}$, then $\boldsymbol{a} \cdot \boldsymbol{c} = 0$. So

$$\boldsymbol{a} \cdot \boldsymbol{b} + \mu \boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} = \lambda,$$

or

$$\boldsymbol{a}\cdot(\boldsymbol{b} + \mu\boldsymbol{c}) = \lambda,$$

giving by the above definition

$$\boldsymbol{b} + \mu\boldsymbol{c} = \lambda/\boldsymbol{a}.$$

In other words, the quotient $\lambda/\boldsymbol{a}$ is not unique since any vector $\boldsymbol{b}+\mu\boldsymbol{c}$ also satisfies the eq. (9.73) provided $\boldsymbol{a}\cdot\boldsymbol{c}=0$.

(ii) Let $\boldsymbol{a}$ and $\boldsymbol{c}$ be two given vectors, then consider a quotient $\boldsymbol{b}=\boldsymbol{a}/\boldsymbol{c}$ defined by the relation

$$\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c},$$

which again is a legitimate vector equation. One trouble with this definition is that $\boldsymbol{a}$ is necessarily perpendicular to $\boldsymbol{c}$, so that the equation does not define a quotient for *any* two given vectors $\boldsymbol{a}$ and $\boldsymbol{c}$. Consequently no general vector $\boldsymbol{b}$ can be found to satisfy this equation when $\boldsymbol{a}$ and $\boldsymbol{c}$ are not at right angles.

Suppose however $\boldsymbol{a}$ and $\boldsymbol{c}$ are at right angles; then again $\boldsymbol{b}+\mu\boldsymbol{c}$ would also satisfy the equation since

$$(\boldsymbol{b} + \mu\boldsymbol{c}) \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c} + \mu\boldsymbol{c} \times \boldsymbol{c},$$

and $\boldsymbol{c}\times\boldsymbol{c}=0$. Thus when a quotient $\boldsymbol{b}$ does exist, it is not unique.

Thus the operation of division by a vector has no place in vector algebra.

§ 6. Products of three or more vectors

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are any three vectors, then $\boldsymbol{b}\times\boldsymbol{c}$ is itself a vector and we can multiply it by $\boldsymbol{a}$ in two different ways, either scalarly as $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ or vectorially as $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$.

§ 6.1. The triple scalar product

The *triple scalar product* $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ of any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ has the following properties

(i) $$\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = 0, \tag{9.74}$$

when $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are all parallel to the same plane.

(ii) $$\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a} \times \boldsymbol{b}), \tag{9.75}$$

$$\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{b}\cdot(\boldsymbol{a} \times \boldsymbol{c}) = -\boldsymbol{c}\cdot(\boldsymbol{b} \times \boldsymbol{a}). \tag{9.76}$$

Property (i) is obvious from the definitions of scalar and vector products, since if we represent $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ by the displacements OA, OB, OC in fig. 9.21, then under the conditions of (i) O, A, B, C are coplanar; thus $\boldsymbol{b}\times\boldsymbol{c}$ is perpendicular to this plane and therefore perpendicular to $\boldsymbol{a}$, making $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ zero.

In particular we note that if two of the vectors in a triple scalar product are equal or parallel, the conditions of (i) are satisfied and the product is zero. For example,

$$\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{c}) = 0, \qquad \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{a}) = 0. \tag{9.77}$$

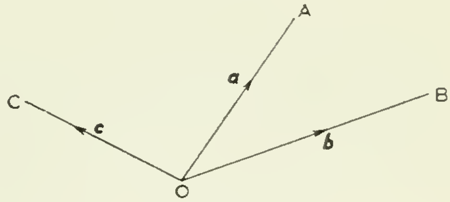

Fig. 9.21

In the case of property (ii), we first note that eq. (9.75) means that the positions of the three vectors in a triple scalar product can be altered provided we preserve their cyclic order, while eq. (9.76) means that if we interchange two of the vectors the product changes sign. Equation (9.75) may be deduced from eq. (9.76) since $\boldsymbol{c}\times\boldsymbol{a}=-\boldsymbol{a}\times\boldsymbol{c}$, and so on.

To prove eq. (9.76) we use the result that

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\{(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}\} = 0,$$

using eq. (9.77). But

$$(\boldsymbol{a}+\boldsymbol{b})\cdot\{(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}\} = (\boldsymbol{a}+\boldsymbol{b})\cdot\{\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}\},$$

and this becomes

$$\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{c}) + \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c}) + \boldsymbol{b}\cdot(\boldsymbol{b}\times\boldsymbol{c}).$$

Since the first and last terms are again zero by eq. (9.77), we get

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = -\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}).$$

The result $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ may be proved in a similar way. Also since scalar products are commutative we may extend eq. (9.75) by writing

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}. \tag{9.78}$$

Thus provided the cyclic order is maintained, a triple scalar product is independent of the position of the dot and cross symbols; hence these symbols may be interchanged. It is therefore usual to denote this product by $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$, which simply indicates the three vectors and their cyclic order.

The geometrical significance of the results in eq. (9.75), (9.76) and

(9.78 can be seen by taking three vector displacements OA, OB, OC representing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ respectively, and forming the parallelipiped of which they are the co-terminous edges as in fig. 9.22. Then $\boldsymbol{a}\times\boldsymbol{b}$ is a vector represented by $OD=\boldsymbol{d}$ (say) perpendicular to the plane OAB and equal in magnitude to the area of the base of the parallelepiped, that is the parallelogram OAC′B. Therefore

$$(\boldsymbol{a} \times \boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{d}\cdot\boldsymbol{c} = dc \cos \widehat{\boldsymbol{cd}}. \tag{9.79}$$

But $\boldsymbol{d}$ being perpendicular to the base, $c \cos \widehat{\boldsymbol{cd}}$ is the height of the parallelepiped; and so $dc \cos \widehat{\boldsymbol{cd}}$ is the volume of the parallelepiped. Thus $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ is numerically equal to the volume of the parallelepiped;

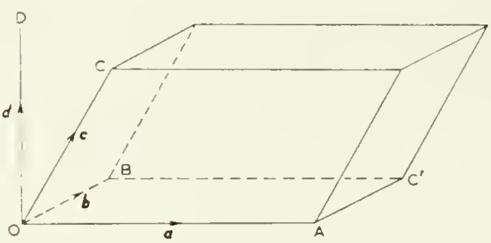

Fig. 9.22

its sign may of course be positive or negative since $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are not necessarily disposed as shown in fig. 9.22, so that $\boldsymbol{d}$ may be in the opposite sense to that shown. Similarly the moduli of the other products in eqs. (9.75), (9.76) and (9.78) are equal to the volume of the parallelepiped.

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are given in terms of their components along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ as

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k},$$
$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k},$$
$$\boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k},$$

then

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1). \tag{9.80}$$

Example 11

Evaluate $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ when

$$\boldsymbol{a} = 9\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k},$$
$$\boldsymbol{b} = -5\boldsymbol{i} + 6\boldsymbol{j} - 11\boldsymbol{k},$$
$$\boldsymbol{c} = 8\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}.$$

Using eq. (9.80) we have

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = 9(6\cdot 7 - 4\cdot 11) + (-11\cdot 8 + 5\cdot 7) - 2(5\cdot 4 - 8\cdot 6) = -15.$$

Example 12

Verify the result $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]=0$ when $\boldsymbol{c}$ is parallel to the plane of $\boldsymbol{a}=2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{b}=4\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$.

If $\boldsymbol{c}$ is parallel to these two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then by Theorem 2 (§ 3.2) it must be of the form $\lambda\boldsymbol{a}+\mu\boldsymbol{b}$. Take $\lambda=2$, $\mu=-1$ in this result for instance and let

$$\boldsymbol{c} = 2\boldsymbol{a} - \boldsymbol{b} = 0\boldsymbol{i} - 4\boldsymbol{j} + 8\boldsymbol{k}.$$

Using eq. (9.80) it is easy to see that $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]=0$.

§ 6.2. The triple vector product

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are any three vectors, then the vector $\boldsymbol{b}\times\boldsymbol{c}$ can be multiplied vectorially by $\boldsymbol{a}$ to give the *vector* $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$. This vector is by definition perpendicular to the vector $\boldsymbol{b}\times\boldsymbol{c}$; this is itself perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$, so that $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ must lie in the plane parallel to $\boldsymbol{b}$ and $\boldsymbol{c}$ as shown in fig. 9.23; so by Theorem 2, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ can be written in the form

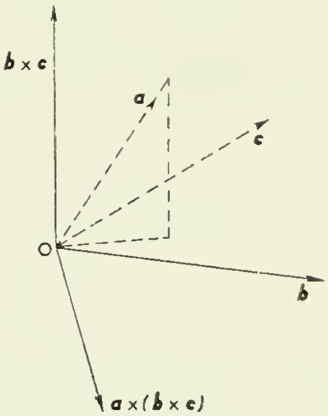

Fig. 9.23

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \lambda\boldsymbol{b} + \mu\boldsymbol{c},$$

where λ and μ are scalars. We shall in fact show that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}, \tag{9.81}$$

the scalars λ, μ being the scalar products $\boldsymbol{a}\cdot\boldsymbol{c}$, $-\boldsymbol{a}\cdot\boldsymbol{b}$ respectively. This is a very important formula, by means of which, together with the other properties of scalar and vectors products which we have already given-we are enabled to simplify all other more complicated products of vectors,

To establish formula (9.81) we express $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in terms of com. ponents along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \qquad \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}, \qquad \boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}.$$

Then since from eq. (9.72)

$$\boldsymbol{b} \times \boldsymbol{c} = (b_2c_3 - b_3c_2)\boldsymbol{i} + (b_3c_1 - b_1c_3)\boldsymbol{j} + (b_1c_2 - b_2c_1)\boldsymbol{k},$$

we have, using eq. (9.72) again

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = &\{a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\}\,\boldsymbol{i} \\ &+ \{a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)\}\,\boldsymbol{j} \\ &+ \{a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\}\,\boldsymbol{k}. \end{aligned} \tag{9.82}$$

Let us now compare the terms in $\boldsymbol{i}$ with those in $\boldsymbol{i}$ on the right hand side of eq. (9.81) which are

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1\boldsymbol{i} - (a_1b_1 + a_2b_2 + a_3b_3)c_1\boldsymbol{i}. \qquad (9.83)$$

It is easy to verify that the terms in $\boldsymbol{i}$ in eq. (9.82) and result (9.83) are the same. The same is true of the terms in $\boldsymbol{j}$ and in $\boldsymbol{k}$ on the right and left hand sides of eq. (9.81).

The formula (9.81) can be proved without resolving the vectors into components †.

Since by eq. (9.66) we have

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{a});$$

so by interchanging $\boldsymbol{c}$ and $\boldsymbol{a}$ in eq. (9.81) we have

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b} - (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}, \qquad (9.84)$$

a result which is quite different from eq. (9.81) for $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$. Thus the associative law of algebra does not hold for vector products and care must be taken in the grouping of the vectors in these products.

Example 13

Evaluate $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ and $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ when

$$\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}, \qquad \boldsymbol{b} = -4\boldsymbol{i} + 7\boldsymbol{j} - 12\boldsymbol{k}, \qquad \boldsymbol{c} = 3\boldsymbol{i} - 8\boldsymbol{j} - \boldsymbol{k}.$$

Using eq. (9.81) for $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$, we have

$$\boldsymbol{a}\cdot\boldsymbol{c} = 2\cdot 3 + (-3)(-8) + 5(-1) = 25,$$

and

$$\boldsymbol{a}\cdot\boldsymbol{b} = 2(-4) + 7(-3) + 5(-12) = -89.$$

Hence

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = 25\boldsymbol{b} - 89\boldsymbol{c},$$

which becomes

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = -367\boldsymbol{i} + 887\boldsymbol{j} - 211\boldsymbol{k}.$$

On the other hand, using eq. (9.84) for $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ we have

$$\boldsymbol{c}\cdot\boldsymbol{b} = 3(-4) - 8\cdot 7 - 1(-12) = -56,$$

and so

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = 25\boldsymbol{b} + 56\boldsymbol{a},$$

which gives

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = 12\boldsymbol{i} + 7\boldsymbol{j} - 20\boldsymbol{k}.$$

† There are, in fact, many proofs of this formula. Two of the simplest are given by Livens [1949] and Talbot [1955].

As we have already stated, eq. (9.81) for the triple vector product enables us to simplify more complicated products involving four or more vectors. We will illustrate the methods of dealing with such products by examples.

Example 14

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are any four vectors, show that

$$(\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}). \tag{9.85}$$

Treating the left hand side of this result as a triple scalar product of the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}\times\boldsymbol{d}$ we can interchange the cross and the dot in this product to give

$$(\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{a}\cdot\{\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})\}.$$

Now using eq. (9.81) on the triple vector product $\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})$, the right hand side becomes

$$\boldsymbol{a}\cdot\{(\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d}\},$$

which is

$$(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}),$$

remembering that $\boldsymbol{b}\cdot\boldsymbol{d}$ and $\boldsymbol{b}\cdot\boldsymbol{c}$ inside the brace brackets are simply scalars or real numbers.

Example 15

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are any four vectors, show that

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d}, \tag{9.86}$$

or

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{a}. \tag{9.87}$$

Using eq. (9.81) on the three vectors $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ we derive the result in eq. (9.86). Alternatively, using eq. (9.84) on the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}\times\boldsymbol{d}$ we derive the result in eq. (9.87).

Then the right hand sides of eqs. (9.86) and (9.87) must be equal. So using eq. (9.75) we see that any four vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ satisfy the following identity

$$[\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{a} - [\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}]\boldsymbol{b} + [\boldsymbol{d}, \boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d} \equiv \boldsymbol{0}; \tag{9.88}$$

this equation has been written so that the cyclic order $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ is the same in each term. The identity can also be written as

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d} \equiv [\boldsymbol{d}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} + [\boldsymbol{d}, \boldsymbol{c}, \boldsymbol{a}]\boldsymbol{b} + [\boldsymbol{d}, \boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}. \tag{9.89}$$

If we suppose that $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] \neq 0$ so that $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are not all parallel to the same plane, then this last identity expresses the vector $\boldsymbol{d}$ in terms of the

three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Another proof of this identity starts from Theorem 3 (§ 3.4) by which we know that $\boldsymbol{d}$ can always be expressed in the form

$$\boldsymbol{d} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c};$$

multiplying scalarly by $\boldsymbol{b}\times\boldsymbol{c}$ the terms containing μ and ν vanish, so λ is given by

$$\boldsymbol{d}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = \lambda\{\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c})\},$$

and similarly for μ and ν.

Example 16

Simplify $(\boldsymbol{a}\times\boldsymbol{b})\cdot\{(\boldsymbol{c}\times\boldsymbol{d})\times(\boldsymbol{e}\times\boldsymbol{f})\}$.

We use eq. (9.84) on the three vectors $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}\times\boldsymbol{f}$ in the brace brackets; this gives

$$(\boldsymbol{a} \times \boldsymbol{b})\cdot\{[\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}]\boldsymbol{d} - [\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}]\boldsymbol{c}\},$$

and again remembering that $[\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}]$, $[\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}]$ are scalar factors, this becomes

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}][\boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}] - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}][\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}].$$

EXERCISE 9.7

1. Prove that the volume of the tetrahedron whose four corners have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ is

$$\tfrac{1}{6}|\boldsymbol{b}\cdot(\boldsymbol{c} \times \boldsymbol{d}) - \boldsymbol{c}\cdot(\boldsymbol{d} \times \boldsymbol{a}) + \boldsymbol{d}\cdot(\boldsymbol{a} \times \boldsymbol{b}) - \boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c})|.$$

2. Prove that the volume of a tetrahedron can be expressed as $\frac{1}{6}abh \sin\theta$, where a, b are the lengths of opposite edges, h is the perpendicular distance between these opposite edges and θ the angle between them.

3. If A, B, C, D are coplanar and A′, B′, C′, D′ are their orthogonal projections on any plane, prove that the tetrahedra AB′C′D′ and A′BCD have the same volume.

4. Three lines OA, OB, OC are drawn from O. Three lines OD, OE, OF are drawn perpendicular to the planes BOC, COA, AOB respectively; then OG, OH, OK are drawn perpendicular to the planes AOD, BOE, COF respectively. Prove that OG, OH, OK are coplanar.

5. ABCD is a tetrahedron and O is any point. AO, BO, CO, DO meet the opposite faces of the tetrahedron in E, F, G, H respectively. Prove that

$$\frac{\mathrm{AO}}{\mathrm{AE}} + \frac{\mathrm{BO}}{\mathrm{BF}} + \frac{\mathrm{CO}}{\mathrm{CG}} + \frac{\mathrm{DO}}{\mathrm{DH}} = 3.$$

6. Prove

(i) $\{(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})\}\cdot\boldsymbol{d} = (\boldsymbol{a}\cdot\boldsymbol{d})[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$,

(ii) $\boldsymbol{a} \times \{\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})\} = (\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{a} \times \boldsymbol{c}) - (\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a} \times \boldsymbol{d})$,

(iii) $(\boldsymbol{a} \times \boldsymbol{b})\cdot\{(\boldsymbol{b} \times \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{a})\} = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^2$,

(iv) $(\boldsymbol{a} \times \boldsymbol{b})^2 = a^2b^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$,

and interpret trigonometrically.

(v) By using (iv) and an alternative form for $\{\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})\}^2$ prove that

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^2 = a^2b^2c^2 - (\boldsymbol{b}\cdot\boldsymbol{c})^2a^2 - (\boldsymbol{c}\cdot\boldsymbol{a})^2b^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2c^2 + 2(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{c}\cdot\boldsymbol{a}).$$

§ 7. Solutions of vector equations

A vector equation is a vector equality from which we can derive information about an unknown vector $\boldsymbol{x}$. In general any vector equation is equivalent to three scalar equations since every vector in the equation can be expressed in terms of its components along any three non-zero vectors which are not all parallel to the same plane (Theorem 3); then like components can be equated. Sometimes however, equations can be solved without resolving the vectors into components; we give here two examples of such equations and indicate methods used in solving them.

Example 17

Find the vector $\boldsymbol{x}$ which satisfies the equation

$$\alpha\boldsymbol{x} + (\boldsymbol{x}\cdot\boldsymbol{b})\boldsymbol{a} = \boldsymbol{c}. \tag{9.90}$$

If there is a solution of this equation, then on scalar multiplication of both sides by $\boldsymbol{b}$ we find that

$$\alpha(\boldsymbol{x}\cdot\boldsymbol{b}) + (\boldsymbol{x}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{b}) = \boldsymbol{c}\cdot\boldsymbol{b}. \tag{9.91}$$

So provided $\alpha+\boldsymbol{a}\cdot\boldsymbol{b}\neq 0$, we have

$$\boldsymbol{x}\cdot\boldsymbol{b} = \frac{\boldsymbol{c}\cdot\boldsymbol{b}}{\{\alpha + \boldsymbol{a}\cdot\boldsymbol{b}\}},$$

and then substitution of this result in the original eq. (9.90) gives

$$\boldsymbol{x} = \frac{\boldsymbol{c}}{\alpha} - \frac{\boldsymbol{c}\cdot\boldsymbol{b}}{\alpha\{\alpha + \boldsymbol{a}\cdot\boldsymbol{b}\}}\boldsymbol{a},$$

which is the required solution.

If $\alpha+\boldsymbol{a}\cdot\boldsymbol{b}=0$, that is $\alpha=-\boldsymbol{a}\cdot\boldsymbol{b}$, then for consistency of eq. (9.91), $\boldsymbol{b}$ and $\boldsymbol{c}$ must be perpendicular, satisfying $\boldsymbol{b}\cdot\boldsymbol{c}=0$; putting $\alpha=-\boldsymbol{a}\cdot\boldsymbol{b}$ and using eq. (9.81), eq. (9.90) becomes

$$\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{x}) = \boldsymbol{c}, \tag{9.92}$$

so that the vector $\boldsymbol{a}\times\boldsymbol{x}$ is also perpendicular to $\boldsymbol{c}$. This means that the vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{x}$ are all perpendicular to the vector $\boldsymbol{a}\times\boldsymbol{x}$, and so are all parallel to the same plane. We can therefore write

$$\boldsymbol{x} = \lambda\boldsymbol{a} + \mu\boldsymbol{c};$$

then by substitution in eq. (9.92) we get

$$\boldsymbol{b} \times \{\boldsymbol{a} \times (\lambda \boldsymbol{a} + \mu \boldsymbol{c})\} = \boldsymbol{c},$$

which becomes

$$\mu\{\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c})\} = \boldsymbol{c},$$

or

$$\mu\{(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} - (\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}\} = \boldsymbol{c}.$$

Since $\boldsymbol{b}\cdot\boldsymbol{c}=0$, this means that $\mu=-1/(\boldsymbol{a}\cdot\boldsymbol{b})$, and the solution is therefore of the form

$$\boldsymbol{x} = \lambda\boldsymbol{a} - \boldsymbol{c}/(\boldsymbol{a}\cdot\boldsymbol{b}). \tag{9.93}$$

This is a solution of eq. (9.92) for all values of λ.

Example 18

Find the vector $\boldsymbol{x}$ satisfying the equation

$$\alpha\boldsymbol{x} + \boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b}. \tag{9.94}$$

Again, if there is a solution then on vector multiplication by $\boldsymbol{a}$, we have

$$\alpha(\boldsymbol{x} \times \boldsymbol{a}) + (\boldsymbol{x} \times \boldsymbol{a}) \times \boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{a},$$

or

$$\alpha(\boldsymbol{x} \times \boldsymbol{a}) + (\boldsymbol{x}\cdot\boldsymbol{a})\boldsymbol{a} - a^2\boldsymbol{x} = \boldsymbol{b} \times \boldsymbol{a}, \tag{9.95}$$

Also, on scalar multiplication of eq. (9.94) by $\boldsymbol{a}$, we have

$$\alpha(\boldsymbol{x}\cdot\boldsymbol{a}) = \boldsymbol{b}\cdot\boldsymbol{a}. \tag{9.96}$$

Thus if $\alpha\neq0$, on substitution for $\boldsymbol{x}\cdot\boldsymbol{a}$ and $\boldsymbol{x}\times\boldsymbol{a}$ from eqs. (9.96) and (9.94) in eq. (9.95) we get

$$\alpha\{\boldsymbol{b} - \alpha\boldsymbol{x}\} - a^2\boldsymbol{x} = \boldsymbol{b} \times \boldsymbol{a} - \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\alpha}\,\boldsymbol{a},$$

giving $\boldsymbol{x}$.

If however $\alpha=0$, eq. (9.94) has the simpler form

$$\boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b}; \tag{9.97}$$

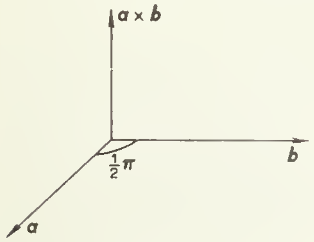

Fig. 9.24

then for consistency $\boldsymbol{a}$ and $\boldsymbol{b}$ must be perpendicular, or $\boldsymbol{a}\cdot\boldsymbol{b}=0$, and again there is no unique solution. In fact eq. (9.97) implies that $\boldsymbol{x}$ is also perpendicular to $\boldsymbol{b}$; it is therefore parallel to the plane which is parallel to the vectors $\boldsymbol{a}$ and $\boldsymbol{a}\times\boldsymbol{b}$ as shown in fig. 9.24. Therefore we can write $x=\lambda\boldsymbol{a}+\mu(\boldsymbol{a}\times\boldsymbol{b})$; substituting this in the eq. (9.97) we get at once $\mu(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}=\boldsymbol{b}$ or using eq. (9.84)

$$\mu a^2\boldsymbol{b} = \boldsymbol{b},$$

since $\boldsymbol{a}\cdot\boldsymbol{b}=0$. Thus $\mu=1/a^2$ and

$$\boldsymbol{x} = \lambda\boldsymbol{a} + (\boldsymbol{a} \times \boldsymbol{b})/a^2; \tag{9.98}$$

once again, λ can take any value.

EXERCISE 9.8

1. Solve the equation $\boldsymbol{x} \times \boldsymbol{a} + (\boldsymbol{x} \cdot \boldsymbol{b})\boldsymbol{c} = \boldsymbol{d}$, for the vector $\boldsymbol{x}$.

2. Solve the simultaneous equations

$$\alpha \boldsymbol{x} + \boldsymbol{p} \times \boldsymbol{y} = \boldsymbol{a},$$
$$\alpha \boldsymbol{y} + \boldsymbol{p} \times \boldsymbol{x} = \boldsymbol{b},$$

for the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

§ 8. Vectors in n-space

Vectors, as already defined, are quite adequate for dealing with all Newtonian dynamics of particles and single rigid bodies, and can be used as we shall see in Chs. 10 and 11, for discussing position vectors of points in two- and three-dimensional analytical geometry. Such vectors, defined in general in terms of three components or coordinates, we will refer to now as vectors in 3-space. However, in dealing, for example, with any mechanical system consisting of more than one rigid body, we find that we would like to express the functions of the system in terms of more than three coordinates. In fact even the *position* of the system of particles or rigid bodies requires in general more than three coordinates to completely define it. We now give some simple examples of systems with more than three coordinates.

(i) The position of a rod is defined by the position of its end point together with the unit vector along its length, which requires two independent components and the sign of the third, as we have seen in § 3.4. Alternatively the position of the rod is determined when the position vectors of its end points are given; but if the length of the rod is specified, there must be a relation between the six components of its two ends which reduces the six to five independent ones.

(ii) The position of a single rigid body is defined by the position of a certain chosen point in it (3 coordinates), together with the direction as in (i) of a certain line in it (2 coordinates), together with the orientation about the line (1 coordinate). This gives six coordinates in all.

(iii) The position of two rods connected together at their ends requires seven coordinates. The position of the first rod requires five as in (i); then since the end of the second rod connected to the first rod is already determined two more coordinates are required to specify its direction through this given point. The linkage of the two rods is known as a *constraint*.

We can describe the motion of a single rigid body entirely in terms of two vectors in 3-space, namely, the velocity of the chosen point, together with the angular velocity about some line through the marked point.

Similarly we could discuss the motion of a general mechanical system consisting of several rigid bodies by discussing the motion of each body of the system in terms of two vectors in 3-space, as above. This process however would be much too complicated if the motion of the system was limited by several constraints.

Description of systems in terms of 3-space vectors is not always satisfactory in other branches of physics besides mechanics, so that we want to extend our definition of a vector to include sets of more than three coordinates or components. If a physical system is described by n coordinates we can say it is defined in n-space and the *whole set* of n components is called a vector. Thus any ordered set of n numbers or components such as $(x_1, x_2, \dots, x_n)$ is called a vector of order n. Again we shall use small letters in heavy type to denote such vectors and write

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n),$$

and

$$\boldsymbol{a} = (a_1, a_2, \dots, a_n).$$

The rules for compounding and multiplying vectors in n-space are essentially the same as those for vectors in 3-space. We summarise them in the following paragraphs.

§ 8.1. Law of addition

If two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ both defined in n-space are given by

$$\boldsymbol{a} = (a_1, a_2, \dots, a_n), \tag{9.99}$$

and

$$\boldsymbol{b} = (b_1, b_2, \dots, b_n), \tag{9.100}$$

then the vector in n-space given by

$$(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n),$$

is called the sum of $\boldsymbol{a}$ and $\boldsymbol{b}$ and is written as $\boldsymbol{a}+\boldsymbol{b}$.

Example 19

$$(2, 3, -1) + (6, -7, 4) = (8, -4, 3).$$

Example 20

$$(5, -1, 6, 4, 3) + (-2, 3, 9, 7, 2) = (3, 2, 15, 11, 5).$$

The particular vector whose components are all zero is called the zero vector and denoted by **0** where $\mathbf{0} = (0, 0, \ldots 0)$.

The properties connected with this law of addition are

(i) $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ (commutative law),

(ii) $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ (associative law).

§ 8.2. Multiplication by a scalar quantity λ

If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ then we define $\lambda \boldsymbol{a}$ to be

$$\lambda \boldsymbol{a} = (\lambda a_1, \lambda a_2, \lambda a_3, \ldots, \lambda a_n);$$

again we see immediately that

(i) $\lambda(\mu \boldsymbol{a}) = \lambda\mu \boldsymbol{a} = \mu\lambda \boldsymbol{a}$,

(ii) $(\lambda + \mu)\boldsymbol{a} = \lambda \boldsymbol{a} + \mu \boldsymbol{a}$,

(iii) $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda \boldsymbol{a} + \lambda \boldsymbol{b}$,

(iv) $\lambda \mathbf{0} = \mathbf{0}, \qquad 0\boldsymbol{a} = \mathbf{0}$.

We also use the following definitions

$$\frac{\boldsymbol{a}}{\lambda} = \frac{1}{\lambda}\boldsymbol{a}, \qquad -\boldsymbol{a} = (-1)\boldsymbol{a},$$

so that

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-1)\boldsymbol{b}.$$

§ 8.3. The inner product of two n-vectors

If two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are both defined in n-space by eqs. (9.99) and (9.100), then the inner product of these two vectors is defined as

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n. \tag{9.101}$$

This is the obvious generalisation of eq. (9.59) for the inner product of two vectors in 3-space. Again we note that the properties in eqs. (9.48), (9.49), (9.50), (9.52) still apply.

Note also that we can only define the inner product of two vectors when they have the same number of components. The sum in eq. (9.101) can be written as

$$\sum_{s=1}^{n} a_s b_s.$$

By analogy with the length of a vector in 3-space, the *length* of vector $\boldsymbol{a}$ in n-space defined by eqs. (9.99) is defined to be

$$(a_1^2 + a_2^2 + \dots + a_n^2)^{\frac{1}{2}} = (\sum_{s=1}^{n} a_s^2)^{\frac{1}{2}}.$$

§ 9. Polar and axial vectors

The form (9.72) for the value of the vector product $\boldsymbol{a}\times\boldsymbol{b}$ in terms of the components does depend on the fact that the triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ has been

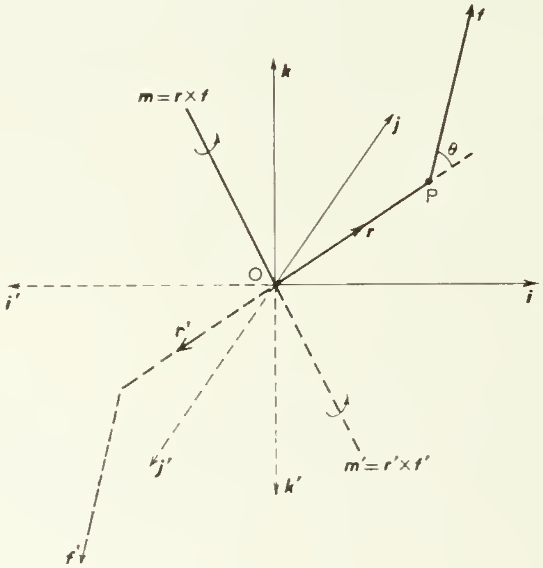

Fig. 9.25

chosen as a R.H. (right-handed) triad of vectors. If the triad were L.H. (left-handed) the results (9.70) would not apply.

As we have already seen, vectors have a meaning independent of the axes of coordinates chosen; this is one of the reasons why they are so useful in physics, since they make it possible to formulate the laws of physics in a manner which is automatically independent of any preferred directions in space. We should also expect these laws to be independent of whether we choose a R.H. or a L.H. system of axes. However under a transformation from R.H. to L.H. axes not all vectors transform in the same manner.

A R.H. system of axes can be transformed into a L.H. system very easily by a reflection in the origin of the axes. Thus if F is the triad or frame $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ then F' is its reflection in the origin indicated by $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ in fig. 9.25. Note that we cannot rotate the frame F into the frame F'.

Suppose then that the displacement vector OP is represented by $\boldsymbol{r}$ in the frame F and by $\boldsymbol{r}'$ in the frame F'. These are obviously related by $\boldsymbol{r}' = -\boldsymbol{r}$.

We now distinguish two types of vector according to the relationship between them under a transformation of axes from R.H. to L.H. such as the reflection shown in fig. 9.25.

Firstly, *polar* vectors $\boldsymbol{a}$ are such that they are transformed in the same way as displacement vectors. That is, $\boldsymbol{a}' = -\boldsymbol{a}$.

Secondly, *axial* vectors $\boldsymbol{b}$ say, are such that they transform by the relation $\boldsymbol{b}' = \boldsymbol{b}$.

Consider, for example, the moment $\boldsymbol{m}$ of the force $\boldsymbol{f}$ acting at the point P in fig. 9.25. We have $\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{f}$ where $\boldsymbol{r}$ and $\boldsymbol{f}$ are polar vectors. We can think of the vector $\boldsymbol{m}$ in two different ways. In component form we have

$$\boldsymbol{m} = (r_2 f_3 - r_3 f_2, r_3 f_1 - r_1 f_3, r_1 f_2 - r_2 f_1),$$

and on reflection $\boldsymbol{r}' = -\boldsymbol{r}$, $\boldsymbol{f}' = -\boldsymbol{f}$ so

$$\boldsymbol{m} = (r_2' f_3' - r_3' f_2', r_3' f_1' - r_1' f_3', r_1' f_2' - r_2' f_1') = \boldsymbol{m}'.$$

Thus $\boldsymbol{m}$ is an axial vector as defined above. Alternatively the vector $\boldsymbol{m}$ has magnitude $rf \sin\theta$ and this is unchanged by reflection. In fact for an axial vector two things are changed by reflection. One is just the 'polar sense' implied by $\boldsymbol{a}' = -\boldsymbol{a}$ for the polar vectors $\boldsymbol{r}$ and $\boldsymbol{f}$, and the other is the 'natural rotation' sense which is defined by the set of axes, either R.H. or L.H. In fig. 9.25 we see that both these are changed for $\boldsymbol{r} \times \boldsymbol{f}$ leaving the actual effect (indicated by $\circlearrowright$) the same. Axial vectors depend on rotation sense rather than on polar sense. It is usual, but perhaps misleading, to indicate axial vectors in a diagram by a directed displacement $\rightarrow$, since the 'natural rotation' sense implied by the right hand rule is not indicated. However since we normally use R.H. axes it is defined implicitly.

Scalar products of polar and axial vectors are axial scalars, more usually called 'pseudoscalars'. Most triple products are of this form. The name 'pseudovector' is also used for an axial vector.

APPLICATIONS OF VECTOR ALGEBRA TO ANALYTICAL GEOMETRY OF STRAIGHT LINES AND PLANES

§ 1. Introduction

Analytical Geometry is concerned with the inter-relations of points satisfying certain geometrical conditions; if one point is chosen as the origin of vectors, then each point is defined by its position vector relative to this origin. The position vector of a point is a localised vector relative to this origin, since it is specifically associated with that point and with the chosen origin, and cannot be moved about at will without altering its significance. So geometrical relations between points can be described in terms of relations between the localised position vectors of the points.

§ 1.1. Coordinates in a plane

Although the position vector of a point relative to a chosen origin gives sufficient definition of the point, it is customary in analysing the arithmetical details of the results and to give analytical precision to the vector, to use the idea of axes and coordinates. In two-dimensional analytical geometry, when we assume that the positive vectors correspond to points lying in a plane, we can introduce at the origin two chosen perpendicular directions in the plane and then resolve all vectors into their components in these two perpendicular directions. These two directions in the plane through the origin are defined by two unit vectors which we denote by $\boldsymbol{i}$ and $\boldsymbol{j}$. The relative senses of these two vectors are usually chosen as shown in fig. 10.1.

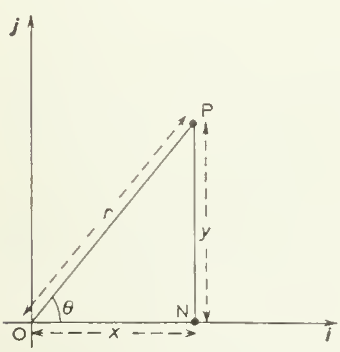

Fig. 10.1

Suppose P is any point in the plane having position vector $\boldsymbol{r}$ relative to O, so that $\boldsymbol{r}=\boldsymbol{OP}$. By drawing PN perpendicular to the direction of $\boldsymbol{i}$

as shown, we have immediately

$$\boldsymbol{r} = \boldsymbol{OP} = \boldsymbol{ON} + \boldsymbol{NP} = x\boldsymbol{i} + y\boldsymbol{j},$$

where x and y are the lengths of ON and NP; these are the two-dimensional *rectangular coordinates* of P with respect to axes Ox in the direction of $\boldsymbol{i}$ and Oy in the direction of $\boldsymbol{j}$. These coordinates written as (x, y) determine the position of P.

Another natural way of specifying the vector $\boldsymbol{OP}$ would be by its modulus r and its direction. If $\boldsymbol{OP}$ makes an angle θ in the positive (or anti-clockwise) sense with the direction of $\boldsymbol{i}$ which is defined as the *initial direction*, we say that (r, θ) are its *polar coordinates*; we see immediately that

$$x = \mathrm{ON} = r\cos\theta \quad \text{and} \quad y = \mathrm{NP} = r\sin\theta.$$

The vector $\boldsymbol{OP}$ may therefore be written as

$$\boldsymbol{r} = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j} = r\{(\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}\}.$$

We note also that

$$r = \sqrt{(x^2 + y^2)}, \qquad \cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r},$$

so that when x and y are given θ is defined in the range $0 \leqslant \theta < 2\pi$. The value of r is, of course, obvious since

$$r^2 = \boldsymbol{r}^2 = (x\boldsymbol{i} + y\boldsymbol{j})^2 = x^2 + y^2.$$

It is also obvious that $(\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ is a unit vector in the direction and sense of $\boldsymbol{OP}$, that is, in the direction making a positive angle θ with the unit vector $\boldsymbol{i}$. In the present chapter we shall confine ourselves to rectangular coordinates (x, y).

§ 1.2. Distance between two points

The distance $\mathrm{P_1P_2}$ between two points $\mathrm{P_1}$, $\mathrm{P_2}$ whose coordinates are (x_1, y_1), (x_2, y_2) respectively follows at once from

$$\boldsymbol{P_1P_2} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j};$$

we have

$$(\mathrm{P_1P_2})^2 = (\boldsymbol{P_1P_2})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

§ 1.3. Point dividing the line P_1P_2 in a given ratio

In fig. 10.2, P is the point on the line P_1P_2 dividing P_1P_2 in the ratio $\lambda : (1-\lambda)$. Using Example 1, Ch. 9, with $p : q = \lambda : (1-\lambda)$ we have immediately

$$OP = (1-\lambda)OP_1 + \lambda OP_2 = (1-\lambda)(x_1\boldsymbol{i} + y_1\boldsymbol{j}) + \lambda(x_2\boldsymbol{i} + y_2\boldsymbol{j}),$$

which becomes

$$OP = \{(1-\lambda)x_1 + \lambda x_2\}\boldsymbol{i} + \{(1-\lambda)y_1 + \lambda y_2\}\boldsymbol{j}.$$

Thus the coordinates of P are (x, y) where

$$x = (1-\lambda)x_1 + \lambda x_2, \qquad y = (1-\lambda)y_1 + \lambda y_2,$$

and therefore

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \lambda.$$

As λ varies, the point P moves along the line, and its coordinates (x, y) are called the *running coordinates* of points on the line. Thus all points on the line have coordinates (x, y) which satisfy the equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}:$$

this is the analytical equation for the line through the two points P_1 and P_2.

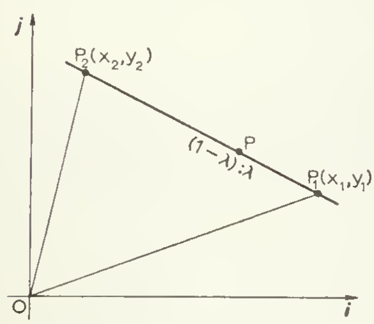

Fig. 10.2

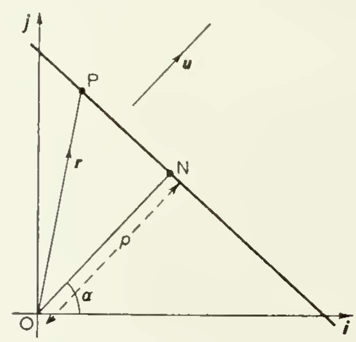

Fig. 10.3

§ 1.4. Equation of a line which is perpendicular to a given direction

In fig. 10.3, let the line be perpendicular to a given direction ON in the plane, where ON is in the direction and sense of the unit vector $\boldsymbol{u}$ and is

of length p, so that

$$ON = p\boldsymbol{u}.$$

Suppose P is any point on the line having position vector $\boldsymbol{r}$ relative to O. The vector $\boldsymbol{r}$ is called the *running vector* of any point on the line. Now since NP is perpendicular to ON, we have

$$NP \cdot ON = 0.$$

But $NP = OP - ON = \boldsymbol{r} - p\boldsymbol{u}$ and thus

$$(\boldsymbol{r} - p\boldsymbol{u}) \cdot p\boldsymbol{u} = 0.$$

Since $p \neq 0$, this becomes

$$\boldsymbol{r} \cdot \boldsymbol{u} - p = 0,$$

or

$$\boldsymbol{r} \cdot \boldsymbol{u} = p; \tag{10.1}$$

this result is also obvious from the fact that ON is the projection of OP on $\boldsymbol{u}$, so that $r \cos \widehat{\boldsymbol{ru}} = p$.

If we assume that a set of rectangular axes are defined by unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and that ON makes an angle α with Ox, then the unit vector $\boldsymbol{u}$ along ON is $\boldsymbol{u} = (\cos \alpha)\, \boldsymbol{i} + (\sin \alpha)\, \boldsymbol{j}$; if $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ then

$$\boldsymbol{r} \cdot \boldsymbol{u} = x \cos \alpha + y \sin \alpha,$$

and eq. (10.1) becomes

$$x \cos \alpha + y \sin \alpha = p, \tag{10.2}$$

a familiar equation in two-dimensional analytical geometry.

Many other results in two-dimensional analytical geometry may be derived in this way using the ideas of vector algebra, remembering all the time that every vector, being a vector in the plane, can be expressed in terms of components along the chosen directions $\boldsymbol{i}$ and $\boldsymbol{j}$. However the corresponding three-dimensional analytical geometry results are so similar that we will now leave any further two-dimensional ones as exercises for the reader and proceed with the less familiar three-dimensional results.

§ 2. Three-dimensional coordinates

Just as coordinate geometry in a plane is based on resolution along two perpendicular vectors $\boldsymbol{i}, \boldsymbol{j}$ in the plane, so in 3-space we introduce three

fixed mutually perpendicular unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, at an origin O and resolve all vectors into their components parallel to these three vectors. The triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are usually chosen to be a right-handed set as explained in Ch. 9 § 3.4 and then they satisfy the eqs. (9.57), (9.58) and (9.70), (9.71).

In analytical geometry, the 3 *coordinate axes* Ox, Oy, Oz are defined to consist of all points with position vectors $\lambda\boldsymbol{i}$, $\mu\boldsymbol{j}$, $\nu\boldsymbol{k}$ respectively, while the 3 coordinate planes xOy, yOz, zOx consist of all points with position vectors of the form $\lambda\boldsymbol{i}+\mu\boldsymbol{j}$, $\mu\boldsymbol{j}+\nu\boldsymbol{k}$, $\nu\boldsymbol{k}+\lambda\boldsymbol{i}$, respectively ($\lambda$, μ, ν arbitrary).

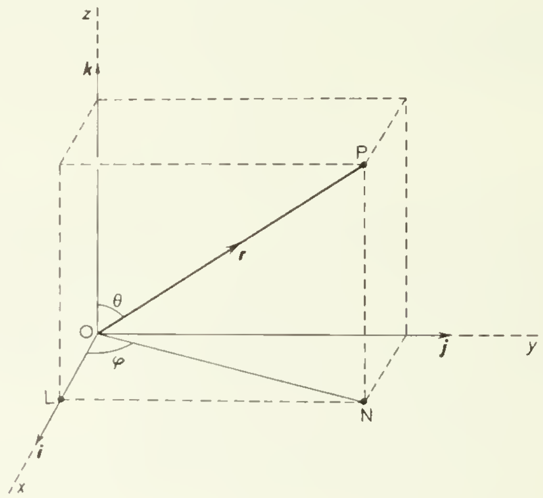

Fig. 10.4

If P is any point in space, the position vector $\boldsymbol{r}$ of P relative to O, can be resolved into components along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, as in fig. 10.4; OP is the diagonal of the rectangular parallelepiped whose edges are parallel to $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, so that

$$\boldsymbol{r} = OP = OL + LN + NP = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{10.3}$$

where x=OL is the projection of OP on the direction $\boldsymbol{i}$, whilst similarly y, z are the projections of OP in the directions $\boldsymbol{j}$, $\boldsymbol{k}$. These values for the projections follow also, without reference to fig. 10.4, when we multiply eq. (10.3) through scalarly by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to give

$$\boldsymbol{r}\cdot\boldsymbol{i} = r\cos\widehat{\boldsymbol{ri}} = x\boldsymbol{i}\cdot\boldsymbol{i} + y\boldsymbol{i}\cdot\boldsymbol{j} + z\boldsymbol{i}\cdot\boldsymbol{k} = x,$$

and similarly

$$\boldsymbol{r}\cdot\boldsymbol{j} = r\cos\widehat{\boldsymbol{rj}} = y,$$

and

$$\boldsymbol{r}\cdot\boldsymbol{k} = r\cos\widehat{\boldsymbol{rk}} = z.$$

The projections (x, y, z) in the directions $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively, are called the *rectangular coordinates* of the point P referred to axes Ox, Oy, Oz.

As the sense of each of the vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is fixed each of the coordinates (x, y, z) may take positive or negative values, depending on the sense of the projection of OP relative to $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. The coordinate y for example will be positive or negative according as the vector $\boldsymbol{LN}=y\boldsymbol{j}$ is in the same or opposite sense to the vector $\boldsymbol{j}$; in fig. 10.4 this means that y is positive for all points to the right of the plane xOz and negative for all points to the left of this plane.

The length of the vector OP is given immediately by

$$\mathrm{OP}^2 = (\boldsymbol{OP})^2 = \boldsymbol{r}^2 = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})^2 = x^2 + y^2 + z^2,$$

which may also be deduced from the geometry of fig. 10.4, using Pythagoras' theorem.

Other ways of specifying the position of P in three dimensions may also be used. Two such ways are especially useful in later work; these are really generalisations of the plane polar coordinates defined in § 1.1 and are known respectively as *cylindrical polar* coordinates and *spherical polar* coordinates.

In the *cylindrical* polar coordinate system, the position of P is determined by its z-coordinate PN in fig. 10.4 and the plane polar coordinates of the point N, the foot of the perpendicular from P on the xOy-plane. Thus if ON$=\rho$, NOL$=\varphi$, then

$$x = \rho \cos \varphi, \qquad y = \rho \sin \varphi; \tag{10.4}$$

the inverse relations are

$$\rho = (x^2 + y^2)^{\frac{1}{2}}, \qquad \cos \varphi = \frac{x}{\rho}, \qquad \sin \varphi = \frac{y}{\rho}, \tag{10.5}$$

and the whole of space is covered by letting (ρ, φ, z) the cylindrical polar coordinates take all values in the ranges

$$0 \leqslant \rho < \infty, \qquad 0 \leqslant \varphi < 2\pi, \qquad -\infty < z < \infty.$$

In the *spherical* polar system, the position of P is determined by (i) its distance $r=$OP from the origin, the magnitude of the vector $\boldsymbol{OP}$, (ii) the angle θ between OP and the z-axis in the direction of the unit vector $\boldsymbol{k}$, (iii) the angle φ defined in the cylindrical polar coordinate system; φ is, of course, the angle between the plane OPN and the plane zOx. Then since $\rho=$ON$=r \sin \theta$ and $z=r \cos \theta$ we have

$$x = r \sin \theta \cos \varphi, \qquad y = r \sin \theta \sin \varphi, \qquad z = r \cos \theta, \tag{10.6}$$

defining the rectangular coordinates (x, y, z) in terms of the spherical polar coordinates (r, θ, φ); conversely

$$\left.\begin{aligned} r &= (x^2 + y^2 + z^2)^{\frac{1}{2}}, \qquad & \tan\theta &= (x^2 + y^2)^{\frac{1}{2}}/z, \\ \cos\varphi &= x/(x^2 + y^2)^{\frac{1}{2}}, & \sin\varphi &= y/(x^2 + y^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{10.7}$$

If φ remains constant r is positive and θ varies from 0 to 2π we see that the point P lies in an infinite plane through the z-axis. If we limit θ to the range $0 \leqslant \theta \leqslant \pi$, P lies in a semi-infinite plane with one edge along the z-axis.

If now this semi-infinite plane is rotated about the z-axis, so that φ varies from 0 to 2π, the whole of space is covered. Thus the whole of space is covered by letting (r, θ, φ) take all values in the ranges

$$0 \leqslant r < \infty, \qquad 0 \leqslant \theta < \pi, \qquad 0 \leqslant \varphi < 2\pi.$$

More general orthogonal coordinates, of which cylindrical and spherical polar coordinates are special cases will be dealt with in Ch. 14. In the present chapter we shall confine ourselves to rectangular coordinates (x, y, z).

§ 2.1. Direction cosines

Certain results given in Ch. 9 § 4.1 are of particular significance in analytical geometry. Suppose $\boldsymbol{u}$, $\boldsymbol{v}$ are any two unit vectors defining two directions in space, and suppose

$$\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}, \tag{10.8}$$

and

$$\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}. \tag{10.9}$$

Then since these are unit vectors we have

$$\boldsymbol{u}^2 = u_1^2 + u_2^2 + u_3^2 = 1, \qquad \boldsymbol{v}^2 = v_1^2 + v_2^2 + v_3^2 = 1. \tag{10.10}$$

But, on scalar multiplication of eq. (10.8) by $\boldsymbol{i}$, we get

$$u_1 = \boldsymbol{u} \cdot \boldsymbol{i} = \cos \widehat{\boldsymbol{ui}}, \tag{10.11}$$

is the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{i}$. Similarly $u_2 = \boldsymbol{u} \cdot \boldsymbol{j} = \cos \widehat{\boldsymbol{uj}}$ and $u_3 = \boldsymbol{u} \cdot \boldsymbol{k} = \cos \widehat{\boldsymbol{uk}}$. Since $\boldsymbol{u}$ defines a direction in 3-space, it is customary in geometrical discussions to call the three components u_1, u_2, u_3 the *direction cosines* in the direction $\boldsymbol{u}$. If a straight line or a vector is in the direction of $\boldsymbol{u}$, we say that (u_1, u_2, u_3) are the direction cosines of the

line or vector. Notice that they satisfy the first of the relations (10.10) so that the sum of the squares of the direction cosines of any line is unity. The two-dimensional analogue of this result is

$$\cos^2 \alpha + \sin^2 \alpha = 1,$$

since for any line in the plane xOy

$$u_1 = \cos \alpha, \qquad u_2 = \cos(\tfrac{1}{2}\pi - \alpha) = \sin \alpha, \qquad u_3 = 0,$$

where α is the angle between the line and the x-axis.

Similarly (v_1, v_2, v_3) are the direction cosines of the unit vector $\boldsymbol{v}$. If now θ is the angle between these two unit vectors $\boldsymbol{u}$, $\boldsymbol{v}$, we have using eq. (9.59)

$$\boldsymbol{u} \cdot \boldsymbol{v} = \cos \theta = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{10.12}$$

In particular, if these two unit vectors are at right angles, then $\cos \theta = 0$, or

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0. \tag{10.13}$$

If the position vector of a point P relative to O is

$$OP = \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},$$

so that

$$\boldsymbol{r}^2 = r^2 = x^2 + y^2 + z^2, \tag{10.14}$$

then the unit vector in the direction of OP is

$$\frac{\boldsymbol{r}}{r} = \frac{x}{r}\boldsymbol{i} + \frac{y}{r}\boldsymbol{j} + \frac{z}{r}\boldsymbol{k},$$

so that its direction cosines are $\left(\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right)$. Note that from eq. (10.14)

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = 1.$$

§ 2.2. Distance between two points

The distance $\mathrm{P_1P_2}$ between two points $\mathrm{P_1}$, $\mathrm{P_2}$ whose coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, again follow at once from

$$P_1P_2 = \boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k};$$

we have

$$(\mathrm{P_1P_2})^2 = (P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

§ 2.3. Point dividing P_1P_2 in a given ratio

If O is the origin in fig. 10.5 and P divides the line joining P_1 and P_2 in the ratio $\lambda : (1-\lambda)$ then from Example 1, Ch. 9 we have

$$\begin{aligned} OP &= (1-\lambda)OP_1 + \lambda OP_2 \\ &= \{(1-\lambda)x_1 + \lambda x_2\}\boldsymbol{i} + \{(1-\lambda)y_1 + \lambda y_2\}\boldsymbol{j} + \{(1-\lambda)z_1 + \lambda z_2\}\boldsymbol{k}, \end{aligned}$$

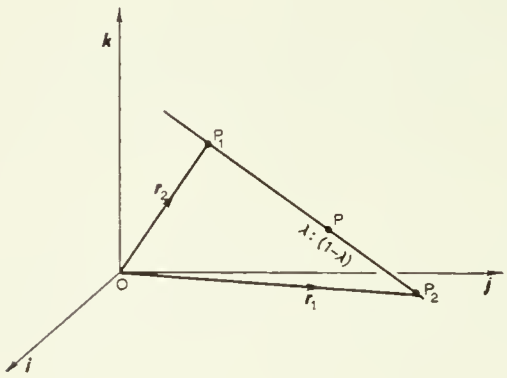

Fig. 10.5

so that if (x, y, z) are the coordinates of P,

$$x = (1-\lambda)x_1 + \lambda x_2,$$
$$y = (1-\lambda)y_1 + \lambda y_2,$$
$$z = (1-\lambda)z_1 + \lambda z_2.$$

We again note that

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = \lambda;$$

as λ varies the point P moves along the line, (x, y, z) being the running coordinates of the points on the line. So the coordinates (x, y, z) of all points on the line satisfy

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}, \tag{10.15}$$

these are the analytical three-dimensional equations of a line through the two points P_1 and P_2. The geometrical significance of these equations will be seen later in § 4.3.

EXERCISE 10.1

1. Let (v_1, v_2) be the components of a vector $\boldsymbol{v}$ referred to one set of rectangular axes in a plane; (v_1', v_2') be the components of the same vector $\boldsymbol{v}$ referred to a second set of axes in the same plane and inclined at an angle θ to the first set. Show by vector methods that

$$\begin{aligned} v_1' &= v_1 \cos\theta + v_2 \sin\theta, \\ v_2' &= -v_1 \sin\theta + v_2 \cos\theta, \end{aligned}$$

or conversely

$$\begin{aligned} v_1 &= v_1' \cos\theta - v_2' \sin\theta, \\ v_2 &= v_1' \sin\theta + v_2' \cos\theta. \end{aligned}$$

2. Find the moduli and direction cosines of each of the following vectors

(i) $3\boldsymbol{i}+7\boldsymbol{j}-4\boldsymbol{k}$, (ii) $\boldsymbol{i}-5\boldsymbol{j}-8\boldsymbol{k}$, (iii) $6\boldsymbol{i}-10\boldsymbol{j}+8\boldsymbol{k}$.

Express the angle between the vectors (ii) and (iii) as an inverse cosine.

3. A point P_1 has coordinates (1, −2, 1), and P_2 has coordinates (0, −2, 3). What is the angle between OP_1 and OP_2? Find the point dividing P_1P_2 in the ratio 2 : 3. What is the equation of the straight line through P_1 and P_2?

4. Find the direction cosines of a line perpendicular to the plane OP_1P_2 of Question 2. (Use the geometrical result that a line perpendicular to a plane is perpendicular to every line in the plane.)

5. Find the length of the line joining the two points (2, 3, −1) and (7, −1, 2). Find the direction cosines of this line.

§ 3. The equation of a plane

The equation of a plane in three dimensions is the analogue of the equation of a line in two dimensions; its equation may be expressed in more than one way according to how the plane is described.

§ 3.1. The equation of a plane which is perpendicular to a given line through the origin and is at a given perpendicular distance from the origin

Suppose in fig. 10.6 that P is any point in the required plane. Then we shall suppose that the position vector of P relative to the origin O is $\boldsymbol{OP}=\boldsymbol{r}$. Then $\boldsymbol{r}$ is called the *running vector* of any point P in the plane. To interpret the results in terms of rectangular coordinates (x, y, z) we write $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$; then (x, y, z) are the running coordinates of P. This notation will be used throughout the following paragraphs on the equation of a plane in various forms.

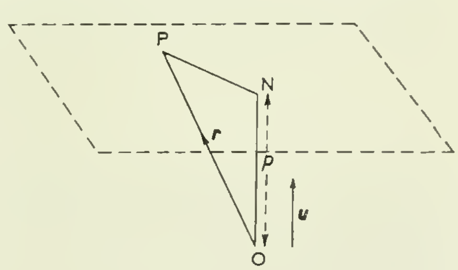

Fig. 10.6

In fig. 10.6, the perpendicular from the origin O on to the plane is ON; we assume that $\boldsymbol{ON}$ is in the direction of a given unit vector $\boldsymbol{u}$ and is of length p, so that

$$\boldsymbol{ON}=p\boldsymbol{u}.$$

To find the equation of the plane, we require a relation between $\boldsymbol{r}$, $\boldsymbol{u}$ and p. Now

$$NP = OP - ON = \boldsymbol{r} - p\boldsymbol{u}.$$

But NP is a line in the plane and must therefore be perpendicular to ON and therefore to $\boldsymbol{u}$. The condition that $\boldsymbol{r}-p\boldsymbol{u}$ and $\boldsymbol{u}$ are perpendicular is

$$(\boldsymbol{r} - p\boldsymbol{u})\cdot\boldsymbol{u} = 0,$$

or

$$\boldsymbol{r}\cdot\boldsymbol{u} = p, \tag{10.16}$$

since $\boldsymbol{u}^2=1$. Equation (10.16) is the vector equation of a plane whose perpendicular or normal from the origin is in the direction of the unit vector $\boldsymbol{u}$ and has length p.

If we write $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ and $\boldsymbol{u}=u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k}$ then eq. (10.16) becomes

$$u_1x + u_2y + u_3z = p; \tag{10.17}$$

this is the equation of the same plane referred to rectangular axes, the direction cosines of its normal being (u_1, u_2, u_3).

Equation (10.17) is the analogue of eq. (10.2) in two dimensions.

Note that if the plane is parallel to the x-axis for example, then $u_1=0$ and the term u_1x is missing from its eq. (10.17). The terms u_2y and u_3z likewise vanish for planes parallel to the y-axis and z-axis respectively.

Conversely any equation of the form

$$\boldsymbol{r}\cdot\boldsymbol{v} = \lambda, \tag{10.18}$$

where λ is a positive real number and $\boldsymbol{v}$ is not necessarily a unit vector, can be written in the form (10.16) as

$$\boldsymbol{r}\cdot\frac{\boldsymbol{v}}{v} = \frac{\lambda}{v}; \tag{10.19}$$

this expresses the condition that the point P($\boldsymbol{r}$) lies in a plane the normal to which is in the direction of $\boldsymbol{v}$, whilst the length of the normal from the origin is λ/v.

§ 3.2. The equation of a plane through the point $P_1(\boldsymbol{r}_1)$ and perpendicular to the vector $\boldsymbol{v}$

Here in fig. 10.7 the line P_1P lies in the plane and therefore the vector P_1P must be perpendicular to the normal to the plane. Hence, since

$P_1P = \boldsymbol{r} - \boldsymbol{r}_1$ we have

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot\boldsymbol{v} = 0. \tag{10.20}$$

This is the vector equation of the plane. In rectangular coordinates with $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ it becomes

$$v_1(x - x_1) + v_2(y - y_1) + v_3(z - z_1) = 0.$$

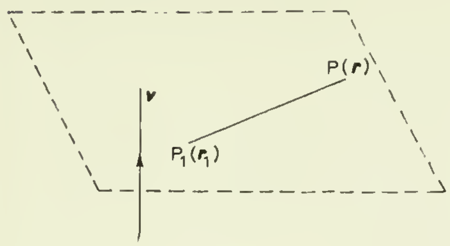

Fig. 10.7

Note that the perpendicular from the origin on to this plane is of length

$$\frac{\boldsymbol{r}_1\cdot\boldsymbol{v}}{v} = \frac{v_1x_1 + v_2x_2 + v_3x_3}{\sqrt{(v_1^2 + v_2^2 + v_3^2)}}.$$

§ 3.3. The equation of a plane through the three points $P_1(\boldsymbol{r}_1)$, $P_2(\boldsymbol{r}_2)$, $P_3(\boldsymbol{r}_3)$

Here the vectors $P_2P_1 = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $P_3P_1 = \boldsymbol{r}_1 - \boldsymbol{r}_3$ both lie in the plane. Hence the vector

$$\boldsymbol{v} = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_3), \tag{10.21}$$

is perpendicular to the plane. Hence the plane goes through the point $\boldsymbol{r}_1$ and is perpendicular to the direction given in eq. (10.21). Therefore from eq. (10.20) its equation is

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot\{(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_3)\} = 0. \tag{10.22}$$

This eq. (10.22) is simple to use in practice, but is not symmetrical between the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$; it is easily seen that it is equivalent to the symmetrical form

$$\boldsymbol{r}\cdot\{\boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1\} = [\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3].$$

The equation of a plane through the three points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ can be expressed in a different form. Since $\boldsymbol{r} - \boldsymbol{r}_1$ is a vector lying in the plane then by Theorem 2 Ch. 9 § 3.2, it can be expressed as a linear combination of the vectors $\boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\boldsymbol{r}_1 - \boldsymbol{r}_3$ also lying in the plane; thus

$$\boldsymbol{r} - \boldsymbol{r}_1 = \lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \mu(\boldsymbol{r}_1 - \boldsymbol{r}_3),$$

or

$$\boldsymbol{r} = (1 + \lambda + \mu)\boldsymbol{r}_1 - \lambda\boldsymbol{r}_2 - \mu\boldsymbol{r}_3. \tag{10.23}$$

So the equation of the plane may be written as

$$\boldsymbol{r} = t_1\boldsymbol{r}_1 + t_2\boldsymbol{r}_2 + t_3\boldsymbol{r}_3, \tag{10.24}$$

where t_1, t_2, t_3 are parameters connected by the relation $t_1+t_2+t_3=1$.

Thus if four points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$ are coplanar, then these vectors satisfy a relation

$$\alpha\boldsymbol{r}_1 + \beta\boldsymbol{r}_2 + \gamma\boldsymbol{r}_3 + \delta\boldsymbol{r}_4 = 0, \tag{10.25}$$

where $\alpha+\beta+\gamma+\delta=0$.

Example 1

Find the equation of the plane through the three points

$$\boldsymbol{r}_1 = 3\boldsymbol{i} + \boldsymbol{j}, \qquad \boldsymbol{r}_2 = 3\boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{r}_3 = \boldsymbol{i} + 3\boldsymbol{k},$$

and also the equation of the plane through the point $\boldsymbol{r}_4=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$, and parallel to the above plane. Find the perpendicular distance between the two planes.

One line in the plane is the line joining $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ which is in the direction of the vector

$$\boldsymbol{r}_1 - \boldsymbol{r}_2 = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}.$$

Similarly a second line in the plane is in the direction of the vector

$$\boldsymbol{r}_1 - \boldsymbol{r}_3 = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}.$$

Thus the normal to the plane, being perpendicular to both these lines is in the direction of the vector $\boldsymbol{v}=(3\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k})\times(2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})=7(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$. But if $\boldsymbol{r}$ is the position vector of any point in the plane $\boldsymbol{r}-\boldsymbol{r}_1$ is a vector in the plane and must be perpendicular to the normal, so that

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0.$$

This equation is just eq. (10.22) of course. Simplifying

$$\boldsymbol{r}\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = \boldsymbol{r}_1\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = (3\boldsymbol{i} + \boldsymbol{j})\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}),$$

so that

$$\boldsymbol{r}\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 4. \tag{10.26}$$

The second plane through $\boldsymbol{r}_4$ being parallel to this plane must also have a normal in the direction of the normal to the above plane namely $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$; since it passes through $\boldsymbol{r}_4$ it has equation

$$(\boldsymbol{r} - \boldsymbol{r}_4)\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0,$$

or

$$\boldsymbol{r}\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 2. \tag{10.27}$$

These two eqs. (10.26) and (10.27) can be written as

$$\boldsymbol{r}\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt{3} = 4/\sqrt{3},$$
$$\boldsymbol{r}\cdot(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt{3} = 2/\sqrt{3},$$

where $(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/\sqrt{3}=\boldsymbol{u}$ is the unit vector in the direction of the common normal to the planes. These equations are now of the form (10.16) and so the length of

the perpendiculars from the origin on to the two planes are $p_1=4/\sqrt{3}$, $p_2=2/\sqrt{3}$. Since the planes are parallel, this means that the perpendicular distance between them is $p_1-p_2=2/\sqrt{3}$.

Example 2

Prove that the four points

$$\boldsymbol{r}_1 = \boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}, \qquad \boldsymbol{r}_2 = -2\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k},$$
$$\boldsymbol{r}_3 = 3\boldsymbol{i} + 9\boldsymbol{j} + 4\boldsymbol{k}, \qquad \boldsymbol{r}_4 = -3\boldsymbol{i} + \boldsymbol{k}$$

are coplanar, and find the equation of the plane in which they lie.

The equation of the plane is eq. (10.22). The vector product of

$$\boldsymbol{r}_1 - \boldsymbol{r}_2 = 3\boldsymbol{i} + 10\boldsymbol{j} + 4\boldsymbol{k},$$

and

$$\boldsymbol{r}_1 - \boldsymbol{r}_3 = -2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k},$$

is $2\boldsymbol{i}-5\boldsymbol{j}+11\boldsymbol{k}$; so the equation of the plane through $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ is

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot(2\boldsymbol{i} - 5\boldsymbol{j} + 11\boldsymbol{k}) = 0,$$

or

$$\boldsymbol{r}\cdot(2\boldsymbol{i} - 5\boldsymbol{j} + 11\boldsymbol{k}) = (\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k})\cdot(2\boldsymbol{i} - 5\boldsymbol{j} + 11\boldsymbol{k}) = 5. \tag{10.28}$$

Substituting $\boldsymbol{r}=\boldsymbol{r}_4=-3\boldsymbol{i}+\boldsymbol{k}$ in eq. (10.28) we see that it is satisfied. This shows that the four points are coplanar and eq. (10.28) is the equation of the plane in which they lie.

Alternatively, the equation of the plane through $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ may be written in the form of eq. (10.23) that is,

$$\boldsymbol{r} = (1 + \lambda + \mu)(\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) - \mu(3\boldsymbol{i} + 9\boldsymbol{j} + 4\boldsymbol{k}),$$

where λ, μ are parameters. If $\boldsymbol{r}_4$ also lies in this plane, then real values of λ, μ can be found such that

$$-3\boldsymbol{i} + \boldsymbol{k} = (1 + \lambda + \mu)(\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) - \mu(3\boldsymbol{i} + 9\boldsymbol{j} + 4\boldsymbol{k}).$$

Equating coefficients of $\boldsymbol{i}$ and $\boldsymbol{j}$ we get

$$-3 = 1 - \lambda - 2\mu,$$
$$0 = 6 + 10\lambda - 3\mu,$$

giving $\lambda=0$, $\mu=2$; since the equation for the coefficient of $\boldsymbol{k}$,

$$1 = 3 + 4\lambda - \mu,$$

is also satisfied by these values of λ and μ, we know that $\boldsymbol{r}_4$ lies in the plane through $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$.

§ 4. The equation of a line

The equation of a line in three dimensions can be expressed in several different ways, according to how the line is described. In the following paragraphs we give several different forms of the equation of a line; in

all of them we use $\boldsymbol{r}$ for the running vector of any point P on the line, and to express the equations in rectangular coordinates we write

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

§ 4.1. The equation of a line through a given point in a given direction

A straight line in space is most simply defined by prescribing a point P_1 on it and by giving its direction. If O is the origin and $OP_1 = \boldsymbol{r}_1$ and the direction of the line is described by the unit vector $\boldsymbol{u}$, then from fig. 10.8 we see that

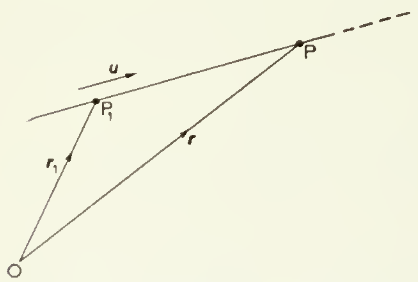

Fig. 10.8

$$\boldsymbol{r} = OP = OP_1 + P_1P = \boldsymbol{r}_1 + s\boldsymbol{u}, \quad (10.29)$$

where s is the length of the vector P_1P. By allowing s to take values from $-\infty$ to $+\infty$, we obtain all points on the line. Eq. (10.29) is therefore the equation of the line through the point $\boldsymbol{r}_1$ in the direction of $\boldsymbol{u}$. Conversely any equation of the form

$$\boldsymbol{r} = \boldsymbol{r}_1 + t\boldsymbol{v}, \quad (10.30)$$

in which t is a variable parameter and $\boldsymbol{v}$ is not necessarily a unit vector, defines the position vector of a point on a line which passes through the point $\boldsymbol{r} = \boldsymbol{r}_1$ ($t = 0$) and is in a direction parallel to the vector $\boldsymbol{v} = (\boldsymbol{r} - \boldsymbol{r}_1)/t$. If $\boldsymbol{v}$ is not a unit vector, t is not equal to the distance between $\boldsymbol{r}$ and $\boldsymbol{r}_1$, as s is in eq. (10.29). This distance will, of course, be $|\boldsymbol{r} - \boldsymbol{r}_1| = t|\boldsymbol{v}|$ which is proportional to t. The *unit* vector defining the direction of the line is $\boldsymbol{v}/v$.

If we write $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, eq. (10.30) becomes

$$x = x_1 + tv_1, \qquad y = y_1 + tv_2, \qquad z = z_1 + tv_3. \quad (10.31)$$

These eqs. (10.31) are the parametric equations of a line in space, t being the parameter; eliminating t we obtain

$$\frac{x - x_1}{v_1} = \frac{y - y_1}{v_2} = \frac{z - z_1}{v_3}, \quad (10.32)$$

as the equations of a line through the point (x_1, y_1, z_1) in the direction of the vector $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$.

§ 4.2. The equation of a line through two points

If the line passes through the two points $P_1(\boldsymbol{r}_1)$ and $P_2(\boldsymbol{r}_2)$ where $\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ then the line must be parallel to the vector

$$\boldsymbol{v} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}.$$

So the vector equation (10.30) can be written as

$$\boldsymbol{r} = \boldsymbol{r}_1 + t(\boldsymbol{r}_2 - \boldsymbol{r}_1), \tag{10.33}$$

and the rectangular coordinate form (10.32) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, \tag{10.34}$$

a form which we derived in § 2.3 eq. (10.15).

§ 4.3. The equation of a line defined by two planes

A straight line can also, of course, be defined as the intersection of two planes. If we take the two planes as

$$\boldsymbol{r}\cdot\boldsymbol{v}_1 = \lambda_1, \tag{10.35}$$

$$\boldsymbol{r}\cdot\boldsymbol{v}_2 = \lambda_2, \tag{10.36}$$

then these equations imply that since the line lies in both planes, its direction must be perpendicular to the normals of both planes, that is, to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The line is therefore in the direction $\boldsymbol{v}_1 \times \boldsymbol{v}_2$. Now $\boldsymbol{r}$ being the position vector of any point on the line, since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ are three vectors which are not all zero and are not all parallel to the same plane, we can, by Theorem 3 Ch. 9, express $\boldsymbol{r}$ in the form

$$\boldsymbol{r} = \alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2 + t\boldsymbol{v}_1 \times \boldsymbol{v}_2, \tag{10.37}$$

where α, β, t are real numbers. But $\boldsymbol{r}$ must satisfy both eqs. (10.35) and (10.36) and by substituting the value of $\boldsymbol{r}$ from eq. (10.37) in these equations we get

$$\alpha\boldsymbol{v}_1^2 + \beta\boldsymbol{v}_1\cdot\boldsymbol{v}_2 = \lambda_1,$$

and

$$\alpha\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + \beta\boldsymbol{v}_2^2 = \lambda_2,$$

these equations determine α, β and thence the equation of the line in the form (10.37) with t a variable parameter.

The use of rectangular coordinates to solve this problem is really more straightforward and simple. The equations of the two planes are assumed to be given as

$$a_1x + b_1y + c_1z + d_1 = 0, \tag{10.38}$$

and

$$a_2x + b_2y + c_2z + d_2 = 0, \tag{10.39}$$

which are the rectangular forms of equations like (10.35) and (10.36). Since any point (x, y, z) satisfying eqs. (10.38) and (10.39) will also satisfy

$$(a_1x + b_1y + c_1z + d_1) + \mu(a_2x + b_2y + c_2z + d_2) = 0, \tag{10.40}$$

where μ is any parameter, then eq. (10.40) must represent the equation of a plane through the line of intersection of the planes (10.38) and (10.39) By choosing μ so that $a_1+\mu a_2=0$ the term in x is eliminated, and we can write eq. (10.40) in the form

$$\frac{y - y_1}{b} = \frac{z - z_1}{c}, \tag{10.41}$$

y_1, z_1, b, c being known constants. Similarly by choosing μ so that the term in z is eliminated we can derive an equation in the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} \tag{10.42}$$

with the same values of y_1, b as in eq. (10.41). The two eqs. (10.41) and (10.42) represent planes parallel to the x-axis and the z-axis respectively. Since points on the line lie on both these planes, they satisfy eqs. (10.41) and (10.42) simultaneously. Therefore

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

is the equation of the line. This particular form of the equation of a line has been given already in §§ 2.3 and 4.2; we now see that it is really the equation of a line defined by a pair of planes, the chosen planes being parallel to a pair of axes.

Example 3

Find the equation of the plane through the line

$$\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}), \tag{10.43}$$

which is perpendicular to the plane

$$\boldsymbol{r}\cdot(5\boldsymbol{i} + 18\boldsymbol{j} + 22\boldsymbol{k}) = 0. \tag{10.44}$$

The eq. (10.43) for the line is of the form (10.30) so that it is in the direction of the vector $\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$. Since the required plane passes through the line, its normal $\boldsymbol{v}$ must be perpendicular to this vector. Since the required plane is perpendicular to the plane defined by eq. (10.44), the normals to the two planes must be perpendicular to each other; so $\boldsymbol{v}$ must also be perpendicular to $5\boldsymbol{i}+18\boldsymbol{j}+22\boldsymbol{k}$. Thus

$$\boldsymbol{v} = (\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) \times (5\boldsymbol{i} + 18\boldsymbol{j} + 22\boldsymbol{k}) = 4(2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}),$$

which is a vector in the direction of $2\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}$.

Also the plane passes through the line (10.43) and so contains the point $\boldsymbol{r}_1=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ which lies on the line. Using eq. (10.20), its equation is therefore

$$\{\boldsymbol{r} - (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})\}\cdot\{2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}\} = 0,$$

or

$$\boldsymbol{r}\cdot(2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) = (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})\cdot(2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) = -3.$$

Example 4

Find the equation of the common line of the two planes

$$x + 2y + 3z + 4 = 0,$$
$$2x + 3y + 4z + 5 = 0.$$

Vector Method. The two planes are given in vector form by

$$\boldsymbol{r}\cdot(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = -4, \tag{10.45}$$
$$\boldsymbol{r}\cdot(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) = -5. \tag{10.46}$$

The direction of the common line, which is perpendicular to both normals is

$$(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \times (2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}.$$

Using eq. (10.37), any point $\boldsymbol{r}$ on the common line can be written as

$$\boldsymbol{r} = \alpha(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + \beta(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) + t(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}). \tag{10.47}$$

Substitution in eqs. (10.45) and (10.46) gives

$$14\alpha + 20\beta = -4,$$
$$20\alpha + 29\beta = -5,$$

which are satisfied by $\alpha=-\frac{8}{3}$, $\beta=\frac{5}{3}$; so the equation of the line can be written as

$$\boldsymbol{r} = \tfrac{2}{3}\boldsymbol{i} - \tfrac{1}{3}\boldsymbol{j} - \tfrac{4}{3}\boldsymbol{k} + t(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}). \tag{10.48}$$

Rectangular Coordinates Method. The two planes are

$$x + 2y + 3z + 4 = 0,$$
$$2x + 3y + 4z + 5 = 0.$$

The line lies in all planes of the form

$$(x + 2y + 3z + 4) + \mu(2x + 3y + 4z + 5) = 0.$$

Choosing $\mu=-\frac{1}{2}$ to eliminate x, we find the plane

$$y + 2z + 3 = 0,$$

which can be written as

$$\frac{y+3}{2} = \frac{z}{-1}. \tag{10.49}$$

Eliminating y by choosing $\mu=-\frac{2}{3}$ we find

$$-x + z + 2 = 0$$

which can be written as

$$\frac{x-2}{-1} = \frac{z}{-1}. \tag{10.50}$$

Points on the line must lie on both the planes (10.49) and (10.50), so the equation of the line is

$$\frac{x-2}{-1} = \frac{y+3}{2} = \frac{z}{-1},$$

this can be put in the vector form

$$\boldsymbol{r} = 2\boldsymbol{i} - 3\boldsymbol{j} + t'(-\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}), \tag{10.51}$$

where t' is a variable parameter. Note that the lines defined by eqs. (10.48) and (10.51) are the same line, since the direction $-\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ is the same and the point $2\boldsymbol{i}-3\boldsymbol{j}$ ($t'=0$) on the line (10.51) also lies on the line (10.48) at $t=-\frac{4}{3}$.

In all such problems in three-dimensional analytical geometry of the straight line and plane, either a vector method or a method using rectangular coordinates may be employed, as in Example 4. In the exercises which follow in this chapter, some are given in vector form and others in coordinate form; ability to change from one form to the other is of value.

EXERCISE 10.2

1. Find the line of intersection of the two planes

$$\boldsymbol{r}\cdot(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 1 \quad \text{and} \quad \boldsymbol{r}\cdot(\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) = 0.$$

Find also the plane through the point $\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ which is perpendicular to this line of intersection.

2. Find the equation of the plane through the line defined by the two planes

$$\boldsymbol{r}\cdot(3\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}) = 10,$$
$$\boldsymbol{r}\cdot(2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) = 4,$$

and parallel to the line

$$\boldsymbol{r} = t(6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}).$$

3. Find the equation of the plane through the origin parallel to both of the lines

(i) $\boldsymbol{r}\cdot(\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}) = 1,$ $\boldsymbol{r}\cdot(2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}) = 2,$ (ii) $\boldsymbol{r}\cdot(\boldsymbol{i}-2\boldsymbol{j}) = -2,$ $\boldsymbol{r}\cdot(2\boldsymbol{j}-3\boldsymbol{k}) = 1.$

4. Find the equation of the common line of the two planes

$$\boldsymbol{r}\cdot(4\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}) = 12, \qquad \boldsymbol{r}\cdot(8\boldsymbol{i}+12\boldsymbol{j}-13\boldsymbol{k}) = 32.$$

5. Find the equation of the plane passing through the origin and the line of intersection of $\boldsymbol{r}\cdot\boldsymbol{u}_1=p_1$ and $\boldsymbol{r}\cdot\boldsymbol{u}_2=p_2$.

6. Find the equation of the plane through the points $(2, -1, 0)$, $(3, -4, 5)$ and parallel to the line

$$2x = 3y = 4z.$$

7. The equations of two planes are $x+2y+3z=4$, $3x-y+2z+1=0$. Find the coordinates of the point of intersection of the planes with another plane through the origin at right angles to their line of intersection.

§ 5. Further results concerning straight lines and planes

In the two previous paragraphs we have simply derived the different forms of the equations of a plane and the equations of a line. There are, of course, many other results in connection with straight lines and planes and in this paragraph we shall derive some of these results.

§ 5.1. The perpendicular distance of the point $P_2(\boldsymbol{r}_2)$ from the line $\boldsymbol{r}=\boldsymbol{r}_1+t\boldsymbol{v}$

In fig. 10.9 let N be the foot of the perpendicular from $P_2(\boldsymbol{r}_2)$ on to the given line which goes through $P_1(\boldsymbol{r}_1)$ and is in the direction of the vector $\boldsymbol{v}$. Then obviously

$$P_2N = P_1P_2 \sin\theta = \left|(\boldsymbol{r}_2-\boldsymbol{r}_1)\times\frac{\boldsymbol{v}}{v}\right|. \tag{10.52}$$

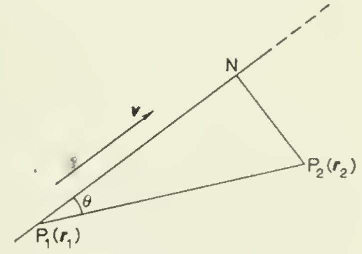

Fig. 10.9

If it is required to find the point N which is the foot of the perpendicular from P_2 on to the line, we proceed as follows. The point N on the line corresponds to some particular value $t=t_1$ (say) of the parameter t. Thus if $\boldsymbol{r}_N$ is the position vector of N, we have

$$\boldsymbol{r}_N = \boldsymbol{r}_1 + t_1\boldsymbol{v}. \tag{10.53}$$

But P_2N is perpendicular to $\boldsymbol{v}$ and therefore

$$\boldsymbol{r}_{\mathrm{N}} - \boldsymbol{r}_2 \equiv \boldsymbol{r}_1 + t_1\boldsymbol{v} - \boldsymbol{r}_2,$$

must be perpendicular to $\boldsymbol{v}$. Therefore

$$(\boldsymbol{r}_1 + t_1\boldsymbol{v} - \boldsymbol{r}_2)\cdot\boldsymbol{v} = 0. \tag{10.54}$$

This eq. (10.54) determines the value of t_1, and then substitution in eq. (10.53) gives the position vector of N. The length of $\mathrm{P_2N}$ can then be found as the modulus of the vector P_2N.

Example 5

Find the distance of the point (5, 4, 2) from the line

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} \tag{10.55}$$

and also the coordinates of the foot of the perpendicular.

Since we are asked to find the coordinates of the foot of the perpendicular we proceed by the second method given above.

The line (10.55) can be written in the vector form

$$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}).$$

Let N be the foot of the perpendicular from the point $\boldsymbol{r}_2 = 5\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ on to the line, having position vector

$$\boldsymbol{r}_{\mathrm{N}} = -\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + t_1(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}). \tag{10.56}$$

Then

$$P_2N = \boldsymbol{r}_{\mathrm{N}} - \boldsymbol{r}_2 = -6\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} + t_1(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}). \tag{10.57}$$

This must be perpendicular to the direction of the line, which is $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$. Thus

$$\{(-6\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) + t_1(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})\}\cdot(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = 0,$$

or

$$-14 + t_1(14) = 0,$$

giving $t_1 = 1$. Therefore from eq. (10.56)

$$\boldsymbol{r}_{\mathrm{N}} = \boldsymbol{i} + 6\boldsymbol{j}$$

and from eq. (10.57)

$$P_2N = -4\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k},$$

so that

$$\mathrm{P_2N} = \sqrt{(16 + 4 + 4)} = 2\sqrt{6};$$

the reader should verify that the result (10.52) gives the same value for $\mathrm{P_2N}$.

§ 5.2. THE NDITIOCON THAT TWO LINES MEET, AND THE POINT OF INTERSECTION

Let the two lines be

$$\boldsymbol{r} = \boldsymbol{r}_1 + t\boldsymbol{v}_1, \tag{10.58}$$

$$\boldsymbol{r} = \boldsymbol{r}_2 + t'\boldsymbol{v}_2 \tag{10.59}$$

where t, t' are variable parameters for the lines. If these two lines meet there must be a real value for $t=t_1$ on the line (10.58) and a real value for $t'=t_2$ on the line (10.59) where the value of $\boldsymbol{r}$ is the same for each, that is

$$\boldsymbol{r}_1 + t_1\boldsymbol{v}_1 = \boldsymbol{r}_2 + t_2\boldsymbol{v}_2. \tag{10.60}$$

If we multiply through scalarly by $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ to eliminate t_1 and t_2, then

$$\boldsymbol{r}_1 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = \boldsymbol{r}_2 \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) \tag{10.61}$$

is the condition that must be satisfied if the two lines meet. To find the point of intersection we need the value of either t_1 or t_2 where eq. (10.60) holds. If we multiply eq. (10.60) through scalarly by $\boldsymbol{r}_2 \times \boldsymbol{v}_2$ we get

$$\boldsymbol{r}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2) + t_1\boldsymbol{v}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2) = 0,$$

or

$$t_1 = -\frac{\boldsymbol{r}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2)}{\boldsymbol{v}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2)}.$$

Substituting this value of t_1 in eq. (10.58), we find that the common point is

$$\boldsymbol{r} = \boldsymbol{r}_1 - \frac{\boldsymbol{r}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2)}{\boldsymbol{v}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2)}\, \boldsymbol{v}_1,$$

which can be written in the form

$$\boldsymbol{r} = -\frac{(\boldsymbol{r}_2 \times \boldsymbol{v}_2) \times (\boldsymbol{r}_1 \times \boldsymbol{v}_1)}{\boldsymbol{v}_1 \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}_2)}.$$

By virtue of eq. (10.61) this can also be written in the form

$$\boldsymbol{r} = -\frac{(\boldsymbol{r}_2 \times \boldsymbol{v}_2) \times (\boldsymbol{r}_1 \times \boldsymbol{v}_1)}{\boldsymbol{v}_2 \cdot (\boldsymbol{r}_1 \times \boldsymbol{v}_1)},$$

and therefore is given by

$$\boldsymbol{r} = \boldsymbol{r}_2 - \frac{\boldsymbol{r}_2 \cdot (\boldsymbol{r}_1 \times \boldsymbol{v}_1)}{\boldsymbol{v}_2 \cdot (\boldsymbol{r}_1 \times \boldsymbol{v}_1)}\, \boldsymbol{v}_2.$$

In this form $\boldsymbol{r}$ is obviously a point on the line (10.59).

Example 6

Find the common point of the two lines

$$\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-3}{4}, \tag{10.62}$$

$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z+1}{1}. \tag{10.63}$$

The vector forms of these lines are

$$\boldsymbol{r} = 3\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}), \tag{10.64}$$

and

$$\boldsymbol{r} = (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + t'(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}). \tag{10.65}$$

Let $t=t_1$ on line (10.64) and $t'=t_2$ on line (10.65) be the common point. Then

$$3\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k} + t_1(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + t_2(\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).$$

To find t_1 multiply through scalarly by

$$(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \times (\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 0\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}.$$

We get

$$-10 - 12 + t_1(-6 - 16) = 0,$$

giving $t_1=-1$. Substituting in eq. (10.64) gives the common point as

$$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}, \tag{10.66}$$

and this obviously lies on the line (10.65) where $t'=t_2=0$.

In rectangular coordinates the result (10.66) is the point $(x, y, z)=(1, 2, -1)$. Note that this could be found as the point satisfying *all* the equations in (10.62) and (10.63).

§ 5.3. The vector moment of a line

Consider the equation of a line given in the form (10.29) which is

$$\boldsymbol{r} = \boldsymbol{r}_1 + s\boldsymbol{u},$$

where $\boldsymbol{u}$ is the unit vector along the line. We notice that at all points $\boldsymbol{r}$ on the line we have

$$\boldsymbol{r} \times \boldsymbol{u} = \boldsymbol{r}_1 \times \boldsymbol{u},$$

so that the vector $\boldsymbol{r}\times\boldsymbol{u}$ has the same value at all points of the line. It is therefore a vector characteristic of the line and is called the *vector moment* of the line about the origin O. We denote this vector by $\boldsymbol{m}$.

The two vectors: $\boldsymbol{u}$, unit vector along the line, and $\boldsymbol{m}=\boldsymbol{r}_1\times\boldsymbol{u}=\boldsymbol{r}\times\boldsymbol{u}$ for any point on the line, are called the *fundamental vectors of the line* and completely define the line. To prove that they are sufficient to define the line, we show that given $\boldsymbol{u}$ and $\boldsymbol{m}$ we can derive the equation of the line

in the more usual simple form. Since

$$r \times u = m, \tag{10.67}$$

we can solve this equation for r by a method given in Ch. 9 Example 18 eq. (9.97). We find that, using $u^2 = 1$,

$$r = u \times m + su, \tag{10.68}$$

where s is an arbitrary parameter. This is the usual form for the equation of a line and we have used s as the parameter since u is the unit vector along the line and therefore s does measure the distance from the point $r_1 = u \times m$ on the line. We note that this particular point $r_1 = u \times m$ on the line is the foot N of the perpendicular from the origin on to the line, since in fig. 10.10 $ON = r_1 = u \times m$ is perpendicular to u and also lies on the line (10.68).

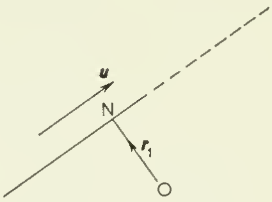

Fig. 10.10

If a line is defined as the intersection of two planes such as

$$r \cdot v_1 = \lambda_1 \quad \text{and} \quad r \cdot v_2 = \lambda_2, \tag{10.69}$$

then the unit vector along the line is

$$u = \frac{v_1 \times v_2}{|v_1 \times v_2|}.$$

So

$$m = r \times u = \frac{r \times (v_1 \times v_2)}{|v_1 \times v_2|},$$

which can be written as

$$m = \frac{(r \cdot v_2)v_1 - (r \cdot v_1)v_2}{|v_1 \times v_2|}.$$

But all points r on the line must satisfy eqs. (10.69), and so

$$m = \frac{\lambda_2 v_1 - \lambda_1 v_2}{|v_1 \times v_2|}.$$

Again, in practice the rectangular coordinate form of the equation of the line defined by eqs. (10.69) can be found as

$$\frac{x - x_1}{u_1} = \frac{y - y_1}{u_2} = \frac{z - z_1}{u_3},$$

where $\boldsymbol{u}=u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k}$ is *unit* vector along the line; its vector moment $\boldsymbol{m}$ is then given

$$\boldsymbol{m} = (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}) \times (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}).$$

§ 5.4. The equation of the plane through the point $\boldsymbol{r}_2$ and the line $(\boldsymbol{u}, \boldsymbol{m})$

In fig. 10.11, suppose $\boldsymbol{r}$ is the position vector of any point P in the plane. Then $\boldsymbol{r}_1=\boldsymbol{u}\times\boldsymbol{m}$ being the foot of the perpendicular from the origin on to the line, we must have the three vectors $\boldsymbol{u}$, $\boldsymbol{r}-\boldsymbol{r}_2$ and $\boldsymbol{r}_1-\boldsymbol{r}_2$ coplanar. Thus

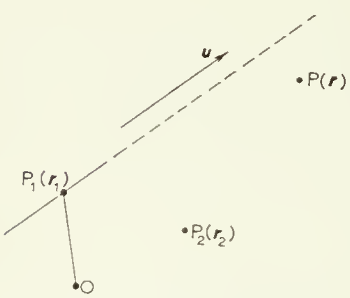

Fig. 10.11

$$(\boldsymbol{r} - \boldsymbol{r}_2)\cdot\{(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{u}\} = 0,$$

and since $\boldsymbol{r}_1\times\boldsymbol{u}=\boldsymbol{m}$, this becomes

$$(\boldsymbol{r} - \boldsymbol{r}_2)\cdot\{\boldsymbol{m} - \boldsymbol{r}_2 \times \boldsymbol{u}\} = 0,$$

or

$$\boldsymbol{r}\cdot\{\boldsymbol{m} - \boldsymbol{r}_2 \times \boldsymbol{u}\} - \boldsymbol{r}_2\cdot\boldsymbol{m} = 0.$$

In particular, the equation of the plane through the origin ($\boldsymbol{r}_2=\boldsymbol{0}$) and the line $(\boldsymbol{u}, \boldsymbol{m})$ is

$$\boldsymbol{r}\cdot\boldsymbol{m} = 0. \tag{10.70}$$

Example 7

Prove that the equation of the plane through the line $(\boldsymbol{u}_1, \boldsymbol{m}_1)$ parallel to the line $(\boldsymbol{u}_2, \boldsymbol{m}_2)$ is

$$\boldsymbol{r}\cdot(\boldsymbol{u}_1 \times \boldsymbol{u}_2) = \boldsymbol{u}_2\cdot\boldsymbol{m}_1.$$

The normal to the required plane must be perpendicular to both the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ along each line. Therefore the normal is in the direction $\boldsymbol{u}_1\times\boldsymbol{u}_2$.

Let $\boldsymbol{r}_1$ be any point on the first line; then $\boldsymbol{r}_1$ lies in the required plane. Therefore, using eq. (10.20) the equation of the plane is

$$(\boldsymbol{r} - \boldsymbol{r}_1)\cdot(\boldsymbol{u}_1 \times \boldsymbol{u}_2) = 0,$$

or

$$\boldsymbol{r}\cdot(\boldsymbol{u}_1 \times \boldsymbol{u}_2) = \boldsymbol{r}_1\cdot(\boldsymbol{u}_1 \times \boldsymbol{u}_2),$$

and using $\boldsymbol{r}_1\times\boldsymbol{u}_1=\boldsymbol{m}_1$ we can write this as

$$\boldsymbol{r}\cdot(\boldsymbol{u}_1 \times \boldsymbol{u}_2) = \boldsymbol{u}_2\cdot(\boldsymbol{r}_1 \times \boldsymbol{u}_1) = \boldsymbol{u}_2\cdot\boldsymbol{m}_1.$$

§ 5.5. The common perpendicular to two skew lines

In general two lines in space do not meet even when they are not parallel. Two lines which are not parallel and which do not meet are said to be *skew* lines. Let the equations of two such lines be

$$\boldsymbol{r} = \boldsymbol{r}_1 + t\boldsymbol{v}_1, \qquad (10.71)$$

and

$$\boldsymbol{r} = \boldsymbol{r}_2 + t'\boldsymbol{v}_2. \qquad (10.72)$$

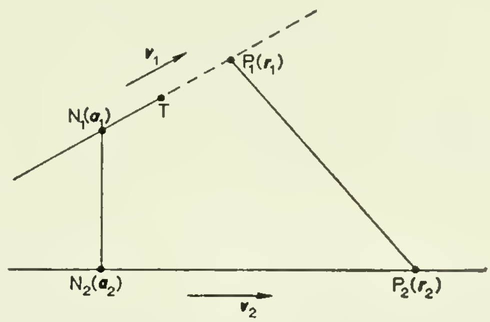

Fig. 10.12

We can now prove by vector methods that there exists a common perpendicular to these two lines, and the end points of this common perpendicular can be found.

In fig. 10.12, which must be pictured as a three-dimensional figure, suppose the line (10.72) is in the plane of the paper and the line (10.71) is thought of as going through the paper at T. We shall suppose that a common perpendicular does exist and that N_1 and N_2 are the end points on the two lines. Let the position vector of N_1 be $\boldsymbol{a}_1$ where $t=t_1$ in eq. (10.71), and the position vector of N_2 be $\boldsymbol{a}_2$ where $t'=t_2$ in eq. (10.72).

Now $N_1N_2=\boldsymbol{a}_2-\boldsymbol{a}_1$ must be perpendicular to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is therefore in the direction $\boldsymbol{v}_1\times\boldsymbol{v}_2$; if it is of length d, then

$$\boldsymbol{a}_2 - \boldsymbol{a}_1 = d(\boldsymbol{v}_1 \times \boldsymbol{v}_2)/|\boldsymbol{v}_1 \times \boldsymbol{v}_2|,$$

or

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 + t_2\boldsymbol{v}_2 - t_1\boldsymbol{v}_1 = \frac{d(\boldsymbol{v}_1 \times \boldsymbol{v}_2)}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|}. \qquad (10.73)$$

Multiplying through scalarly by $\boldsymbol{v}_1\times\boldsymbol{v}_2$ gives

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1)\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = d|\boldsymbol{v}_1 \times \boldsymbol{v}_2|,$$

or

$$d = \frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2)}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|}. \qquad (10.74)$$

This result is also obvious from the fact that N_1N_2 is the projection of $P_1P_2=\boldsymbol{r}_2-\boldsymbol{r}_1$ on to the unit vector $(\boldsymbol{v}_1\times\boldsymbol{v}_2)/|\boldsymbol{v}_1\times\boldsymbol{v}_2|$ along the common perpendicular N_1N_2. To find the position vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of the ends, we require the values of t_1 and t_2. These can be determined from eq.

(10.73) by multiplying through scalarly by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively, giving

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1)\cdot\boldsymbol{v}_1 + t_2(\boldsymbol{v}_1\cdot\boldsymbol{v}_2) - t_1\boldsymbol{v}_1^2 = 0, \tag{10.75}$$

and

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1)\cdot\boldsymbol{v}_2 + t_2\boldsymbol{v}_2^2 - t_1(\boldsymbol{v}_1\cdot\boldsymbol{v}_2) = 0, \tag{10.76}$$

which can be solved for t_1 and t_2. If real values of t_1 and t_2 can be found to satisfy these equations, then a common perpendicular to the two lines does exist. Since the coefficients of t_1 and t_2 and the first terms in each of the eqs. (10.75) and (10.76) are real, the values of t_1, t_2 will be real and finite provided

$$\boldsymbol{v}_1^2\boldsymbol{v}_2^2 - (\boldsymbol{v}_1\cdot\boldsymbol{v}_2)^2 \neq 0$$

or

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2 \neq 0.$$

Now, unless the two lines are parallel, so that $\boldsymbol{v}_1\times\boldsymbol{v}_2=\boldsymbol{0}$, the value of $(\boldsymbol{v}_1\times\boldsymbol{v}_2)^2$ is essentially positive and so a real and unique common perpendicular can be found. When the end points have been determined the equation and also the length of the common perpendicular can be found from them. Note that the length is zero, which means that the lines meet, when $d=0$ in eq. (10.74). This means that

$$\boldsymbol{r}_2\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = \boldsymbol{r}_1\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2),$$

which is equivalent to the condition given in eq. (10.61).

Example 8

Find the coordinates of the ends of the common perpendicular to the two lines

$$x = 3t, \qquad y = 9 - t, \qquad z = 2 + t,$$

and

$$x = -6 - 3t', \qquad y = -5 + 2t', \qquad z = 10 + 4t'.$$

Here the lines have been given in the parametric form of eq. (10.31). The vector forms are found as

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = 9\boldsymbol{j} + 2\boldsymbol{k} + t(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}), \tag{10.77}$$

and

$$\boldsymbol{r} = -6\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k} + t'(-3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}). \tag{10.78}$$

Let the ends of the common perpendicular be given by

$$\boldsymbol{r} = \boldsymbol{a}_1 = 9\boldsymbol{j} + 2\boldsymbol{k} + t_1(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}), \tag{10.79}$$

on line (10.77), and

$$\boldsymbol{r} = \boldsymbol{a}_2 = -6\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k} + t_2(-3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}), \tag{10.80}$$

on line (10.78).

The direction of line (10.77) is $\boldsymbol{v}_1 = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ while the direction of line (10.78) is $\boldsymbol{v}_2 = -3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$. Therefore the direction of the common perpendicular is along

$$(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \times (-3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) = 3(-2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}). \tag{10.81}$$

Note that the scalar product of this vector with $\boldsymbol{v}_1$ and also with $\boldsymbol{v}_2$ is zero, as required. So, using eq. (10.73), we have

$$\begin{aligned} \boldsymbol{a}_1 - \boldsymbol{a}_2 = 6\boldsymbol{i} + 14\boldsymbol{j} - 8\boldsymbol{k} + t_1(3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) \\ - t_2(-3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}) = \pm \frac{d(-2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k})}{\sqrt{30}}, \end{aligned} \tag{10.82}$$

where d is the length of the common perpendicular. Multiply eq. (10.82) through scalarly by $3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and scalarly by $-3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ to give

$$\begin{aligned} -4 + 11t_1 + 7t_2 &= 0, \\ -22 - 7t_1 - 29t_2 &= 0, \end{aligned}$$

with solutions $t_1 = 1$, $t_2 = -1$. Thus the ends of the common perpendicular are, from eqs. (10.79) and (10.80),

$$\boldsymbol{a}_1 = 3\boldsymbol{i} + 8\boldsymbol{j} + 3\boldsymbol{k} \quad \text{and} \quad \boldsymbol{a}_2 = -3\boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k},$$

giving

$$\boldsymbol{a}_1 - \boldsymbol{a}_2 = 3(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}),$$

so that

$$d = |\boldsymbol{a}_1 - \boldsymbol{a}_2| = 3\sqrt{(4 + 25 + 1)} = 3\sqrt{30}.$$

This result does satisfy eq. (10.74). The equation of the common perpendicular is

$$\boldsymbol{r} = 3\boldsymbol{i} + 8\boldsymbol{j} + 3\boldsymbol{k} + t(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}).$$

The reader should verify by vector methods, that the common perpendicular of two skew lines is the shortest distance between them.

EXERCISE 10.3

1. Find the equations of the line through $(-1, 0, 1)$ which cuts at right angles the line

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z-3}{3}.$$

Find also the length of this line, and the equation of the plane containing the two lines.

2. Prove that the lines

(i) $2x + y + 3z - 1 = 0 = x + 10y - 21$,

(ii) $2x - y = 0 = 7x + z - 6$,

intersect. Find the coordinates of their point of intersection and the equation of the plane containing them.

3. Show that the lines

$$\frac{x-3}{1} = \frac{y+9}{-2} = \frac{z-2}{1}, \qquad \frac{x-4}{5} = \frac{y}{1} = \frac{z+5}{-3},$$

intersect at right angles. Find the equations of the line through their common point perpendicular to both, and the equation of the plane containing the given lines.

4. Find the foot of the perpendicular from the point (1, 2, 1) to the line joining the origin to the point (2, 2, 5).

5. Find the distance of the point (−1, 2, 5) from the line through the point (3, 4, 5) whose direction cosines are in the ratio (2, −3, 6).

6. A straight line is drawn parallel to the plane $x+y+z=0$ and intersects the two straight lines $x=a$, $y=z$; $x=-a$, $y=-z$ in P, Q. Show that the locus of a point which divides PQ in the ratio $\lambda : (1-\lambda)$ is a straight line.

7. Through the point P($\boldsymbol{a}$) a plane is drawn at right angles to OP to meet the coordinate axes in A, B, C. Prove that the area of the triangle ABC is

$$\tfrac{1}{2}a^5/(\boldsymbol{a}\cdot\boldsymbol{i})(\boldsymbol{a}\cdot\boldsymbol{j})(\boldsymbol{a}\cdot\boldsymbol{k}).$$

8. Find the length and equation of the line which is the common perpendicular to the two lines

$$\frac{x-4}{2} = \frac{y+2}{1} = \frac{z-3}{-1}, \qquad \frac{x+7}{3} = \frac{y+2}{2} = \frac{z-1}{1}.$$

9. Find the length and coordinates of the ends of the common perpendicular to the two lines

$$\frac{x-3}{1} = \frac{y-4}{1} = \frac{z+1}{-1}, \qquad \frac{x+6}{2} = \frac{y+5}{4} = \frac{z-1}{-1}.$$

10. Find the direction, length and ends of the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-4}{3}, \qquad \frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

11. Find the equation of the plane through the line

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-4}{3}$$

and parallel to the line $6x=3y=2z$. Hence or otherwise find the shortest distance between the two lines.

12. A rectangular parallelepiped has edges $a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$. Show by vector methods that the shortest distances between a diagonal and the edges that do not meet it are

$$\frac{bc}{\sqrt{(b^2+c^2)}}, \qquad \frac{ca}{\sqrt{(c^2+a^2)}}, \qquad \frac{ab}{\sqrt{(a^2+b^2)}}.$$

13. Find the equation of the plane containing the lines

$$\frac{x-2}{3}=\frac{y-1}{-6}=\frac{z+1}{2}, \qquad \frac{x-2}{1}=\frac{y-1}{2}=\frac{z+1}{2},$$

and find the locus of points equidistant from these two lines.

14. Find the equation of the line through the point $(4, -3, 7)$ perpendicular to the plane $3x-5y+4z=5$. Find also the coordinates of the foot of this perpendicular and the coordinates of the image of the point in the plane. (The image of a point in a plane is the point on the other side of the plane, on its normal, and equidistant from it.)

15. Show that the angle between the line

$$\frac{x-4}{2}=\frac{y-2}{3}=\frac{z-8}{6},$$

and the plane $18x-5y+22z=85$ is $\cos^{-1}(32/49)$. Find the coordinates of the point of intersection and the equations of the projection of the line on the plane.

16. Find the equation of the line of intersection of the planes

$$\boldsymbol{r}\cdot(2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})=5, \qquad \boldsymbol{r}\cdot(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})=4.$$

Verify that its fundamental vectors are

$$\boldsymbol{u}=(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})/\sqrt{6}, \qquad \boldsymbol{m}=(3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})/\sqrt{6}.$$

Write down the equation of the plane through the origin and the line.

17. Show that the equation of the plane through the line $\boldsymbol{u}$, $\boldsymbol{m}$ and perpendicular to the plane $\boldsymbol{r}\cdot\boldsymbol{v}=p$ is

$$\boldsymbol{r}\cdot(\boldsymbol{u}\times\boldsymbol{v})=\boldsymbol{v}\cdot\boldsymbol{m}.$$

18. Prove that the projection of the line $(\boldsymbol{u}, \boldsymbol{m})$ on the plane $\boldsymbol{r}\cdot\boldsymbol{v}=p$ is defined by the vectors

$$k\boldsymbol{v}\times(\boldsymbol{u}\times\boldsymbol{v}), \qquad k\{p(\boldsymbol{v}\times\boldsymbol{u})+(\boldsymbol{v}\cdot\boldsymbol{m})\boldsymbol{v}\}$$

where $u=1$, $v=1$ and $k|\boldsymbol{u}\times\boldsymbol{v}|=1$.

19. Find the equations of the image of the line of intersection of the planes

$$2x-3y-z=0, \qquad 3x-4y-3z=0,$$

in the plane $x+y+z=2$.

20. Prove that the straight line $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{u}$, $(\boldsymbol{u}, \boldsymbol{m}=\boldsymbol{a}\times\boldsymbol{u})$ meets the coordinate planes in the points A, B, C where

$$\boldsymbol{A} = \frac{\boldsymbol{i}\times\boldsymbol{m}}{\boldsymbol{u}\cdot\boldsymbol{i}}, \qquad \boldsymbol{B} = \frac{\boldsymbol{j}\times\boldsymbol{m}}{\boldsymbol{u}\cdot\boldsymbol{j}}, \qquad \boldsymbol{C} = \frac{\boldsymbol{k}\times\boldsymbol{m}}{\boldsymbol{u}\cdot\boldsymbol{k}}.$$

If α, β, γ are the angles $\widehat{\mathrm{BOC}}$, $\widehat{\mathrm{COA}}$, $\widehat{\mathrm{AOB}}$ prove that

$$\tan\alpha = \frac{|\boldsymbol{m}|(\boldsymbol{m}\cdot\boldsymbol{i})}{(\boldsymbol{m}\cdot\boldsymbol{j})(\boldsymbol{m}\cdot\boldsymbol{k})}, \qquad \tan\beta = \frac{|\boldsymbol{m}|(\boldsymbol{m}\cdot\boldsymbol{j})}{(\boldsymbol{m}\cdot\boldsymbol{k})(\boldsymbol{m}\cdot\boldsymbol{j})}, \qquad \tan\gamma = \frac{|\boldsymbol{m}|(\boldsymbol{m}\cdot\boldsymbol{k})}{(\boldsymbol{m}\cdot\boldsymbol{i})(\boldsymbol{m}\cdot\boldsymbol{j})}.$$

Hence prove that if these angles are fixed the point $\boldsymbol{a}$ lies on the plane

$$\boldsymbol{r}\cdot\{\sqrt{(\tan\alpha)}\boldsymbol{i} + \sqrt{(\tan\beta)}\boldsymbol{j} + \sqrt{(\tan\gamma)}\boldsymbol{k}\} = 0.$$

CHAPTER 11

VECTOR FUNCTIONS. DIFFERENTIAL GEOMETRY OF CURVES. LINE INTEGRALS

§ 1. Introduction

If P is a point on a given curve and $\boldsymbol{r}$ is its localised position vector relative to a chosen origin O, then $\boldsymbol{r}$ depends on the position of the point on the curve. If the position of the point P is defined by its distance s measured along the curve from some fixed point P_1 on it, then we can say that the position vector $\boldsymbol{r}$ of the point P is given as a *vector function* of the distance s, meaning that it is a vector whose magnitude and direction vary in a specific manner with s. A particular example of such a vector is that defining the position of a point on a straight line in eq. (10.29):

$$\boldsymbol{r} = \boldsymbol{r}_1 + s\boldsymbol{u}, \tag{11.1}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{u}$ are given vectors and $\boldsymbol{r}$ varies in a definite manner as s varies, and so in this eq. (11.1) $\boldsymbol{r}$ is given as a function of the scalar parameter s. If $\boldsymbol{u}$ is a unit vector we know that s measures the distance of P from the point $P_1(\boldsymbol{r}_1)$ on the line.

Similarly the velocity or the acceleration of a point moving along a given curve will vary in magnitude and direction during the motion and may depend therefore on the distance s measured along the curve or on the time T at which the particle is at the point. Thus the velocity and acceleration vectors for any particular motion will be given as functions of s or T.

Likewise the position vector of a point in a given plane depends on the position of the point in the plane; if we define the position of the point in the plane by its rectangular coordinates (x, y) or its polar coordinates (r, θ) we can write the position vector $\boldsymbol{r}$ relative to the origin in the form

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}, \tag{11.2}$$

or

$$\boldsymbol{r} = (r\cos\theta)\boldsymbol{i} + (r\sin\theta)\boldsymbol{j}. \tag{11.3}$$

Thus in eq. (11.2) $\boldsymbol{r}$ is given as a function of the scalar parameters (x, y), whilst in eq. (11.3) $\boldsymbol{r}$ is given as a function of the scalar parameters (r, θ).

We have thus introduced the concept of a vector $\boldsymbol{v}$ say, being given as a variable function of certain scalar parameters. The dependence of a vector on scalar parameters is expressed symbolically as $\boldsymbol{v}=\boldsymbol{V}(s)$ or $\boldsymbol{v}=\boldsymbol{V}(T)$ or $\boldsymbol{r}=\boldsymbol{f}(x, y)$ as the case may be.

Example 1

The equation of a circle of radius a in the Ox, Oy plane shown in fig. 11.1 can be written parametrically in the form

$$x = a\cos\theta, \qquad y = a\sin\theta.$$

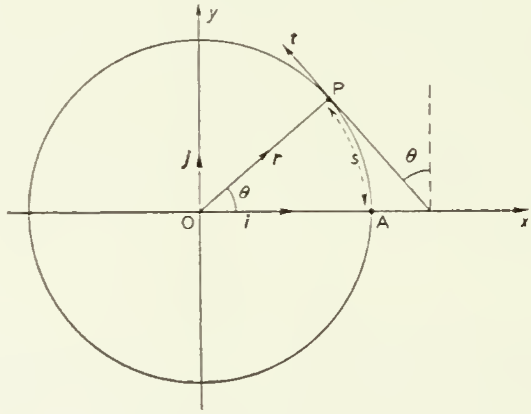

Fig. 11.1

Thus if $\boldsymbol{r}$ is the position vector of any point P on the circle, we have

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = (a\cos\theta)\boldsymbol{i} + (a\sin\theta)\boldsymbol{j}. \tag{11.4}$$

In this equation $\boldsymbol{r}$ is given as a function of the scalar parameter θ. If we write $\theta=s/a$ then s is the distance along the curve from the point A to the point P, and

$$\boldsymbol{r} = a\left(\cos\frac{s}{a}\right)\boldsymbol{i} + a\left(\sin\frac{s}{a}\right)\boldsymbol{j}, \tag{11.5}$$

so that $\boldsymbol{r}$ is given as a function of the arc length s from A.

Example 2

The curve known as the cycloid, being the curve traced out by a point on the circumference of a circle which rolls in contact with a fixed straight line Ox in fig. 11.2, can be expressed parametrically in the form

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

Thus the position vector of P is

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = a(\theta - \sin\theta)\boldsymbol{i} + a(1 - \cos\theta)\boldsymbol{j}, \tag{11.6}$$

so that $\boldsymbol{r}$ is given as a function of the parameter θ.

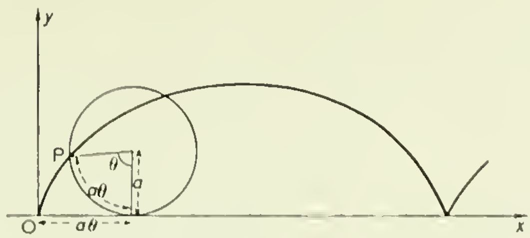

Fig. 11.2

Example 3

The equation of a sphere of radius a, centre the origin, is

$$x^2 + y^2 + z^2 = a^2.$$

Using spherical polar coordinates (r, θ, φ) we can express the values of x, y, z on this sphere in fig. 11.3 as

$$x = a \sin\theta \cos\varphi,$$
$$y = a \sin\theta \sin\varphi,$$
$$z = a \cos\theta.$$

Using the triad $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along the axes, we have the position vector of any point P on the sphere as

$$\boldsymbol{r} = (a \sin\theta \cos\varphi)\boldsymbol{i} + (a \sin\theta \sin\varphi)\boldsymbol{j} + (a \cos\theta)\boldsymbol{k}. \tag{11.7}$$

Here $\boldsymbol{r}$ is given as a function of the two scalar parameters θ, φ.

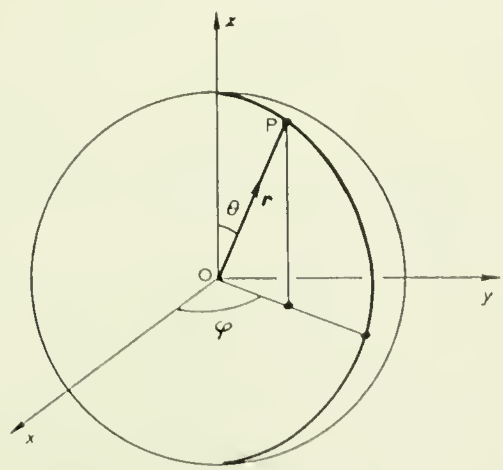

Fig. 11.3

§ 1.1. Differentiation with respect to a scalar parameter

The fact that a vector varies for example from one point of a curve to another, or from one time to another, leads us to consider the rate of change of the vector at a point on a curve or at a particular time. Taking a quite general case, let us suppose that a vector $\boldsymbol{v}$ is given as a function of any scalar parameter u by

$$\boldsymbol{v} = V(u);$$

let $V(u)$, $V(u+\delta u)$ represent the vector functions corresponding to the

values u, $u+\delta u$ of the independent variable parameter. If we now draw from a common origin O (fig. 11.4) two vectors OP, OP' such that

$$OP = V(u), \qquad OP' = V(u + \delta u),$$

then these define another vector PP' given by

$$PP' = OP' - OP = V(u + \delta u) - V(u),$$

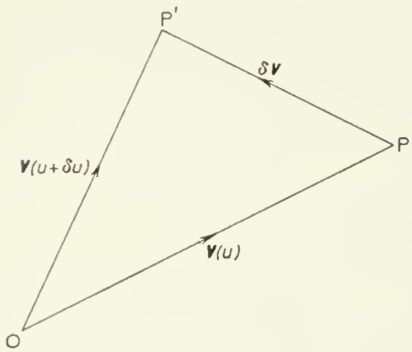

Fig. 11.4

which represents the change in the value of $\boldsymbol{v}$ as u changes to $u+\delta u$. Calling this change in the value of $\boldsymbol{v}$ the vector $\delta\boldsymbol{v}$ we can form another vector parallel to PP' and of magnitude $PP'/\delta u$ and represented therefore by

$$\frac{\delta \boldsymbol{v}}{\delta u} = \frac{PP'}{\delta u} = \frac{V(u + \delta u) - V(u)}{\delta u}.$$

This vector we call the average rate of change of the vector $\boldsymbol{v}$ over the interval δu of the parameter u.

Defining, as for scalar rates of change, the limit of this ratio as $\delta u \to 0$, we obtain the vector

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u} = \lim_{\delta u \to 0} \frac{\delta \boldsymbol{v}}{\delta u} = \lim_{\delta u \to 0} \frac{V(u + \delta u) - V(u)}{\delta u} = V'(u). \qquad (11.8)$$

Provided this limit exists it defines the instantaneous rate of change of the vector $\boldsymbol{v}$. Its direction is the limiting direction of PP' which is not in general in the direction of either OP or OP', and its magnitude is the limiting value of the ratio $PP'/\delta u$.

This introduces the concept of the derivative of a vector with respect to a scalar parameter u. The operation of differentiation in this way obviously obeys all the rules of ordinary analysis for differentiation of functions. For convenience, we give here a list of the rules, which are relevant to vectors.

If $\boldsymbol{v}$, $\boldsymbol{w}$ are given as two vector functions of a scalar parameter u, and m is a scalar function of the same parameter u, then

(i) $$\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{v} + \boldsymbol{w}) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u} + \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}u}. \qquad (11.9)$$

(ii) $$\frac{\mathrm{d}}{\mathrm{d}u}(m\boldsymbol{v}) = \frac{\mathrm{d}m}{\mathrm{d}u}\boldsymbol{v} + m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}, \tag{11.10}$$

(iii) $$\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{v}\cdot\boldsymbol{w}) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\cdot\boldsymbol{w} + \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}u}, \tag{11.11}$$

(iv) $$\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{v}\times\boldsymbol{w}) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\times\boldsymbol{w} + \boldsymbol{v}\times\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}u}. \tag{11.12}$$

These results are deduced by the usual methods: for example in eqs. (11.11), if $\delta\boldsymbol{v}$, $\delta\boldsymbol{w}$ are the increments in the vectors $\boldsymbol{v}$, $\boldsymbol{w}$ corresponding to the increment δu in u then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{v}\cdot\boldsymbol{w}) &= \lim_{\delta u\to 0}\frac{(\boldsymbol{v}+\delta\boldsymbol{v})\cdot(\boldsymbol{w}+\delta\boldsymbol{w}) - \boldsymbol{v}\cdot\boldsymbol{w}}{\delta u}\\ &= \lim_{\delta u\to 0}\left(\frac{\delta\boldsymbol{v}}{\delta u}\cdot\boldsymbol{w} + \boldsymbol{v}\cdot\frac{\delta\boldsymbol{w}}{\delta u} + \frac{\delta\boldsymbol{v}}{\delta u}\cdot\delta\boldsymbol{w}\right).\end{aligned}$$

As $\delta u\to 0$,

$$\lim_{\delta u\to 0}\left(\frac{\delta\boldsymbol{v}}{\delta u}\cdot\boldsymbol{w}\right) = \left(\lim_{\delta u\to 0}\frac{\delta\boldsymbol{v}}{\delta u}\right)\cdot\boldsymbol{w} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\cdot\boldsymbol{w},$$

likewise the second term, and in the last term $\delta\boldsymbol{w}\to 0$, so that we get the required result (iii).

In (iv) care must be taken in preserving the order of the vectors, as in all vector product formulae.

We note that there is no quotient rule since division of vectors has not been defined.

If m is independent of u in (ii) we have

$$\frac{\mathrm{d}}{\mathrm{d}u}(m\boldsymbol{v}) = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u},$$

while if the vector $\boldsymbol{v}$ is independent of u but m is not, we have

$$\frac{\mathrm{d}}{\mathrm{d}u}(m\boldsymbol{v}) = \frac{\mathrm{d}m}{\mathrm{d}u}\boldsymbol{v}. \tag{11.13}$$

A particular example of this result is when any vector $\boldsymbol{v}$ is expressed in terms of its components along $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. If the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are in *fixed* directions and

$$\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}, \tag{11.14}$$

then using eq. (11.13)

$$\frac{d\boldsymbol{v}}{du} = \frac{dv_1}{du}\boldsymbol{i} + \frac{dv_2}{du}\boldsymbol{j} + \frac{dv_3}{du}\boldsymbol{k}, \tag{11.15}$$

$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ being independent of u. There is also one important result to note in connection with this vector $d\boldsymbol{v}/du$. Its modulus is given by

$$\left|\frac{d\boldsymbol{v}}{du}\right| = \left\{\left(\frac{dv_1}{du}\right)^2 + \left(\frac{dv_2}{du}\right)^2 + \left(\frac{dv_3}{du}\right)^2\right\}^{\frac{1}{2}}. \tag{11.16}$$

Now

$$v^2 = v_1^2 + v_2^2 + v_3^2,$$

and differentiating this result with respect to u, we get

$$v\frac{dv}{du} = v_1\frac{dv_1}{du} + v_2\frac{dv_2}{du} + v_3\frac{dv_3}{du},$$

so that

$$\frac{dv}{du} = \frac{v_1\dfrac{dv_1}{du} + v_2\dfrac{dv_2}{du} + v_3\dfrac{dv_3}{du}}{(v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}}}. \tag{11.17}$$

which is obviously quite different from that given in eq. (11.16). Thus in general $|d\boldsymbol{v}/du|$ is *not equal* to dv/du.

When $\boldsymbol{v}=\boldsymbol{w}$, (iii) gives

$$\frac{d}{du}(\boldsymbol{v}^2) = \frac{d}{du}(\boldsymbol{v}\cdot\boldsymbol{v}) = 2\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt};$$

but $\boldsymbol{v}^2=v^2$ and $dv^2/du=2v\, dv/du$, thus

$$v\frac{dv}{du} = \boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{du},$$

or

$$\frac{dv}{du} = \frac{1}{v}\left(\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{du}\right). \tag{11.18}$$

The result given in eq. (11.17) is this result (11.18) for the vector $\boldsymbol{v}$ in eq. (11.14).

If the modulus of $\boldsymbol{v}$ is constant, so that $dv/du=0$ then

$$\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{du} = 0,$$

therefore either $\mathrm{d}\boldsymbol{v}/\mathrm{d}u=0$ so that $\boldsymbol{v}$ is a fixed vector, or $\mathrm{d}\boldsymbol{v}/\mathrm{d}u$ is perpendicular to $\boldsymbol{v}$.

Example 4

Consider the position vector $\boldsymbol{r}$ for the point on a circle, given in eq. (11.4). We have

$$\boldsymbol{r} = (a\cos\theta)\boldsymbol{i} + (a\sin\theta)\boldsymbol{j},$$

and then

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta} = -(a\sin\theta)\boldsymbol{i} + (a\cos\theta)\boldsymbol{j}. \tag{11.19}$$

Thus

$$\boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta} = [(a\cos\theta)\boldsymbol{i} + (a\sin\theta)\boldsymbol{j}]\cdot[(-a\sin\theta)\boldsymbol{i} + (a\cos\theta)\boldsymbol{j}] = 0;$$

since $\boldsymbol{r}$ is not a fixed vector, $\mathrm{d}\boldsymbol{r}/\mathrm{d}\theta$ must be perpendicular to $\boldsymbol{r}$. From the form of $\mathrm{d}\boldsymbol{r}/\mathrm{d}\theta$ in eq. (11.19) we see that it is a vector in the direction of the tangent to the circle at P in fig. 11.1, that is at right angles to $\boldsymbol{r}=\boldsymbol{OP}$. Note also that $\boldsymbol{r}^2=a^2$, so that $\mathrm{d}r/\mathrm{d}\theta=0$, whilst $|\mathrm{d}\boldsymbol{r}/\mathrm{d}\theta|=a$.

Derivatives of several vector expressions and derivatives of products of three or more vectors are found by a combination of the rules (i)–(iv) as in ordinary analysis.

Example 5

Using rule (iii) we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\{\boldsymbol{r}\cdot(\boldsymbol{v}\times\boldsymbol{w})\} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) + \boldsymbol{r}\cdot\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{v}\times\boldsymbol{w}),$$

and applying rule (iv) this becomes

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) + \boldsymbol{r}\cdot\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\times\boldsymbol{w}\right) + \boldsymbol{r}\cdot\left(\boldsymbol{v}\times\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}u}\right).$$

Example 6

Differentiate the expression $r^2\boldsymbol{r}+(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b}$ where $\boldsymbol{r}$ is given as a function of a parameter u, $r=|\boldsymbol{r}|$ and $\boldsymbol{a}$, $\boldsymbol{b}$ are constant vectors.

We have, using eq. (11.10)

$$\frac{\mathrm{d}}{\mathrm{d}u}r^2\boldsymbol{r} = 2r\frac{\mathrm{d}r}{\mathrm{d}u}\boldsymbol{r} + r^2\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b} = \left\{\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{a}\cdot\boldsymbol{r})\right\}\boldsymbol{b};$$

then using eq. (11.11) this last expression is

$$\left(\boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\right)\boldsymbol{b}.$$

Example 7

With $\boldsymbol{r}$, $\boldsymbol{a}$, $\boldsymbol{b}$ as given in Example 6, differentiate $\boldsymbol{rb}/(\boldsymbol{a}\cdot\boldsymbol{r})$.

Here the term $\boldsymbol{a}\cdot\boldsymbol{r}$ is a scalar quantity which is a function of u, and we treat it as $(\boldsymbol{a}\cdot\boldsymbol{r})^{-1}$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\boldsymbol{rb}}{\boldsymbol{a}\cdot\boldsymbol{r}}\right)=\frac{1}{\boldsymbol{a}\cdot\boldsymbol{r}}\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{rb})-\frac{\boldsymbol{rb}}{(\boldsymbol{a}\cdot\boldsymbol{r})^2}\frac{\mathrm{d}}{\mathrm{d}u}(\boldsymbol{a}\cdot\boldsymbol{r}),$$

which becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\boldsymbol{rb}}{\boldsymbol{a}\cdot\boldsymbol{r}}\right)=\frac{\mathrm{d}\boldsymbol{r}/\mathrm{d}u}{\boldsymbol{a}\cdot\boldsymbol{r}}\boldsymbol{b}-\frac{\boldsymbol{r}(\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}/\mathrm{d}u)}{(\boldsymbol{a}\cdot\boldsymbol{r})^2}\boldsymbol{b}.$$

§ 1.2. Higher order derivatives

The derivative $\mathrm{d}\boldsymbol{v}/\mathrm{d}u$ of a vector $\boldsymbol{v}$ is, in general, itself given as a function of the parameter u. Its derivative is called the second derivative with respect to u, and it is written as usual in the form

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\right)=\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}u^2},$$

and so on for higher derivatives. The derivative of order n of a vector is always a vector.

We can adapt such general theorems as the mean value theorem and Taylor's theorem to obtain the expansion of a vector function. The conditions of these theorems must be satisfied by the vector function but since any vector can be resolved into components along three fixed vectors such as the triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, this means that eq. (11.15) applies to the first and all higher derivatives, so in stating the conditions to be satisfied by the vector function we are really stating the conditions for all the component functions. If a vector function $\boldsymbol{V}(u)$ and all its derivatives up to and including $\boldsymbol{V}^{(n-1)}(u)$ are continuous in the interval $a\leqslant u\leqslant b$, $\boldsymbol{V}^{(n)}(u)$ exists in the interval $a<u<b$ and h is any scalar such that $a<u+h<b$ then Taylor's theorem states that

$$\boldsymbol{V}(u+h)=\boldsymbol{V}(u)+h\boldsymbol{V}'(u)+\frac{h^2}{2!}\boldsymbol{V}''(u)+\dots+\frac{h^{n-1}}{(n-1)!}\boldsymbol{V}^{(n-1)}(u)+\frac{h^n}{n!}\boldsymbol{V}^{(n)}(u+\theta h), \tag{11.20}$$

where $0<\theta<1$. The Taylor series can be continued to infinity provided it is a convergent series.

When the expansion is about the value $u=0$, we obtain the result corresponding to Maclaurin's series:

$$V(u) = V(0) + uV'(0) + \ldots + \frac{u^n}{n!} V^{(n)}(\theta u), \tag{11.21}$$

where $0<\theta<1$. Again we can let $n\to\infty$ to obtain an infinite series, provided that it is convergent.

An example of the use of Taylor's theorem will be given in § 2.4.

§ 1.3. Partial derivatives. Total differential

If a vector $\boldsymbol{v}$ is given as a function of several independent scalar parameters $u, w, x, \ldots$ we can let the first variable u increase to $u+\delta u$ while the others remain constant; if $\delta\boldsymbol{v}$ is the corresponding change in $\boldsymbol{v}$, the limiting value of $\delta\boldsymbol{v}/\delta u$ as $\delta u\to 0$ is called the *partial derivative* of $\boldsymbol{v}$ with respect to u; it is written $\partial\boldsymbol{v}/\partial u$. Similarly we define $\partial\boldsymbol{v}/\partial w$, $\partial\boldsymbol{v}/\partial x, \ldots$. These derivatives will be given as functions of the same set of variables, and can be differentiated partially to give second partial derivatives $\partial^2\boldsymbol{v}/\partial u^2$, $\partial^2\boldsymbol{v}/\partial u\partial w, \ldots$. Then under certain conditions similar to those for derivatives of scalar functions

$$\frac{\partial^2\boldsymbol{v}}{\partial u\partial w} = \frac{\partial^2\boldsymbol{v}}{\partial w\partial u}.$$

We can also introduce the concept of differentials; the *total differentia* of $\boldsymbol{v}$ is given by the formula

$$\mathrm{d}\boldsymbol{v} = \frac{\partial\boldsymbol{v}}{\partial u}\,\mathrm{d}u + \frac{\partial\boldsymbol{v}}{\partial w}\,\mathrm{d}w + \ldots. \tag{11.22}$$

Example 8

Using eq. (11.7) we have for the position vector $\boldsymbol{r}$ of any point on the surface of a sphere of radius a,

$$\boldsymbol{r} = (a\sin\theta\cos\varphi)\boldsymbol{i} + (a\sin\theta\sin\varphi)\boldsymbol{j} + (a\cos\theta)\boldsymbol{k};$$

then

$$\frac{\partial\boldsymbol{r}}{\partial\theta} = (a\cos\theta\cos\varphi)\boldsymbol{i} + (a\cos\theta\sin\varphi)\boldsymbol{j} - (a\sin\theta)\boldsymbol{k},$$

and

$$\frac{\partial\boldsymbol{r}}{\partial\varphi} = -(a\sin\theta\sin\varphi)\boldsymbol{i} + (a\sin\theta\cos\varphi)\boldsymbol{j}.$$

Also

$$\frac{\partial^2 \boldsymbol{r}}{\partial\theta\partial\varphi} = \frac{\partial^2 \boldsymbol{r}}{\partial\varphi\partial\theta} = -(a\cos\theta\sin\varphi)\boldsymbol{i} + (a\cos\theta\cos\varphi)\boldsymbol{j},$$

while

$$\frac{\partial^2 \boldsymbol{r}}{\partial\theta^2} = -(a\sin\theta\cos\varphi)\boldsymbol{i} - (a\sin\theta\sin\varphi)\boldsymbol{j} - (a\cos\theta)\boldsymbol{k},$$

and so on for higher partial derivatives. The total differential $\mathrm{d}\boldsymbol{r}$ is given by

$$\begin{aligned}\mathrm{d}\boldsymbol{r} = {} & (a\cos\theta\cos\varphi\,\mathrm{d}\theta - a\sin\theta\sin\varphi\,\mathrm{d}\varphi)\boldsymbol{i} \\ & + (a\cos\theta\sin\varphi\,\mathrm{d}\theta + a\sin\theta\cos\varphi\,\mathrm{d}\varphi)\boldsymbol{j} - (a\sin\theta\,\mathrm{d}\theta)\boldsymbol{k}.\end{aligned}$$

§ 1.4. Integration of a vector function with respect to a scalar parameter

Given a vector function $\boldsymbol{V}(u)$ of a scalar parameter u, the process of finding another vector function $\boldsymbol{F}(u)$ which is such that its derivative with respect to u is $\boldsymbol{V}(u)$ is called integration, as in ordinary analysis; we write

$$\boldsymbol{F}(u) = \int \boldsymbol{V}(u)\,\mathrm{d}u, \tag{11.23}$$

and this means that

$$\frac{\mathrm{d}}{\mathrm{d}u}\{\boldsymbol{F}(u)\} = \boldsymbol{V}(u). \tag{11.24}$$

The function $\boldsymbol{F}(u)$ is not unique since any other function $\boldsymbol{F}(u)+\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary constant vector also satisfies eq. (11.24). In any problem, $\boldsymbol{c}$ is determined by other conditions on $\boldsymbol{F}(u)$. The function $\boldsymbol{F}(u)$ in eq. (11.23) is called an indefinite integral of $\boldsymbol{V}(u)$.

From the results in § 1.1 we can write down some indefinite integrals. We note that in some of these integrals, the integrands while involving vector functions are themselves scalar functions, being scalar products. Thus the indefinite integrals of these functions are also scalar quantities, and any arbitrary constant of integration required in any problem must also be a scalar.

$$\int \left\{\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}u} + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\cdot\boldsymbol{w}\right\}\mathrm{d}u = \boldsymbol{v}\cdot\boldsymbol{w},$$

$$\int 2\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}\,\mathrm{d}u = \boldsymbol{v}^2,$$

$$\int \boldsymbol{v}\times\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}u^2}\,\mathrm{d}u = \boldsymbol{v}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}u}.$$

If $\boldsymbol{a}$ is a constant vector

$$\int \boldsymbol{a} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\,\mathrm{d}u = \boldsymbol{a} \times \boldsymbol{r}.$$

EXERCISE 11.1

Differentiate the expressions in Nos. 1–6, in which $\boldsymbol{r}$ is a function of u, $r=|\boldsymbol{r}|$ and the other quantities are constants.

1. $r^3\boldsymbol{r} + \boldsymbol{a} \times \boldsymbol{r}$.

2. $|a\boldsymbol{r} + r\boldsymbol{b}|^2$.

3. $\dfrac{\boldsymbol{r}}{r^3}$.

4. $\boldsymbol{r}^2 + \dfrac{1}{\boldsymbol{r}^2}$.

5. $\frac{1}{2}m\left(\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\right)^2$.

6. $\dfrac{\boldsymbol{r}\cdot\boldsymbol{b}}{(\boldsymbol{a}\cdot\boldsymbol{r})^2}$.

7. Find the first and second derivatives of

$$\text{(i)}\quad \boldsymbol{r}\cdot(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}), \qquad \text{(ii)}\quad \boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}),$$

where $\dot{\boldsymbol{r}}=\mathrm{d}\boldsymbol{r}/\mathrm{d}u$ and $\ddot{\boldsymbol{r}}=\mathrm{d}^2\boldsymbol{r}/\mathrm{d}u^2$.

8. If n, $\boldsymbol{a}$, $\boldsymbol{b}$ are constants, and

$$\boldsymbol{r} = \boldsymbol{a}\cos nu + \boldsymbol{b}\sin nu,$$

prove that

$$\ddot{\boldsymbol{r}} + n^2\boldsymbol{r} = 0, \qquad \boldsymbol{r} \times \dot{\boldsymbol{r}} = n\boldsymbol{a} \times \boldsymbol{b}.$$

9. Find $\boldsymbol{r}_u=\partial\boldsymbol{r}/\partial u$ and $\boldsymbol{r}_\theta=\partial\boldsymbol{r}/\partial\theta$ when

$$\boldsymbol{r} = u(\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta) + (1 - u^2)\boldsymbol{k}.$$

Evaluate $\boldsymbol{r}_u \times \boldsymbol{r}_\theta$ and its magnitude when $u=1$, $\theta=\frac{1}{4}\pi$.

Calculate $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{u\theta}$, $\boldsymbol{r}_{\theta\theta}$ at this point.

10. Find values of $\boldsymbol{r}$ satisfying the equations

$$\text{(i)}\quad \ddot{\boldsymbol{r}} = \boldsymbol{a}, \qquad \text{(ii)}\quad \boldsymbol{a} \times \ddot{\boldsymbol{r}} = \boldsymbol{b}, \qquad \text{(iii)}\quad \ddot{\boldsymbol{r}} = \boldsymbol{a}u + \boldsymbol{b},$$

where $\dot{\boldsymbol{r}}=\mathrm{d}\boldsymbol{r}/\mathrm{d}u$, $\ddot{\boldsymbol{r}}=\mathrm{d}^2\boldsymbol{r}/\mathrm{d}u^2$; $\boldsymbol{a}$, $\boldsymbol{b}$ are constant vectors, and $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ both vanish when $u=0$.

§ 2. Differential geometry of plane curves

If P is any point on a plane curve as shown in fig. 11.5 and O is the origin of vectors in the plane of the curve, then the position vector $\boldsymbol{OP}=\boldsymbol{r}$ is, of course, a function of the position P on the curve. To define

the position of P we can specify the arc distance s measured along the curve from some fixed point A on the curve; then $\boldsymbol{r}$ will be given as a function of s, as in eq. (11.5) for the circle. If a parametric form of the equation of the curve is given, we can use the parameter itself to define the position vector of P, as in eqs. (11.4) and (11.6) for the circle and cycloid.

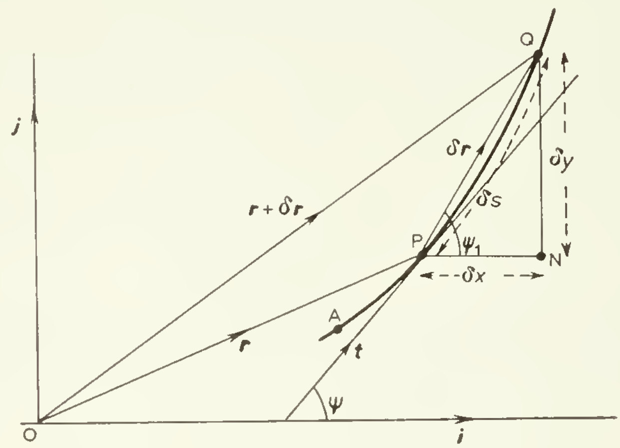

Fig. 11.5

The change in the value of $\boldsymbol{r}$ as we proceed along the curve is determined by the difference

$$\delta \boldsymbol{r} = \boldsymbol{OQ} - \boldsymbol{OP} = \boldsymbol{PQ},$$

between the position vectors of the points P and Q at arc distances s and $s+\delta s$ respectively from A; the arc distance PQ is δs. Dividing through by δs we have

$$\frac{\delta \boldsymbol{r}}{\delta s} = \frac{\boldsymbol{PQ}}{\delta s}.$$

In the limit as $\delta s \to 0$ the length of the vector $\boldsymbol{PQ}$, that is the chord PQ, approximates more and more closely to the arc length δs, so that the magnitude of the vector $\delta \boldsymbol{r}/\delta s$ tends to unity. Also as Q moves back to P, the direction of the chord PQ approximates more and more closely to the direction of the tangent to the curve at P. If we denote differentiation with respect to s by a prime, we see that the vector

$$\boldsymbol{r}' = \frac{d\boldsymbol{r}}{ds} = \lim_{\delta s \to 0} \frac{\boldsymbol{PQ}}{\delta s},$$

is a unit vector along the direction of the tangent to the curve at P in

the sense of s increasing. Calling this unit vector along the tangent $\boldsymbol{t}$, we have $\boldsymbol{t}=\boldsymbol{r}'$.

If we now introduce fixed rectangular axes in the directions $\boldsymbol{i}, \boldsymbol{j}$ in the plane of the curve, then the components (x, y) of $\boldsymbol{r}$ along them will be given as functions of s. Thus

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} \quad \text{and} \quad \boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}s}\boldsymbol{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\boldsymbol{j}. \tag{11.25}$$

As $\boldsymbol{t}$ is a unit vector we must be able to express it in the form

$$(\cos\psi)\boldsymbol{i} + (\sin\psi)\boldsymbol{j}$$

where ψ is the angle between $\boldsymbol{i}$ and $\boldsymbol{t}$. Thus

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\psi, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi, \tag{11.26}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\psi. \tag{11.27}$$

Eq. (11.26) implies also that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1. \tag{11.28}$$

Relations (11.26) and (11.27) are the limiting forms of the equalities $\delta x/\delta s=\cos\psi_1$, $\delta y/\delta s=\sin\psi_1$ and $\delta y/\delta x=\tan\psi_1$ in the right-angled triangle formed by the chord PQ and lines PN, QN drawn parallel to the axes, as shown in fig. 11.5.

Note also that in the notation of differentials

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2,$$

so that if x and y are given as functions of a parameter u, then

$$\left(\frac{\mathrm{d}s}{\mathrm{d}u}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2. \tag{11.29}$$

Example 9

The position vector $\boldsymbol{r}$ of any point P on the circle in fig. 11.1 is given by eq. (11.5) as

$$\boldsymbol{r} = \left(a\cos\frac{s}{a}\right)\boldsymbol{i} + \left(a\sin\frac{s}{a}\right)\boldsymbol{j}.$$

Thus

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \left(-\sin\frac{s}{a}\right)\boldsymbol{i} + \left(\cos\frac{s}{a}\right)\boldsymbol{j} = (-\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j},$$

and is obviously a unit vector in the direction of the tangent at P in the sense of s increasing.

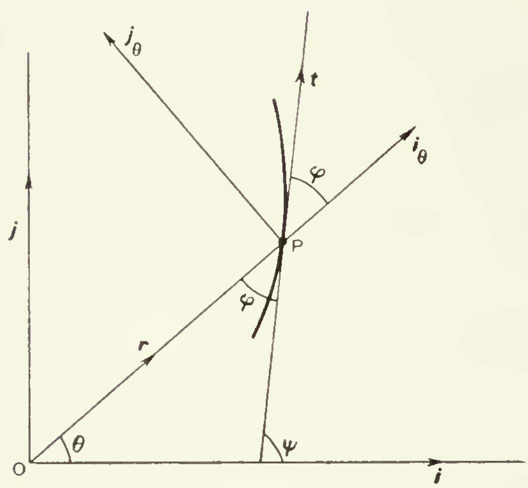

Fig. 11.6

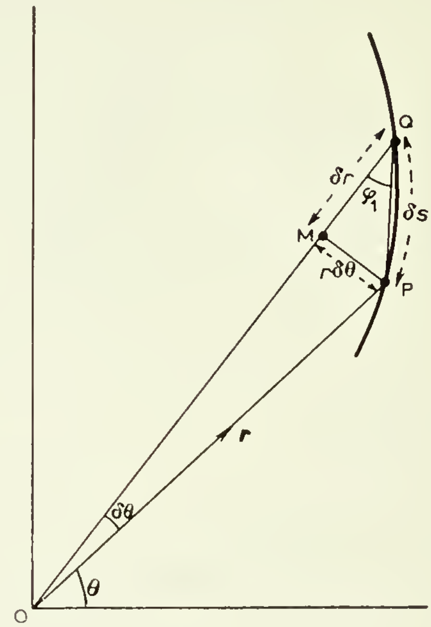

Fig. 11.7

If we use plane polar coordinates to define the position of P, then from fig. 11.6

$$\boldsymbol{r} = r[(\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}].$$

Thus

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}r}{\mathrm{d}s}(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}) + r\frac{\mathrm{d}\theta}{\mathrm{d}s}(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j});$$

we write this as

$$\boldsymbol{t} = \frac{\mathrm{d}r}{\mathrm{d}s}\boldsymbol{i}_\theta + r\frac{\mathrm{d}\theta}{\mathrm{d}s}\boldsymbol{j}_\theta, \tag{11.30}$$

where $\boldsymbol{i}_\theta = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ is a unit vector making an angle θ with $\boldsymbol{i}$, while $\boldsymbol{j}_\theta = (-\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j}$ is a unit vector making an angle θ with $\boldsymbol{j}$. These unit vectors are marked in fig. 11.6. Again since $\boldsymbol{t}$ is a unit vector, we can write it as

$$\boldsymbol{t} = \cos\varphi\,\boldsymbol{i}_\theta + \sin\varphi\,\boldsymbol{j}_\theta,$$

where φ is the angle between $\boldsymbol{t}$ and $\boldsymbol{i}_\theta$ as shown. Comparing with eq. (11.30), we find

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \cos\varphi, \qquad r\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sin\varphi, \tag{11.31}$$

so that

$$r\frac{\mathrm{d}\theta}{\mathrm{d}r} = \tan\varphi. \tag{11.32}$$

These formulae together with

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 1,$$

or

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2, \tag{11.33}$$

are limits of simple geometrical relations in the right-angled triangle formed by the chord PQ, the radius OQ and the perpendicular PM from P on to OQ in fig. 11.7. In this diagram OM$=$OP $\cos\delta\theta = r\cos\delta\theta \approx r$ so that MQ$\approx r+\delta r-r=\delta r$; also PM$=$OP $\sin\delta\theta\approx r\delta\theta$, all to the first order in $\delta\theta$ and δr. As Q$\to$P so $\varphi_1\to\varphi$.

Notice also from fig. 11.6 that $\psi=\theta+\varphi$.

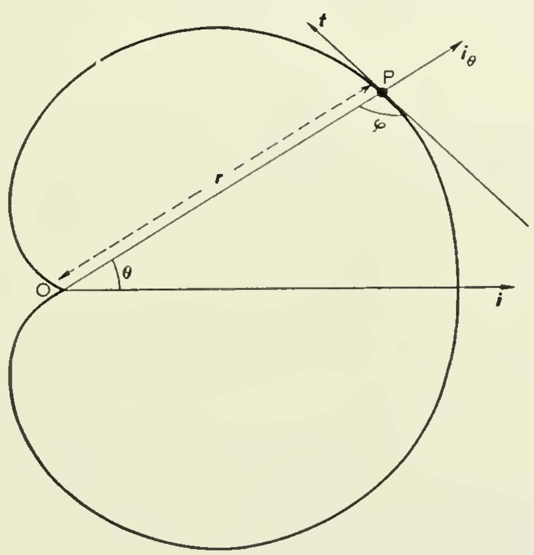

Fig. 11.8

Example 10

The cardioid shown in fig. 11.8 is best described in the polar coordinate form

$$r = a(1+\cos\theta).$$

We have

$$\tan\varphi = \frac{r}{\mathrm{d}r/\mathrm{d}\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta},$$

which simplifies to

$$\tan\varphi = -\tan\tfrac{1}{2}\theta = \tan(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta).$$

Also the arc length ds along the curve is given by eq. (11.33)

$$\mathrm{d}s^2 = a^2\sin^2\theta\,\mathrm{d}\theta^2 + a^2(1+\cos\theta)^2\,\mathrm{d}\theta^2,$$

which becomes

$$\mathrm{d}s^2 = 4a^2\cos^2\tfrac{1}{2}\theta\,\mathrm{d}\theta^2,$$

or $\mathrm{d}s = 2a\cos\frac{1}{2}\theta\,\mathrm{d}\theta$ if s is measured in the sense of θ increasing.

§ 2.1. The equations of the tangent and normal at any point on a plane curve

The equations of the tangent and normal at P on a plane curve may be found by vector methods.

The unit vector $\boldsymbol{t}$ along the tangent at P is

$$\boldsymbol{t} = \cos\psi\,\boldsymbol{i} + \sin\psi\,\boldsymbol{j}, \tag{11.34}$$

where $\tan\psi = \mathrm{d}y/\mathrm{d}x$.

If now we use $\boldsymbol{R}$ for the running vector along the tangent at P, as shown in fig. 11.9, $\boldsymbol{r}$ being the position vector of P on the curve, the

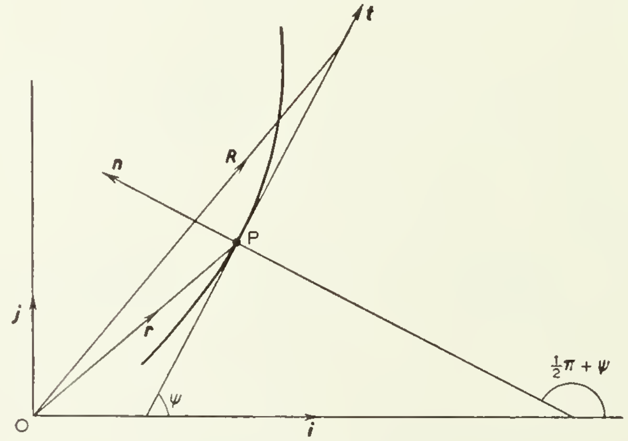

Fig. 11.9

equation of the tangent is

$$\boldsymbol{R} = \boldsymbol{r} + \lambda\boldsymbol{t}, \tag{11.35}$$

where λ is a scalar parameter.

If $\boldsymbol{i}$, $\boldsymbol{j}$ define a set of rectangular axes at O, and we write

$$\boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j},$$

and

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j},$$

then in rectangular coordinates, eq. (11.35) becomes

$$\frac{X - x}{\cos\psi} = \frac{Y - y}{\sin\psi},$$

or

$$Y - y = (\tan\psi)(X - x) = \frac{\mathrm{d}y}{\mathrm{d}x}(X - x).$$

The unit vector $\boldsymbol{n}$ along the normal to the curve is perpendicular to $\boldsymbol{t}$ as shown in fig. 11.9; hence

$$\boldsymbol{n} = \cos(\tfrac{1}{2}\pi + \psi)\,\boldsymbol{i} + \sin(\tfrac{1}{2}\pi + \psi)\,\boldsymbol{j},$$

or

$$\boldsymbol{n} = -\sin\psi\,\boldsymbol{i} + \cos\psi\,\boldsymbol{j}. \tag{11.36}$$

Again using $\boldsymbol{R}$ for the running vector along the normal, its equation is

$$\boldsymbol{R} = \boldsymbol{r} + \lambda\boldsymbol{n};$$

referred to rectangular axes, this is

$$\frac{X - x}{-\sin\psi} = \frac{Y - y}{\cos\psi},$$

or

$$Y - y = (-\cot\psi)(X - x) = -\frac{\mathrm{d}x}{\mathrm{d}y}(X - x).$$

Examples of tangents and normals to plane curves have been given in Ch. 1.

§ 2.2. Lemma

Let $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ be two unit vectors represented by displacements OA, OB drawn from the same point O in fig. 11.10.

The internal bisector OC of the angle between these two vectors is in the direction of the unit vector $(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2)/2\cos\frac{1}{2}\theta$, whilst the external bisector OD is the direction of the unit vector $(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)2\sin\frac{1}{2}\theta$. Note that AB is parallel to OD. Using the second of these results we see that when any unit vector $\boldsymbol{\nu}$ is varying continuously in direction so that two consecutive directions can be represented by $\boldsymbol{\nu}_1=\boldsymbol{\nu}$ and $\boldsymbol{\nu}_2=\boldsymbol{\nu}+\delta\boldsymbol{\nu}$ as in

fig. 11.11 the angle between them being $\delta\theta$, then in the limit as $\delta\theta \to 0$, the unit vector perpendicular to $\boldsymbol{v}$ in a plane parallel to the consecutive direction $\boldsymbol{v}+\delta\boldsymbol{v}$ is given by

$$\lim_{\delta\theta\to 0} \frac{\delta \boldsymbol{v}}{2 \sin \frac{1}{2}\delta\theta} = \lim_{\delta\theta\to 0} \left(\frac{\delta \boldsymbol{v}}{\delta\theta}\right)\left(\frac{\frac{1}{2}\delta\theta}{\sin \frac{1}{2}\delta\theta}\right) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\theta}. \tag{11.37}$$

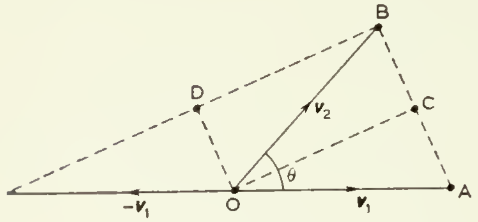

Fig. 11.10

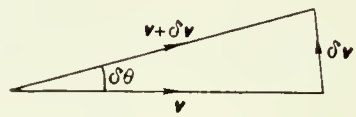

Fig. 11.11

§ 2.3. Curvature of a plane curve

The unit vector $\boldsymbol{t}$ along the tangent to a curve at P given by eqs. (11.25) and (11.26) as

$$\boldsymbol{t} = \boldsymbol{r}' = \cos \psi\, \boldsymbol{i} + \sin \psi\, \boldsymbol{j}, \tag{11.38}$$

is a vector which changes its direction but not its magnitude, which is always unity, as the point P moves along the curve. The rate of change of its direction is measured by the rate of change $\mathrm{d}\psi/\mathrm{d}s$ of the angle ψ; this is called the *curvature* of the curve at the point P. The reciprocal ρ of κ given by

$$\rho = \frac{1}{\kappa} = \frac{\mathrm{d}s}{\mathrm{d}\psi}, \tag{11.39}$$

is, for a reason which will appear later, called the *radius of curvature* at the point P. From eq. (11.38) by differentiation with respect to s, we have

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \boldsymbol{t}' = \boldsymbol{r}'' = (-\sin \psi\, \boldsymbol{i} + \cos \psi\, \boldsymbol{j}) \frac{\mathrm{d}\psi}{\mathrm{d}s} = \kappa \boldsymbol{n}, \tag{11.40}$$

where $\boldsymbol{n}$ is a unit vector along the normal to the curve defined by eq. (11.36) and shown in fig. 11.9.

Note that $\boldsymbol{t}$ is a special case of the vector $\boldsymbol{v}$ in the lemma in § 2.2. Thus $\delta\psi$ being the angle between consecutive tangents, we know that $\mathrm{d}\boldsymbol{t}/\mathrm{d}\psi$ is a unit vector perpendicular to $\boldsymbol{t}$ in the plane of the curve, and must there-

fore be $\boldsymbol{n}$; that is

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}s} = \kappa\boldsymbol{n}.$$

The result that $\boldsymbol{t}$ and $\boldsymbol{t}'$ are perpendicular vectors follows also by differentiation of the equation $\boldsymbol{t}^2=1$ with respect to s, giving

$$\boldsymbol{t}\cdot\boldsymbol{t}' = 0.$$

From the eq. (11.40) we write

$$\boldsymbol{r}'' = \kappa\boldsymbol{n}, \tag{11.41}$$

and by squaring both sides of eq. (11.41) we deduce that

$$\kappa^2 = \frac{1}{\rho^2} = (\boldsymbol{r}'')^2.$$

In rectangular coordinates this relation becomes

$$\kappa^2 = \frac{1}{\rho^2} = (x''\boldsymbol{i} + y''\boldsymbol{j})^2,$$

which is

$$\kappa^2 = \frac{1}{\rho^2} = (x'')^2 + (y'')^2 = \left(\frac{\mathrm{d}^2x}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2y}{\mathrm{d}s^2}\right)^2. \tag{11.42}$$

Example 11

For the circle given in Example 1 we have

$$\boldsymbol{r} = a\cos\frac{s}{a}\,\boldsymbol{i} + a\sin\frac{s}{a}\,\boldsymbol{j},$$

and we can easily deduce that

$$\boldsymbol{r}'' = -\frac{1}{a}\cos\frac{s}{a}\,\boldsymbol{i} - \frac{1}{a}\sin\frac{s}{a}\,\boldsymbol{j},$$

which is obviously in the opposite sense to $\boldsymbol{r}$, that is along the inward normal. Also

$$(\boldsymbol{r}'')^2 = \frac{1}{a^2}\left(\cos^2\frac{s}{a} + \sin^2\frac{s}{a}\right) = \frac{1}{a^2},$$

so that

$$\kappa^2 = \frac{1}{\rho^2} = \frac{1}{a^2},$$

and the curvature of the circle is $1/a$ where a is the radius. This result is, of course, obvious from the fact that $\psi=\frac{1}{2}\pi+\theta=\frac{1}{2}\pi+s/a$ so that

$$\kappa = \frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{1}{a}.$$

We have seen that for the circle the result (11.42) is very simple. In general however it is not easy to express the position vector $\boldsymbol{r}$ for a point P on a curve as a function of the arc length s, so the result (11.42) is not a very useful one. However from this result we can derive other more useful expressions for the curvature of a curve.

When the position vector $\boldsymbol{r}$ on the curve is given as a function of a parameter u, then using a dot to denote differentiation with respect to u, we have

$$\boldsymbol{r}' = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\dot{\boldsymbol{r}}}{\dot{s}}.$$

Thus

$$\dot{\boldsymbol{r}} = \dot{s}\boldsymbol{r}' = \dot{s}\boldsymbol{t}, \tag{11.43}$$

and

$$\ddot{\boldsymbol{r}} = \ddot{s}\boldsymbol{t} + \dot{s}\boldsymbol{t}'\dot{s} = \ddot{s}\boldsymbol{t} + \dot{s}^2\kappa\boldsymbol{n}. \tag{11.44}$$

From eq. (11.29)

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2. \tag{11.45}$$

Since $\boldsymbol{t}$ and $\boldsymbol{n}$ are perpendicular unit vectors in the senses shown in fig. 11.9 then the vector $\boldsymbol{t}\times\boldsymbol{n}$ will be a unit vector perpendicular to the plane of the curve; so if $\boldsymbol{i}$, $\boldsymbol{j}$ define rectangular axes as in fig. 11.9 we have $\boldsymbol{k}=\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{t}\times\boldsymbol{n}$. So forming the vector product of $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ in eqs. (11.43) and (11.44) we have

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = \dot{s}^3\kappa\boldsymbol{k},$$

and therefore

$$\kappa = \frac{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|}{\dot{s}^3}. \tag{11.46}$$

But if $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}$ we have $\dot{\boldsymbol{r}}=\dot{x}\boldsymbol{i}+\dot{y}\boldsymbol{j}$ and $\ddot{\boldsymbol{r}}=\ddot{x}\boldsymbol{i}+\ddot{y}\boldsymbol{j}$ so that using eqs. (11.45) and (11.46)

$$\kappa = \left|\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}\right|. \tag{11.47}$$

A curve given in the form $y=f(x)$ may be written parametrically in the form $x=u$, $y=f(u)$ so that $\dot{x}=1$, $\ddot{x}=0$ and

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \ddot{y} = \frac{\mathrm{d}^2y}{\mathrm{d}u^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

The result in eq. (11.47) then reduces to

$$\kappa = \left|\frac{\mathrm{d}^2y/\mathrm{d}x^2}{\{1 + (\mathrm{d}y/\mathrm{d}x)^2\}^{\frac{3}{2}}}\right|. \tag{11.48}$$

§ 2.4. The circle of curvature

If we consider any circle centre C($\boldsymbol{c}$) passing through the point P($\boldsymbol{r}$) on a curve as in fig. 11.12, then using $\boldsymbol{R}$ for the running vector round the circle its equation is

$$(\boldsymbol{R} - \boldsymbol{c})^2 = (\boldsymbol{r} - \boldsymbol{c})^2, \tag{11.49}$$

expressing the fact that $TC^2=PC^2$. This eq. (11.49) can be written in the form

$$(\boldsymbol{R} - \boldsymbol{r})\cdot(\boldsymbol{R} + \boldsymbol{r} - 2\boldsymbol{c}) = 0. \tag{11.50}$$

Let us now suppose that the position vector $\boldsymbol{r}$ on the curve is given as a function of the arc length s by the equation

$$\boldsymbol{r} = \boldsymbol{V}(s). \tag{11.51}$$

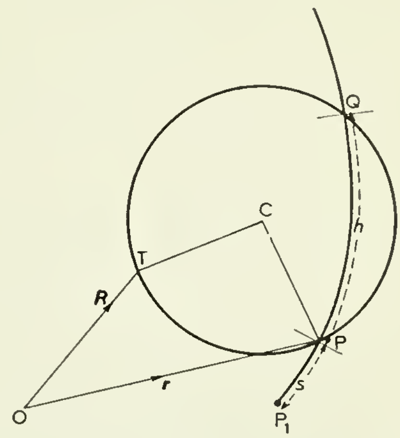

Fig. 11.12

The circle defined by eq. (11.50) will cut the curve again at the point Q where $\boldsymbol{R}=\boldsymbol{V}(s+h)$, arc PQ$=h$, if $\boldsymbol{R}=\boldsymbol{V}(s+h)$ satisfies eq. (11.50), that is, if

$$\{\boldsymbol{V}(s + h) - \boldsymbol{V}(s)\}\cdot\{\boldsymbol{V}(s + h) + \boldsymbol{V}(s) - 2\boldsymbol{c}\} = 0. \tag{11.52}$$

Using Taylor's expansion, eq. (11.20), on $\boldsymbol{V}(s+h)$ we have

$$\boldsymbol{V}(s + h) = \boldsymbol{V}(s) + h\boldsymbol{V}'(s) + \tfrac{1}{2}h^2\boldsymbol{V}''(s) + \ldots,$$

or, using $\boldsymbol{r}=\boldsymbol{V}(s)$, $\boldsymbol{r}'=\boldsymbol{V}'(s)$ and so on, we may rewrite eq. (11.52) in the form

$$\{h\boldsymbol{r}' + \tfrac{1}{2}h^2\boldsymbol{r}'' + \ldots\}\cdot\{2(\boldsymbol{r} - \boldsymbol{c}) + h\boldsymbol{r}' + \tfrac{1}{2}h^2\boldsymbol{r}'' + \ldots\} = 0,$$

and collecting terms in powers of h, this is

$$2h\boldsymbol{r}'\cdot(\boldsymbol{r} - \boldsymbol{c}) + h^2\{\boldsymbol{r}''\cdot(\boldsymbol{r} - \boldsymbol{c}) + (\boldsymbol{r}')^2\} + h^3\{\boldsymbol{r}'\cdot\boldsymbol{r}'' + \ldots\} + \ldots = 0. \tag{11.53}$$

This equation obviously has one root $h=0$ since we did make the circle cut the curve once at P. It will have two roots $h=0$, that is it will cut the curve twice at P, if the equation has no term in h, which means that the coefficient of h must be zero. This gives

$$\boldsymbol{r}'\cdot(\boldsymbol{r} - \boldsymbol{c}) = 0, \tag{11.54}$$

which means that $CP=\boldsymbol{r}-\boldsymbol{c}$ is perpendicular to $\boldsymbol{r}'=\boldsymbol{t}$ the direction of the

tangent to the curve. The circle and the curve then have a common tangent and have what is called *two-point* contact. The circle and the curve will have *three-point* contact, if the eq. (11.53) has three roots $h=0$, which means that the coefficient of h^2 must also be zero, that is

$$\boldsymbol{r}''\cdot(\boldsymbol{r}-\boldsymbol{c}) = -(\boldsymbol{r}')^2 = -\boldsymbol{t}^2 = -1. \tag{11.55}$$

But because of eq. (11.54) we know that $\boldsymbol{r}-\boldsymbol{c}$ is perpendicular to $\boldsymbol{t}$ and therefore along the normal $\boldsymbol{n}$ to the curve, so that we can write

$$\boldsymbol{r}-\boldsymbol{c} = \lambda\boldsymbol{n}. \tag{11.56}$$

Further since $\boldsymbol{r}''=\kappa\boldsymbol{n}$, eq. (11.55) gives

$$\kappa\lambda\boldsymbol{n}^2 = -1,$$

or $\lambda=-1/\kappa$. Therefore substituting in eq. (11.56) we have

$$\boldsymbol{c} = \boldsymbol{r} + \frac{1}{\kappa}\boldsymbol{n}, \tag{11.57}$$

so that PC, the radius of this particular circle, is given by $\boldsymbol{c}-\boldsymbol{r}=\boldsymbol{n}/\kappa$ and is of magnitude $1/\kappa$. Thus $1/\kappa$ is called the *radius of curvature* ρ and it is the radius of the circle having three-point contact with the curve at the point. This circle obviously has the same curvature as the curve at the point P. This circle is called the *circle of curvature* at the point P, and its centre is called the *centre of curvature* of the curve at P.

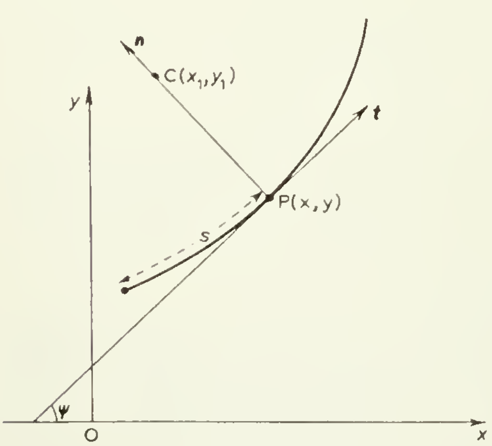

Fig. 11.13

The centre of curvature can also be proved to be the point of intersection of consecutive normals to the curve at P, but we leave this as an exercise for the reader.

The position of the centre of curvature is given by eq. (11.36) as

$$\boldsymbol{c} = \boldsymbol{r} + \rho\boldsymbol{n}. \tag{11.58}$$

If ρ is positive, the value of $ds/d\psi$ is positive as in fig. 11.13, then the coordinates (x_1, y_1) of C the centre of curvature are given by

$$\boldsymbol{c} = x_1\boldsymbol{i} + y_1\boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j} + \rho\{-(\sin\psi)\,\boldsymbol{i} + (\cos\psi)\,\boldsymbol{j}\},$$

using eq. (11.58). Or

$$x_1 = x - \rho \sin \psi, \qquad y_1 = y + \rho \cos \psi. \tag{11.59}$$

If ρ is negative as in fig. 11.14, then x_1, y_1 are given by

$$\boldsymbol{c} = x_1\boldsymbol{i} + y_1\boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j} - |\rho|\{-(\sin \psi)\, \boldsymbol{i} + (\cos \psi)\, \boldsymbol{j}\},$$

or

$$x_1 = x + |\rho| \sin \psi, \qquad y_1 = y - |\rho| \cos \psi. \tag{11.60}$$

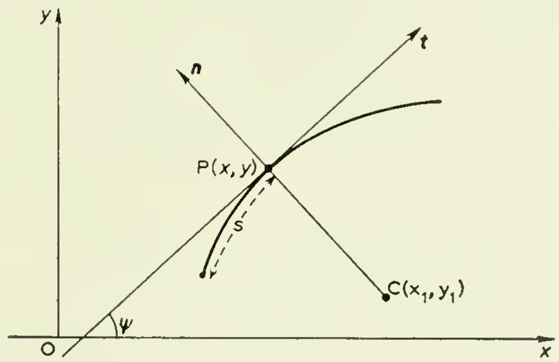

Fig. 11.14

Example 12

We can illustrate the evaluation of some of these formulae for curvature, centre of curvature, arc length etc., by considering the cycloid, fig. 11.2, given in Example 2. We have

$$x = a(\theta - \sin \theta), \qquad y = a(1 - \cos \theta)$$

where θ is the parameter; so we will use a dot to denote differentiation with respect to θ. Thus

$$\dot{x} = a(1 - \cos \theta), \qquad \dot{y} = a \sin \theta,$$

and

$$\ddot{x} = a \sin \theta, \qquad \ddot{y} = a \cos \theta;$$

therefore from eq. (11.45) we have

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 = 4a^2 \sin^2 \tfrac{1}{2}\theta. \tag{11.61}$$

We use eq. (11.61) to find the arc length of the curve as a function of θ. It gives

$$\dot{s} = \pm 2a \sin \tfrac{1}{2}\theta,$$

and choosing $\dot{s}$ to be positive when $\sin \frac{1}{2}\theta$ is positive, that is for $0 \leqslant \theta \leqslant \pi$, we have

$$\dot{s} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = +2a \sin \tfrac{1}{2}\theta.$$

This result can be integrated to give

$$s = \int_0^\theta 2a \sin \tfrac{1}{2}\theta \, \mathrm{d}\theta$$

measuring s from the point on the curve where $\theta=0$. Therefore

$$s = 4a(1 - \cos \tfrac{1}{2}\theta). \tag{11.62}$$

To find the curvature of the curve, we can use eq. (11.47) which gives the magnitude only as

$$\frac{1}{\kappa} = \rho = |4a \sin \tfrac{1}{2}\theta|.$$

To find whether ρ is positive or negative we note that

$$\tan \psi = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin \theta}{1 - \cos \theta} = \cot \tfrac{1}{2}\theta = \tan(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta).$$

Thus $\psi=\frac{1}{2}(\pi-\theta)$ and substituting in eq. (11.62) we have

$$s = 4a(1 - \sin \psi),$$

so that

$$\rho = \frac{ds}{d\psi} = -4a \cos \psi = -4a \sin \tfrac{1}{2}\theta.$$

From eq. (11.60), the coordinates of the centre of curvature at the point 'θ' are

$$x_1 = a(\theta - \sin \theta) + 4a \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta = a(\theta + \sin \theta),$$
$$y_1 = a(1 - \cos \theta) - 4a \sin^2 \tfrac{1}{2}\theta = -a(1 - \cos \theta).$$

§ 2.5. Other well-known formulae

The length of the perpendicular OL from the origin O on to the tangent at any point P on a plane curve is denoted by p; from fig. 11.15 it is clear that

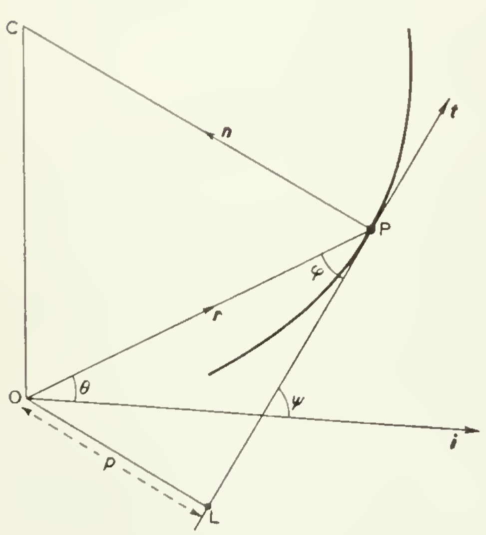

Fig. 11.15

$$p = r \sin \varphi, \tag{11.63}$$

since from eq. (11.32)

$$\cot \varphi = \frac{1}{r} \frac{dr}{d\theta},$$

we deduce the identity

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \varphi,$$

or

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \tag{11.64}$$

Write $u=1/r$, so that $\mathrm{d}u/\mathrm{d}\theta=-\mathrm{d}r/r^2\,\mathrm{d}\theta$, eq. (11.64) can be written in the form

$$\frac{1}{p^2}=u^2+\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2. \tag{11.65}$$

Using these two eqs. (11.63) and (11.64) we can derive two well-known formulae for the radius of curvature ρ.

From fig. 11.15 we have

$$\psi=\theta+\varphi;$$

thus

$$\frac{\mathrm{d}\psi}{\mathrm{d}s}=\frac{\mathrm{d}\theta}{\mathrm{d}s}+\frac{\mathrm{d}\varphi}{\mathrm{d}s}=\frac{\mathrm{d}\theta}{\mathrm{d}s}+\frac{\mathrm{d}\varphi}{\mathrm{d}r}\,\frac{\mathrm{d}r}{\mathrm{d}s}.$$

From eqs. (11.31) we have

$$\frac{\mathrm{d}r}{\mathrm{d}s}=\cos\varphi,\qquad r\frac{\mathrm{d}\theta}{\mathrm{d}s}=\sin\varphi,$$

and hence

$$r\frac{\mathrm{d}\psi}{\mathrm{d}s}=\sin\varphi+r\cos\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}r}=\frac{\mathrm{d}(r\sin\varphi)}{\mathrm{d}r}=\frac{\mathrm{d}p}{\mathrm{d}r},$$

using eq. (11.63). Thus

$$\rho=\frac{\mathrm{d}s}{\mathrm{d}\psi}=r\frac{\mathrm{d}r}{\mathrm{d}p}. \tag{11.66}$$

Further, from eq. (11.65), differentiating with respect to u, we get

$$-\frac{2}{p^3}\,\frac{\mathrm{d}p}{\mathrm{d}u}=2u+2\frac{\mathrm{d}u}{\mathrm{d}\theta}\,\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\,\frac{\mathrm{d}\theta}{\mathrm{d}u}=2\left\{u+\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\right\}. \tag{11.67}$$

But

$$-\frac{1}{p^3}\,\frac{\mathrm{d}p}{\mathrm{d}u}=-\frac{1}{p^3}\,\frac{\mathrm{d}p}{\mathrm{d}r}\,\frac{\mathrm{d}r}{\mathrm{d}u}=\frac{r^2}{p^3}\,\frac{\mathrm{d}p}{\mathrm{d}r},$$

and using eq. (11.66), this becomes $r^3/p^3\rho$. Thus substituting in eq. (11.67)

$$\frac{1}{\rho}=u^3p^3\left(u+\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\right),$$

and using eq. (11.65), this becomes

$$\frac{1}{\rho}=\frac{u+\mathrm{d}^2u/\mathrm{d}\theta^2}{\left\{1+\left(\dfrac{1}{u}\,\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2\right\}^{\frac{3}{2}}}. \tag{11.68}$$

This is the formula required for ρ when the equation of a curve is given in polar coordinate form, so that $u=r^{-1}$ is a known function of θ.

Example 13

For the cardioid $r=a(1+\cos\theta)$ prove that

$$r^3 = 2ap^2, \qquad \rho = \tfrac{4}{3}a\cos\tfrac{1}{2}\theta.$$

The cardioid is shown in fig. 11.8, and we have already shown in Example 10 that $\tan\varphi=\tan\frac{1}{2}(\pi+\theta)$. Thus $\varphi=\frac{1}{2}(\pi+\theta)$ and therefore

$$p = r\sin\varphi = r\cos\tfrac{1}{2}\theta, \tag{11.69}$$

and

$$p^2 = r^2\cos^2\tfrac{1}{2}\theta = \tfrac{1}{2}r^2(1+\cos\theta) = \frac{r^3}{2a},$$

or

$$r^3 = 2ap^2. \tag{11.70}$$

Differentiating eq. (11.70) with respect to p, we get

$$3r^2\frac{dr}{dp} = 4ap$$

so that using eqs. (11.66) and (11.69) we have

$$\rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3}\cos\tfrac{1}{2}\theta.$$

EXERCISE 11.2

1. For the curve defined by the parametric equations

$$x = a(\cos u + \log\tan\tfrac{1}{2}u),$$
$$y = a(1-\sin u),$$

show that $ds/du=a\cot u$, and find the length of arc between the points for which $u=\frac{1}{3}\pi$, $u=\frac{1}{2}\pi$.

2. If ρ is the radius of curvature at a point P on the parabola $y^2=4ax$, and S the focus, prove that $a\rho^2=4(\mathrm{SP})^3$. If P is the point $(am^2, 2am)$ and the centre of curvature is also on the parabola, find the value of $|m|$.

3. If x_1, y_1 are the rectangular coordinates, referred to the pole as origin and to the initial line as axis of x, of the centre of curvature of the cardioid $r=a(1+\cos\theta)$, at the point (r, θ), show that

$$3(x_1+iy_1) - 2a = a(1-\cos\theta)e^{i\theta}.$$

4. For the curve

$$r = a(1-3\tan^2\theta)\sec\theta,$$

show that

$$\frac{ds}{d\theta} = a \sec^2 \theta(1 + 9 \tan^2 \theta).$$

5. The parametric equations of an astroid are

$$x = a \cos^3 \theta, \qquad y = a \sin^3 \theta.$$

Show that the coordinates x_1, y_1 of the centre of curvature at the point 'θ' satisfy the equations

$$x_1 + y_1 = a(\cos \theta + \sin \theta)^3, \qquad x_1 - y_1 = a(\cos \theta - \sin \theta)^3.$$

6. For the catenary $y = c \cosh(x/c)$ obtain the intrinsic equation in the form $s = c \tan \psi$ and show that the radius of curvature at any point is equal to the length of the normal from the curve to the x-axis.

7. Prove that for the curve $r^n = a^n \sin n\theta$ the angle φ between the radius and the tangent is $n\theta$ and

$$pa^n = r^{n+1}, \qquad \rho = a^n/(n + 1)r^{n-1}.$$

§ 3. Differential geometry of twisted curves

A twisted curve, as distinct from a plane curve, is a curve in space which does not lie in a plane. It is defined as the locus of a point P whose coordinates (x, y, z) are known functions of a single parameter which we will denote by u. Since x, y, z are given as functions of u, we may eliminate u to obtain two relations of the form

$$f(x, y) = 0, \qquad g(x, z) = 0,$$

between x, y, z. These two relations define two surfaces in space; the curve in space may therefore be thought of as the intersection of these two surfaces.

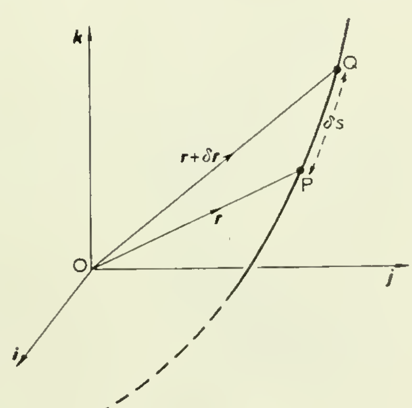

Fig. 11.16

Introducing the triad $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ along the axes at a chosen origin O, we can write the position vector $\boldsymbol{r}$ for the point P on the curve in the form

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k};$$

$\boldsymbol{r}$ is then given as a vector function of the parameter u.

The arc length s measuring the distance of P from some fixed point on the curve may be used as the parameter u. If $\boldsymbol{r}$ and $\boldsymbol{r}+\delta\boldsymbol{r}$ are the po-

sition vectors of P and Q on the curve where arc $PQ = \delta s$ then by the same analysis as in § 2 for the plane curve we can define the vector

$$\frac{d\boldsymbol{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta \boldsymbol{r}}{\delta s} = \lim_{\delta s \to 0} \frac{PQ}{\delta s}.$$

Thus as Q moves up to P and $\delta s \to 0$, we see that $d\boldsymbol{r}/ds = \boldsymbol{r}'$ is again the unit vector along the tangent to the curve at P and we denote it by $\boldsymbol{t}$. Thus

$$\boldsymbol{r}' = \boldsymbol{t}; \qquad \left|\frac{d\boldsymbol{r}}{ds}\right| = 1 \quad \text{or} \quad |d\boldsymbol{r}| = |ds|,$$

and since

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},$$

we have

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k},$$

so that

$$ds = |d\boldsymbol{r}| = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}},$$

or

$$\dot{s} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}, \tag{11.71}$$

where dots denote differentiation with respect to the parameter u. This formula again enables us to find the arc length of any curve in space given its parametric form.

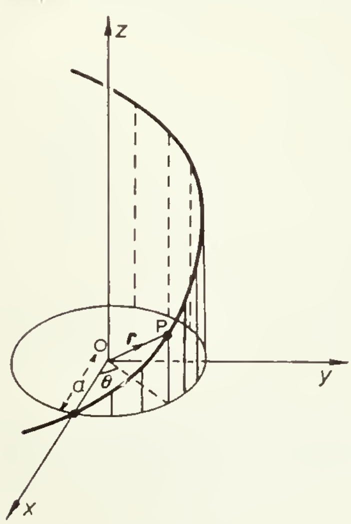

Fig. 11.17

Example 14

One of the simplest examples of a twisted curve is a circular helix. It is a curve drawn on the surface of a circular cylinder making a constant angle with the direction of the axis of the cylinder. Suppose that the cylinder has radius a; we take rectangular axes with Oz along the axis of the cylinder and Ox through some point on the curve, as shown in fig. 11.17. The equation of the helix has the parametric form

$$x = a\cos\theta,$$
$$y = a\sin\theta,$$
$$z = c\theta,$$

where c is a constant, and θ is a parameter. Thus

$$\dot{x} = -a\sin\theta, \qquad \dot{y} = a\cos\theta, \qquad \dot{z} = c,$$

so by eq. (11.71) we have

$$\frac{ds}{d\theta} = (c^2 + a^2)^{\frac{1}{2}},$$

and the arc length measured from the point on the curve where $\theta=0$, that is on the x-axis, is given by

$$s = \int_0^\theta (c^2 + a^2)^{\frac{1}{2}} \mathrm{d}\theta = (c^2 + a^2)^{\frac{1}{2}}\theta. \tag{11.72}$$

Also the unit vector along the tangent is

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\dot{\boldsymbol{r}}}{\dot{s}} = \frac{1}{(c^2 + a^2)^{\frac{1}{2}}} \{-a \sin\theta\, \boldsymbol{i} + a\cos\theta\, \boldsymbol{j} + c\boldsymbol{k}\}.$$

Since

$$\boldsymbol{t}\cdot\boldsymbol{k} = \frac{c}{(c^2 + a^2)^{\frac{1}{2}}},$$

the tangent to the curve does make a constant angle with the direction $\boldsymbol{k}$ of the axis. If $c=a \cot\alpha$, then $c/(c^2+a^2)^{\frac{1}{2}}=\cos\alpha$, or $\boldsymbol{t}\cdot\boldsymbol{k}=\cos\alpha$, so the constant angle is then α. The parametric equations are then

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad z = a\theta\cot\alpha,$$

and the arc length in eq. (11.72) is

$$s = c\theta \sec\alpha.$$

The circular helix is an important curve because of its occurrence in practical solids. The thread of a circular screw is a circular helix; the inner and outer curves of a spiral staircase are circular helices; also the surface formed by lines perpendicular to the z-axis and intersecting the curve, known as a right helicoid, is the shape of a screw propeller.

§ 3.1. The equation of the tangent to a twisted curve

Since $\boldsymbol{t}=\boldsymbol{r}'$ is the unit vector along the tangent at the point P on the curve, then using $\boldsymbol{R}$ as the running vector along the tangent, its equation is

$$\boldsymbol{R} = \boldsymbol{r} + \lambda\boldsymbol{r}'. \tag{11.73}$$

Writing

$$\boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k},$$

the rectangular coordinates (X, Y, Z) satisfy

$$X = x + \lambda x', \qquad Y = y + \lambda y', \qquad Z = z + \lambda z'.$$

Eliminating λ, we obtain the equation of the tangent in the form

$$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'}.$$

But now since $x' : y' : z'=\dot{x} : \dot{y} : \dot{z}$, this is more conveniently written as

$$\frac{X-x}{\dot{x}} = \frac{Y-y}{\dot{y}} = \frac{Z-z}{\dot{z}}, \tag{11.74}$$

the derivatives at the point x, y, z on the curve being taken with respect to an arbitrary parameter u rather than with respect to the arc length s.

§ 3.2. The principal normal. The osculating plane. Curvature

The direction of the unit vector $\boldsymbol{t}$ along the tangent to a curve at P varies continuously as the point P moves along the curve. To find the rate at which it turns we use the lemma given in § 2.2 replacing $\boldsymbol{v}$ by $\boldsymbol{t}$. If $\delta\theta$ is the small angle between the tangents at the two neighbouring points P, Q on the curve in fig. 11.16, then in the limit as Q moves up to P and $\delta\theta \to 0$, the unit vector perpendicular to $\boldsymbol{t}$ in the plane parallel to the direction of the consecutive tangent is given by $\mathrm{d}\boldsymbol{t}/\mathrm{d}\theta$. There is, of course, a whole plane in three dimensions which is perpendicular to $\boldsymbol{t}$ at P; this plane is called the *normal* plane, and this particular vector $\mathrm{d}\boldsymbol{t}/\mathrm{d}\theta$ therefore lies along a line in the normal plane. This line is called the *principal normal* to the curve at P and the unit vector along it is denoted by $\boldsymbol{n}$, so that $\mathrm{d}\boldsymbol{t}/\mathrm{d}\theta = \boldsymbol{n}$. The plane of $\boldsymbol{t}$ and $\boldsymbol{n}$ is called the *osculating plane* of the curve; for a plane curve, it is obvious from the definition of $\boldsymbol{n}$ that the principal normal is the normal to the curve in its own plane, and hence the osculating plane is the plane in which the curve lies. Further we can write

$$\boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\theta} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}\,\frac{\mathrm{d}s}{\mathrm{d}\theta}, \tag{11.75}$$

and if we use $\kappa = 1/\rho = \mathrm{d}\theta/\mathrm{d}s$ as a measure of the rate of turning of the tangent, we have from this equation

$$\boldsymbol{r}'' = \boldsymbol{t}' = \kappa\boldsymbol{n}. \tag{11.76}$$

By analogy with the plane curve, κ is called the *curvature* of the curve at P.

The equation of the principal normal at P is

$$\boldsymbol{R} = \boldsymbol{r} + \lambda\boldsymbol{n},$$

and can be written in the rectangular coordinate form

$$\frac{X - x}{x''} = \frac{Y - y}{y''} = \frac{Z - z}{z''},$$

since the components of $\boldsymbol{n}$ are proportional to the components of $\boldsymbol{r}''$ from eq. (11.76).

§ 3.3. The binormal

One other normal to the curve at P is chosen for special designation. It is called the *binormal* and denoted by $\boldsymbol{b}$. The binormal $\boldsymbol{b}$ is the unit vector perpendicular to the osculating plane at P, the sense of $\boldsymbol{b}$ being chosen to make $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ in this order, a right-handed triad of unit vectors as shown in fig. 11.18. These three unit vectors therefore have the following properties.

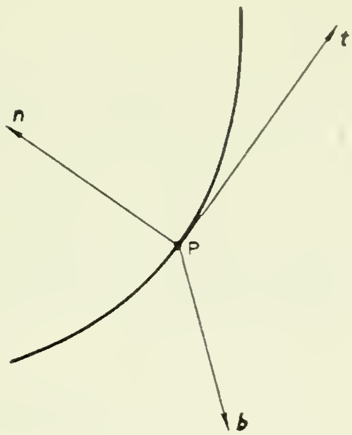

Fig. 11.18

$$\boldsymbol{t}^2 = \boldsymbol{n}^2 = \boldsymbol{b}^2 = 1; \qquad \boldsymbol{t}\cdot\boldsymbol{n} = \boldsymbol{n}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{t} = 0; \tag{11.77}$$

$$\boldsymbol{t}\times\boldsymbol{n} = \boldsymbol{b}, \qquad \boldsymbol{n}\times\boldsymbol{b} = \boldsymbol{t}, \qquad \boldsymbol{b}\times\boldsymbol{t} = \boldsymbol{n}. \tag{11.78}$$

§ 3.4. The Serret–Frenet formulae. The torsion

We have already seen in eq. (11.76) that

$$\boldsymbol{t}' = \kappa\boldsymbol{n}. \tag{11.79}$$

This result expresses the derivative of $\boldsymbol{t}$ with respect to s in terms of the unit vector $\boldsymbol{n}$. Similar results can be derived for the values of $\boldsymbol{n}'$ and $\boldsymbol{b}'$ in terms of $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$; these with (11.79) are known as the Serret–Frenet formulae.

From the relations $\boldsymbol{b}^2=1$ and $\boldsymbol{b}\cdot\boldsymbol{t}=0$ in eq. (11.77), we find by differentiation with respect to s that

$$\boldsymbol{b}\cdot\boldsymbol{b}' = 0 \quad \text{and} \quad \boldsymbol{b}\cdot\boldsymbol{t}' + \boldsymbol{b}'\cdot\boldsymbol{t} = 0.$$

Since $\boldsymbol{b}\cdot\boldsymbol{t}'=\kappa(\boldsymbol{b}\cdot\boldsymbol{n})=0$ these imply that

$$\boldsymbol{b}\cdot\boldsymbol{b}' = 0 \quad \text{and} \quad \boldsymbol{b}'\cdot\boldsymbol{t} = 0.$$

So $\boldsymbol{b}'$ is a vector perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{t}$ and is therefore in the direction of $\boldsymbol{n}$. We therefore write

$$\boldsymbol{b}' = -\tau\boldsymbol{n}; \tag{11.80}$$

the significance of the constant τ will be seen later.

Further, from the relation $\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{t}$ we have by differentiation with respect to s,

$$\boldsymbol{n}' = \boldsymbol{b} \times \boldsymbol{t}' + \boldsymbol{b}' \times \boldsymbol{t} = \kappa(\boldsymbol{b} \times \boldsymbol{n}) - \tau(\boldsymbol{n} \times \boldsymbol{t}),$$

or

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}. \tag{11.81}$$

Relations (11.79), (11.80) and (11.81) are the Serret–Frenet formulae.

The significance of the constant τ or its reciprocal $\sigma=1/\tau$ can be seen by considering the binormal $\boldsymbol{b}$ as P moves along the curve. It is again a unit vector varying in a continuous manner, so if $\delta\varphi$ is the angle between $\boldsymbol{b}$ and $\boldsymbol{b}+\delta\boldsymbol{b}$ then using the lemma given in § 2.2, the unit vector perpendicular to $\boldsymbol{b}$ in the plane parallel to the consecutive direction of the binormal is

$$\lim_{\delta\varphi\to 0} \frac{\delta\boldsymbol{b}}{\delta\varphi} = \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}\varphi} = \boldsymbol{b}'\frac{\mathrm{d}s}{\mathrm{d}\varphi} = -\tau\boldsymbol{n}\frac{\mathrm{d}s}{\mathrm{d}\varphi}.$$

Since this is a unit vector, $\tau=\mathrm{d}\varphi/\mathrm{d}s$, and measures the angle turned through per unit length. But the binormal $\boldsymbol{b}$ is perpendicular to the osculating plane of the curve, and the angle between adjacent binormals is therefore the angle between adjacent osculating planes; so $\mathrm{d}\varphi/\mathrm{d}s$ is a measure of the rate of turning of the osculating plane of the curve. This tells us to what extent the curve is departing from a plane curve at the point; it measures what is called the twist or *torsion* of the curve. Thus $\tau=\mathrm{d}\varphi/\mathrm{d}s$ is the torsion of the curve; by analogy with the definition $\rho=1/\kappa$, we call $\sigma=1/\tau$ the *radius of torsion.*

§ 3.5. Formulae for the curvature and torsion of a twisted curve

The formulae for calculating the curvature and torsion of a twisted curve are derived by methods very similar to those for the plane curve. The first Serret–Frenet formula

$$\boldsymbol{r}'' = \boldsymbol{t}' = \kappa\boldsymbol{n},$$

gives immediately, by squaring both sides

$$\kappa^2 = (\boldsymbol{r}'')^2 = (x'')^2 + (y'')^2 + (z'')^2;$$

but again, as for the plane curve, this is not a very convenient formula to use in practice. However, we have again, as in eqs. (11.43) and (11.44)

$$\dot{\boldsymbol{r}} = \dot{s}\boldsymbol{t}, \tag{11.82}$$

$$\boldsymbol{r} = \ddot{s}\boldsymbol{t} + \dot{s}^2\kappa\boldsymbol{n}, \tag{11.83}$$

with $\dot{s}=|\dot{\boldsymbol{r}}|$, and from these two equations we get

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = \dot{s}^3\kappa \boldsymbol{t} \times \boldsymbol{n} = \dot{s}^3\kappa \boldsymbol{b}. \tag{11.84}$$

Hence

$$\kappa = \frac{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|}{\dot{s}^3}. \tag{11.85}$$

Further by differentiating eq. (11.84) with respect to the parameter u, we get

$$\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}} = -\dot{s}^4\kappa\tau \boldsymbol{n} + \boldsymbol{b}\frac{\mathrm{d}}{\mathrm{d}u}(\dot{s}^3\kappa) \tag{11.86}$$

and from the two eqs. (11.83) and (11.86) by scalar multiplication, we have

$$\ddot{\boldsymbol{r}}\cdot(\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}) = -\dot{s}^6\kappa^2\tau,$$

or using eq. (11.85) this gives

$$\tau = \frac{[\dot{\boldsymbol{r}},\ \ddot{\boldsymbol{r}},\ \dddot{\boldsymbol{r}}]}{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|^2}. \tag{11.87}$$

Example 15

The circular helix described in Example 14 is given by

$$\boldsymbol{r} = a\cos\theta\,\boldsymbol{i} + a\sin\theta\,\boldsymbol{j} + a\theta\cot\alpha\,\boldsymbol{k},$$

so that we have the following results immediately:

$$\begin{aligned}
\dot{\boldsymbol{r}} &= -a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j} + a\cot\alpha\,\boldsymbol{k},\\
\dot{s} &= |\dot{\boldsymbol{r}}| = a\operatorname{cosec}\alpha,\\
\ddot{\boldsymbol{r}} &= -a\cos\theta\,\boldsymbol{i} - a\sin\theta\,\boldsymbol{j},\\
\dddot{\boldsymbol{r}} &= a\sin\theta\,\boldsymbol{i} - a\cos\theta\,\boldsymbol{j}.
\end{aligned}$$

Hence

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = a^2\{\sin\theta\cot\alpha\,\boldsymbol{i} - \cos\theta\cot\alpha\,\boldsymbol{j} + \boldsymbol{k}\},$$

and using eq. (11.85) this gives

$$\kappa = \frac{1}{a}\sin^2\alpha.$$

Also

$$\dddot{\boldsymbol{r}}\cdot(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) = a^3\cot\alpha,$$

so that using eq. (11.87), we have

$$\tau = \frac{a^3\cot\alpha}{a^4\operatorname{cosec}^2\alpha} = \frac{1}{a}\sin\alpha\cos\alpha.$$

Thus a circular helix is a curve for which both the curvature and the torsion are constants.

Example 16

Find the direction cosines of the tangent at the point $u=1$ on the twisted cubic

$$x = 6u, \qquad y = 3u^2, \qquad z = u^3.$$

Show that the tangent at $u=1$ meets the osculating plane of the curve at the origin in the point

$$x = 4, \qquad y = 1, \qquad z = 0.$$

We have

$$\boldsymbol{r} = 6u\boldsymbol{i} + 3u^2\boldsymbol{j} + u^3\boldsymbol{k},$$

and

$$\dot{\boldsymbol{r}} = 6\boldsymbol{i} + 6u\boldsymbol{j} + 3u^2\boldsymbol{k}. \tag{11.88}$$

From eq. (11.82) we see that $\boldsymbol{t}$ is in the direction of $\dot{\boldsymbol{r}}$; thus at $u=1$, $\boldsymbol{t}$ is in the direction of the vector

$$6\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k} = 3(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}).$$

The direction cosines of the vector $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ are $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$. Thus unit vector along the tangent at this point is

$$\boldsymbol{t} = \tfrac{1}{3}(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}).$$

The osculating plane at any point $\boldsymbol{r}$ on the curve is perpendicular to $\boldsymbol{b}$ and its equation is therefore

$$(\boldsymbol{R} - \boldsymbol{r})\cdot\boldsymbol{b} = 0.$$

But from eq. (11.84) we see that $\boldsymbol{b}$ is in the direction of the vector $\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}$. Thus the equation of the osculating plane at any point $\boldsymbol{r}$ is

$$(\boldsymbol{R} - \boldsymbol{r})\cdot(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) = 0.$$

Differentiating eq. (11.88) we have

$$\ddot{\boldsymbol{r}} = 6\boldsymbol{j} + 6u\boldsymbol{k}, \tag{11.89}$$

and at the origin where $u=0$, eqs. (11.88) and (11.89) give $\dot{\boldsymbol{r}}=6\boldsymbol{i}$, $\ddot{\boldsymbol{r}}=6\boldsymbol{j}$ so that $\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}=36\boldsymbol{k}$. Also at the origin $\boldsymbol{r}=\boldsymbol{0}$ so the equation of the osculating plane at the origin is

$$\boldsymbol{R}\cdot\boldsymbol{k} = 0. \tag{11.90}$$

The equation of the tangent at $u=1$ where $\boldsymbol{r}=6\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ is

$$\boldsymbol{R} = 6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}). \tag{11.91}$$

This line meets the plane defined by eq. (11.90) where

$$\{6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})\}\cdot\boldsymbol{k} = 0,$$

or

$$1 + \lambda = 0,$$

giving $\lambda=-1$. Thus substituting $\lambda=-1$ in eq. (11.91), the point of intersection is

$$\boldsymbol{R} = 4\boldsymbol{i} + \boldsymbol{j},$$

which is the required result.

EXERCISE 11.3

1. Find the equation of the osculating plane at an arbitrary point of the curve

$$x = u^3, \qquad y = 3u^2 - 2u, \qquad z = 2u - 3.$$

Find also the equation of the normal plane at the point $u = 1$.

2. For the space curve

$$x = a\theta \cos \theta, \qquad y = a\theta \sin \theta, \qquad z = b\theta,$$

find the equations of the osculating plane, the curvature κ and the torsion τ at the point θ.

3. For the curve

$$\boldsymbol{r} = a(u^3\boldsymbol{i} + 3u^2\boldsymbol{j} + 6u\boldsymbol{k}),$$

show that the equation of the osculating plane at the point $\boldsymbol{r}$ is

$$\boldsymbol{R}\cdot\{2\boldsymbol{i} - 2u\boldsymbol{j} + u^2\boldsymbol{k}\} = 2au^3.$$

4. Find the equation of the tangent to the curve

$$x = \tfrac{1}{4}u^4, \qquad y = \tfrac{1}{3}u^3, \qquad z = \tfrac{1}{2}u^2,$$

at the point $u=u_0$ $(\neq 0)$, and determine two points on the curve where the tangents are parallel to the plane

$$x + 3y + 2z = 0.$$

5. Find the osculating plane of the curve

$$\boldsymbol{r} = 3u\boldsymbol{i} + 3u^2\boldsymbol{j} + 2u^3\boldsymbol{k},$$

at the point 'u', show that it intersects the osculating plane at $u=0$ in the line $\boldsymbol{r}=(u-\lambda)\boldsymbol{i}-\lambda u\boldsymbol{j}$. Prove that $\kappa=\tau$ for this curve.

6. For the helix

$$\boldsymbol{r} = \alpha(3\boldsymbol{i}\cos\theta + 3\boldsymbol{j}\sin\theta + 4\boldsymbol{k}\theta),$$

show that the length of arc from $\theta=0$ to 'θ' is $5\alpha\theta$.

7. For the curve

$$y^2 = 2ax - x^2, \qquad z = -a \log\left(1 - \frac{x}{2a}\right),$$

find the tangent at the point where $x=x_0$. Find further the length of the arc of the curve when x varies from 0 to x_0.

(Hint: put $x=u$, the curve is then defined parametrically with u as parameter.)

§ 4. Curvilinear or line integrals

Suppose C, shown in fig. 11.19 is a curve in space defined as in § 3. It can therefore be represented parametrically by the equations

$$x = f_1(u), \qquad y = f_2(u), \qquad z = f_3(u), \tag{11.92}$$

and the position vector $\boldsymbol{r}$ of any point on the curve is given as a vector function of u by the equation

$$\boldsymbol{r} = \boldsymbol{f}(u) = f_1(u)\boldsymbol{i} + f_2(u)\boldsymbol{j} + f_3(u)\boldsymbol{k}. \tag{11.93}$$

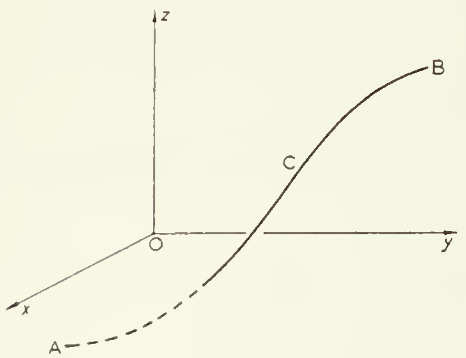

Fig. 11.19

We shall suppose (i) that C does not cross itself, so that $\boldsymbol{f}(u)$ is a single valued vector function of u; (ii) that u varies from u_0 to u_1 as the point varies on the curve from A to B; (iii) that it has a unique tangent at every point along its length, so that $\boldsymbol{f}(u)$ has a definite derivative with respect to u for all values of u between u_0 and u_1.

If then $F(x, y, z)$ is a continuous function of x, y, z on the curve C, we define the curvilinear integral with respect to x along the curve C from A to B and denoted by

$$\int_{\mathrm{C}} F(x, y, z)\,\mathrm{d}x \quad \text{or} \quad \int_{\mathrm{A}}^{\mathrm{B}} F(x, y, z)\,\mathrm{d}x,$$

as the single integral

$$\int_{u_0}^{u_1} G(u)\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u, \tag{11.94}$$

where, by means of the equations (11.92), we write

$$F(x, y, z) = F\{f_1(u), f_2(u), f_3(u)\} \equiv G(u).$$

Similarly we define the curvilinear integrals

$$\int_{\mathrm{C}} F(x, y, z)\,\mathrm{d}y = \int_{u_0}^{u_1} G(u)\,\frac{\mathrm{d}y}{\mathrm{d}u}\,\mathrm{d}u, \tag{11.95}$$

and

$$\int_C F(x, y, z)\,\mathrm{d}z = \int_{u_0}^{u_1} G(u)\,\frac{\mathrm{d}z}{\mathrm{d}u}\,\mathrm{d}u. \tag{11.96}$$

Note that in these integrals the curve C has an assigned direction namely from A to B. The function $F(x, y, z)$ is said to be integrated along the curve C in this direction.

§ 4.1. Two-dimensional line integrals

When the curve C is a plane curve defined by the single valued function

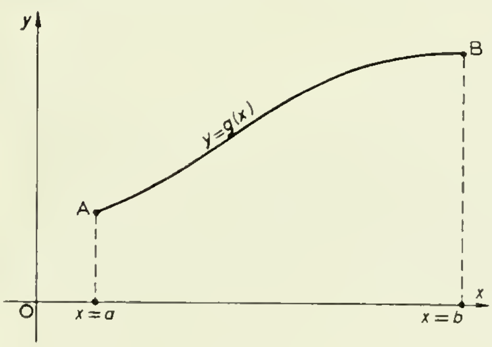

Fig. 11.20

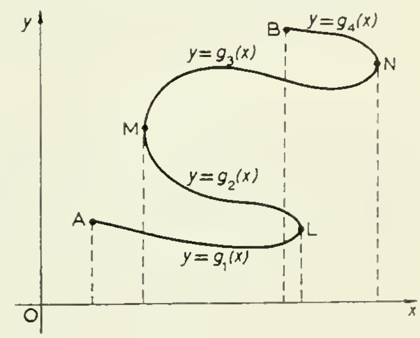

Fig. 11.21

$y=g(x)$ in the xy-plane, as shown in fig. 11.20 and the function $F(x, y, z)$ is also a continuous function $F(x, y) \equiv F(x, y, 0)$ of x, y in this plane, then we can use $x = u$ as the parameter and write

$$\int_C F(x, y)\,\mathrm{d}x = \int_a^b F\{x, g(x)\}\,\mathrm{d}x,$$

where $x=a$ at A, and $x=b$ at B.

If however, the curve C is as shown in fig. 11.21, then it must be divided into segments AL, LM, MN,... such that for each segment only one value of y corresponds to each value of x. Denoting these values of y by $g_1(x)$, $g_2(x)$, $g_3(x)$,... respectively, the integral

$$\int_C F(x, y)\,\mathrm{d}x,$$

is defined to be the following sum of integrals

$$\int_A^L F\{x, g_1(x)\}\,\mathrm{d}x + \int_L^M F\{x, g_2(x)\}\,\mathrm{d}x + \int_M^N F\{x, g_3(x)\}\,\mathrm{d}x + \ldots.$$

This method must also be used if the curve C is a closed curve, so that B coincides with A, the value of y on the upper part of the curve having a different value from that on the lower part, as in fig. 11.22. We then have, taking C in the anti-clockwise sense

$$\oint_C F(x, y)\,\mathrm{d}x = \int_{x_1}^{x_2} F\{x, g_1(x)\}\,\mathrm{d}x + \int_{x_2}^{x_1} F\{x, g_2(x)\}\,\mathrm{d}x,$$

where the notation on the left hand side in this equation is used to denote an integral round a closed curve in the anti-clockwise sense.

In practice however, it is often best to evaluate integrals along closed or open curves by using an auxiliary parameter u, even in the two-dimensional case.

Similarly we can define the integral

$$\int_C F(x, y)\,\mathrm{d}y, \tag{11.97}$$

using either y or u as parameter; if it is more convenient we can use x again as the parameter by writing the integral (11.97) as

$$\int_C F\{x, g(x)\}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x.$$

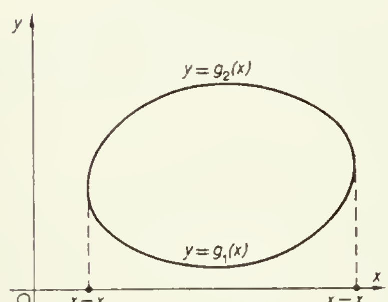

Fig. 11.22

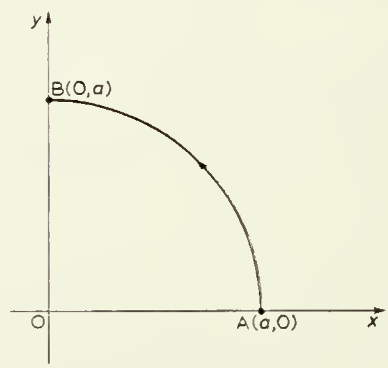

Fig. 11.23

Example 17

Evaluate

$$\int_C x^2 y\,\mathrm{d}x,$$

where C is the part of the circle $x^2+y^2=a^2$ which lies in the positive quadrant, as shown in fig. 11.23. Here we can write on C

$$y = \sqrt{(a^2 + x^2)},$$

and at A, $x=a$ while at B, $x=0$, so that the integral becomes

$$\int_a^0 x^2 \sqrt{(a^2 - x^2)}\, \mathrm{d}x. \tag{11.98}$$

This integral can be evaluated as in Ch. 5 § 5.3 by the substitution $x=a\cos\theta$. If however, we use the usual parametric equations of this circle, namely

$$x = a\cos\theta, \qquad y = a\sin\theta, \tag{11.99}$$

then at A, $\theta=0$, while at B, $\theta=\frac{1}{2}\pi$, and the integral would be immediately

$$\int_0^{\frac{1}{2}\pi} a^3 \cos^2\theta \sin\theta\, \mathrm{d}(a\cos\theta). \tag{11.100}$$

This is precisely the integral found from eq. (11.98) with the substitution $x=a\cos\theta$ and so the parametric form is really used here in any case. The integral (11.100) becomes

$$-a^4 \int_0^{\frac{1}{2}\pi} \cos^2\theta \sin^2\theta\, \mathrm{d}\theta = -\frac{\pi a^4}{16}, \tag{11.101}$$

using the reduction formula in eq. (5.28). The negative sign occurs in the result because $\mathrm{d}x$ in (11.98) is negative as x varies from $A(x=a)$ to $B(x=0)$.

If, in this example, C is the complete circle $x^2+y^2=a^2$ taken in the anti-clockwise sense, the simplest way of evaluating the integral is to use the parametric equations (11.99), with θ varying from 0 to 2π. The integral becomes

$$-a^4 \int_0^{2\pi} \cos^2\theta \sin^2\theta\, \mathrm{d}\theta,$$

,and can be evaluated either by dividing the range 0 to 2π into the ranges $(0, \frac{1}{2}\pi)$ $(\frac{1}{2}\pi, \pi)$, $(\pi, \frac{3}{2}\pi)$, $(\frac{3}{2}\pi, 2\pi)$ and using methods suggested in Ch. 5 § 2 or since the powers of $\cos\theta$, $\sin\theta$ are small, proceeding as follows:

$$-a^4 \int_0^{2\pi} \cos^2\theta \sin^2\theta\, \mathrm{d}\theta = -\frac{a^4}{4} \int_0^{2\pi} \sin^2 2\theta\, \mathrm{d}\theta,$$

which can be written as

$$-\frac{a^4}{8} \int_0^{2\pi} (1 - \cos 4\theta)\, \mathrm{d}\theta = -\frac{a^4}{8} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi},$$

which has the value $-\frac{1}{4}\pi a^4$. This result is four times the value of the integral in eq. (11.101) as we would expect, since the integrand is even in $\cos\theta$ and $\sin\theta$.

§ 4.2. Three-dimensional examples

The following example will illustrate the methods used in three dimensions.

Example 18

Find the value of

$$\int_C (y^2\,dx + xy\,dy + zx\,dz),$$

along the following curves C joining the point (0, 0, 0) to the point (1, 1, 1):

(i) the straight line between the two points,
(ii) the curve $x = u$, $y = u^2$, $z = u^3$,
(iii) along the x-axis to the point (1, 0, 0) labelled C_1 in fig. 11.24, then along the quarter circle C_2 in the xz-plane, from (1, 0, 0) to (0, 0, 1), and then along the straight line C_3 from (0, 0, 1) to (1, 1, 1).

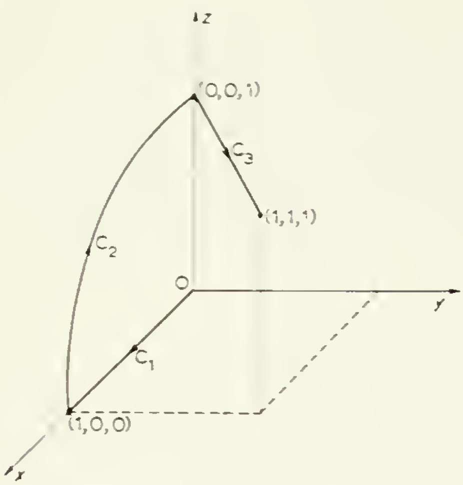

Fig. 11.24

The straight line in (i) is given by

$$\boldsymbol{r} = u(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}),$$

or in parametric form

$$x = u, \qquad y = u, \qquad z = u,$$

with u varying from 0 to 1. Thus the integral becomes

$$\int_0^1 (u^2\,du + u^2\,du + u^2\,du)$$
$$= 3\int_0^1 u^2\,du = 1.$$

The curve in (ii) given by

$$x = u, \qquad y = u^2, \qquad z = u^3,$$

between the values $u=0$ and $u=1$, gives for the integral

$$\int_0^1 (u^4\,du + 2u^4\,du + 3u^6\,du) = \left[\frac{3u^5}{5} + \frac{3u^7}{7}\right]_0^1 = \frac{36}{35}.$$

The curve C in (iii) has to be treated in the separate parts C_1, C_2, C_3. Along C_1, we have $y=0$, $z=0$ so that $dy=0$, $dz=0$ and each term of the integral is zero. Along the circle C_2 in the xz-plane, we can choose z as the parameter, with $x=\sqrt{(1-z^2)}$, $y=0$ and $dy=0$; then the integral is

$$\int_0^1 z\sqrt{(1-z^2)}\,dz = \left[-\tfrac{1}{3}(1-z^2)^{\frac{3}{2}}\right]_0^1 = \tfrac{1}{3}.$$

Along C_3, the equation of the straight line joining (0, 0, 1) to (1, 1, 1) is

$$\boldsymbol{r} = \boldsymbol{k} + u(\boldsymbol{i} + \boldsymbol{j}),$$

or in parametric form

$$x = u, \qquad y = u, \qquad z = 1,$$

so the integral along C_3 is

$$\int_0^1 (u^2\,du + u^2\,du + 0) = \tfrac{2}{3}.$$

Thus altogether the integral for the curve in (iii) is

$$0 + \tfrac{1}{3} + \tfrac{2}{3} = 1.$$

If a curve in three dimensions is not expressed parametrically by equations of the form (11.92), but is defined as the intersection of the two surfaces

$$\varphi(x, y, z) = 0, \qquad \psi(x, y, z) = 0, \tag{11.102}$$

then we can either use one of the variables, say x, as a parameter and try to solve the eqs. (11.102) for y, z as functions of x; or look for a parametric form as in the following example.

Example 19

Evaluate

$$\int_C [y\,dx - y(x-1)\,dy + y^2z\,dz]$$

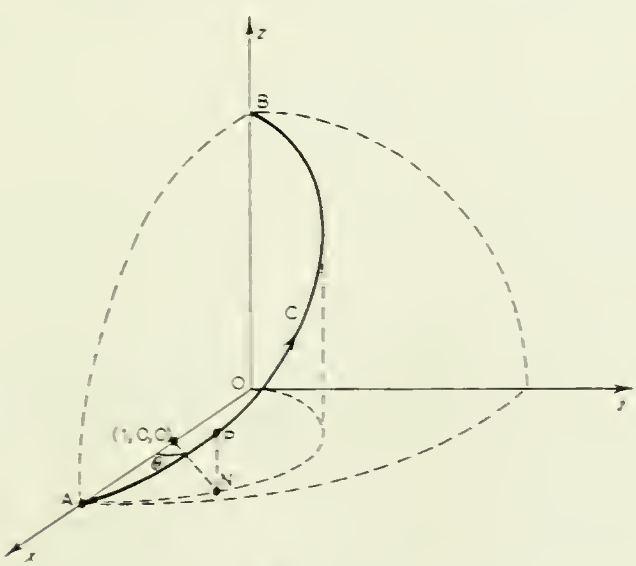

Fig. 11.25

where C is the positive octant part of the curve given by the intersection of the sphere

$$x^2 + y^2 + z^2 = 4 \tag{11.103}$$

and the cylinder

$$(x-1)^2 + y^2 = 1, \tag{11.104}$$

between the points (2, 0, 0) and (0, 0, 2).

The curve C is shown by the solid line in fig. 11.25. Points (x, y, z) on the curve must satisfy eqs. (11.103) and (11.104) simultaneously. On the cylinder in eq. (11.104) we can write

$$x - 1 = \cos\theta, \qquad y = \sin\theta, \tag{11.105}$$

and since x and y are both positive and go round the semi-circle ANO in the xy-plane, θ varies from 0 to π. Substituting the values (11.105) in the eq. (11.103) we get

$$1 + 2\cos\theta + \cos^2\theta + \sin^2\theta + z^2 = 4,$$

giving $z^2 = 4\sin^2\frac{1}{2}\theta$. Since z must be positive and θ varies from 0 to π, we must have $z = 2\sin\frac{1}{2}\theta$ on the curve and then z does vary from the value 0 at A to the value 2 at B. The values (x, y) similarly change from (2, 0) to (0, 0) as required. Thus the integral becomes

$$\int_0^\pi \{\sin\theta\, \mathrm{d}(1 + \cos\theta) - \sin\theta\cos\theta\, \mathrm{d}(\sin\theta) + 2\sin^2\theta\sin\tfrac{1}{2}\theta\, \mathrm{d}(\sin\tfrac{1}{2}\theta)\},$$

which becomes

$$\int_0^\pi (-\sin^2\theta - \sin\theta\cos^2\theta + \sin^3\theta)\,\mathrm{d}\theta,$$

and the reader can easily verify that this has the value $\frac{2}{3} - \frac{1}{2}\pi$.

EXERCISE 11.4

1. Evaluate

$$\int_C (xy\,\mathrm{d}x + x\,\mathrm{d}y),$$

where C is the unit circle, centre the origin, taken in the anti-clockwise sense.

2. Find the value of the integral

$$\int \{y^2\,\mathrm{d}x + (xy - x^2)\,\mathrm{d}y\},$$

ntegrated from the point (0, 0) to the point (1, 3) when the path is (i) along the ine $y = 3x$, and (ii) along the parabola $y^2 = 9x$.

3. Evaluate

$$\oint_C \{(y^2 - 5x)\,\mathrm{d}x + (x^2 - 5y)\,\mathrm{d}y\},$$

where C is the closed contour made up of:

(i) the circular quadrant centre the origin, joining the points (−5, 0) and (0, 5);
(ii) the line $y = 5$ from (0, 5) to (−5, 5);
(iii) the line $x = -5$ from (−5, 5) to (−5, 0).

4. Evaluate $\oint_C (y\,\mathrm{d}x - x\,\mathrm{d}y)$, along the curve $(x-2)^2 + y^2 = 2$.

5. Evaluate

$$\oint_C \{(x^3 + 6x^2y + y^2 - 3y)\,dx + (2x^3 - 4xy^2 - 2xy + 3y)\,dy\},$$

where C is the circle $(x-2)^2+(y-2)^2=4$.

6. Evaluate

$$\oint \{yz\,dx + (y + zx)\,dy + xz\,dz\},$$

along the path formed by the straight lines joining O(0, 0, 0), A(1, 0, 0), B(1, 2, 0), C(1, 2, 1), D(0, 0, 1) and O.

7. Evaluate

$$\int \{(x^2 + yz)\,dx + (z^2 - x^2)\,dy + xy\,dz\},$$

from O(0, 0, 0) to A(1, 1, 1) when taken along the path followed by the curve

$$x = 1 - \cos\theta, \qquad y = \sin\theta, \qquad z = 2\theta/\pi.$$

8. Evaluate

$$\int_C \{(x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz\},$$

where C is (i) the straight line joining A(0, 0, 0) to B(1, 1, 1). Show that the integral has the same value when C is formed by (ii) lines parallel to the axes Ox, Oy, Oz in turn joining these two points.

9. Evaluate

$$\int_C \{y^2\,dx + xy\,dy + xz\,dz\},$$

along the path (ii) of Example 8.

10. Evaluate

$$\int \left\{\left(\frac{2x}{1 + x^2 + y^2} + yz\right)dx + \left(\frac{2y}{1 + x^2 + y^2} + zx\right)dy + xy\,dz\right\},$$

along any path joining the point (0, 0, 0) to (1, 2, -1).

MATRICES, DETERMINANTS AND LINEAR DEPENDENCE

§ 1. Notation

In Ch. 9 § 8 we introduced the concept of a vector in n-space. The components of a vector $\boldsymbol{v}$ in 3-space are (v_1, v_2, v_3), and are referred to collectively as v_r, the suffix r taking values 1, 2, 3. Likewise the components of a vector $\boldsymbol{x}=(x_1, x_2, \dots x_n)$ in n-space are referred to as x_ρ, the suffix ρ taking values $1, 2, \dots n$. In many branches of mathematics and physics, sets of numbers occur which can most conveniently be expressed algebraically by using two suffices, ρ and σ say, *each* suffix having a certain range of values. For example, if $x_1, x_2, \dots x_n$ are n unknowns and are related by a set of m linear equations, these equations can be written in the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1\sigma}x_\sigma + \dots + a_{1n}x_n &= k_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2\sigma}x_\sigma + \dots + a_{2n}x_n &= k_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{\rho 1}x_1 + a_{\rho 2}x_2 + \dots + a_{\rho\sigma}x_\sigma + \dots + a_{\rho n}x_n &= k_\rho \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{m\sigma}x_\sigma + \dots + a_{mn}x_n &= k_m.
\end{aligned} \tag{12.1}$$

The coefficients on the left of equations (12.1) are written as *two-suffix quantities* $a_{\rho\sigma}$. The first suffix ρ tells us in which of the m equations the coefficient $a_{\rho\sigma}$ occurs, so that ρ may take any integral value between 1 and m; the second suffix σ tells us that $a_{\rho\sigma}$ multiplies the unknown x_σ, and σ takes values between 1 and n. The two-suffix notation enables us to identify immediately any of the mn coefficients; for example, a_{32} we know at once to be the coefficient of x_2 in the third equation. We note that there is no reason why the ranges of values of the two suffixes should be the same; in this example, the number of equations m need not be the same as the number of unknowns n.

A two-suffix set of numbers $a_{\rho\sigma}$, with ρ and σ taking values independently over given ranges, is known as a *matrix*; the laws of addition and multiplication which we ascribe to matrices are discussed in the next section. The numbers $a_{\rho\sigma}$, known as the *elements* of the matrix, can in general be complex numbers; if *all* the elements of a matrix are real, the matrix is called a *real matrix*. If we wish to write out the matrix in full, we set it out as a rectangular array in curved brackets:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\sigma} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\sigma} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{\rho 1} & a_{\rho 2} & \cdots & a_{\rho\sigma} & \cdots & a_{\rho n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n\sigma} & \cdots & a_{nn} \end{pmatrix}; \tag{12.2}$$

this array is often abbreviated to $(a_{\rho\sigma})$. If we do not wish to specify the elements of a matrix, it is usual to denote it simply by a solid capital letter such as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}, \ldots$; it is often convenient to associate the same capital and small letters with a given matrix, so that $\boldsymbol{A} = (a_{\rho\sigma})$, for instance.

In the matrix (12.2), the elements $a_{\rho\sigma}$ with a fixed value of ρ form a *row* of the matrix; for example $a_{21}, a_{22}, \ldots, a_{2n}$, with $\rho = 2$, form the second row of the matrix. Likewise elements with a given value of σ form a *column*. We can say therefore that the suffix ρ labels the rows while the suffix σ labels the columns of the matrix. A particular element $a_{\rho\sigma}$ is the element in the ρth row and the σth column, and is referred to as the (ρ, σ) element. If a matrix has m rows and n columns, it is called an $(m \times n)$ matrix.

Example 1

Two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in 3-space can be expressed in terms of their components referred to given axes. If the components of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are (u_{11}, u_{12}, u_{13}) and (u_{21}, u_{22}, u_{23}) respectively, then the (2×3) matrix

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}$$

consists of all components of the two vectors. The first suffix of an element $u_{\rho\sigma}$ tells us to which vector the component belongs, while the second suffix tells us which of the three components it is. If the vectors were written as $\boldsymbol{u}_1 = (u_{11}, u_{21}, u_{31})$ and $\boldsymbol{u}_2 = (u_{12}, u_{22}, u_{32})$ with the *second* suffix telling us to which vector the com-

ponents belong, then the matrix would be written as

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix};$$

the vectors now appear as columns of a matrix instead of as rows.

Note that the components of a single vector $\boldsymbol{u}$ can be regarded as a (1×3) matrix $(u_1\, u_2\, u_3)$, known as a *row matrix*, or alternately as a (3×1) *column matrix*, consisting of a single column.

Example 2

Each man of a group of n men pays a certain sum of money to each of the other $n-1$ men in the group. Suppose that the men are numbered 1, 2,... n, and that the amount paid by the ρth man to the σth is called $a_{\rho\sigma}$. Then these sums can be written conveniently as an $(n\times n)$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

in which the elements a_{11}, a_{22},... a_{nn} are all zero. Any matrix in which the number of rows equals the number of columns is known as a *square matrix*; the elements a_{11}, a_{22},... a_{nn} form the *leading diagonal* of a square matrix, and are called *diagonal elements*. So the matrix defined here is a square matrix with all diagonal elements equal to zero. We note that the rows and columns have a particular significance: the elements in the ρth row are the amounts paid out by the ρth man, while the elements in the σth column are the amounts received by the σth man.

§ 2. General properties of matrices

§ 2.1. Matrix addition and subtraction

The laws of matrix addition and subtraction are simple generalisations of the corresponding laws for vectors. Vectors can be added only when they have the same numbers of components, since they would otherwise be vectors in spaces with different numbers of dimensions. Likewise we assume that matrices can be added only when the ranges of *both* the suffixes coincide. Then the sum of an $(m\times n)$ matrix $(a_{\rho\sigma})$ and a second $(m\times n)$ matrix $(b_{\rho\sigma})$ is defined as $(a_{\rho\sigma}+b_{\rho\sigma})$, formed by adding the corresponding components of the two matrices. The difference of the two matrices is likewise defined by

$$(a_{\rho\sigma}) - (b_{\rho\sigma}) = (a_{\rho\sigma} - b_{\rho\sigma});$$

that is, we subtract each element of $(b_{\rho\sigma})$ from the corresponding element of $(a_{\rho\sigma})$.

The zero $(m \times n)$ matrix, denoted by $\mathbf{0}$, is the matrix with every element zero. Clearly, if A is any $(m \times n)$ matrix,

$$A + \mathbf{0} = \mathbf{0} + A = A.$$

Example 3

The sum of the matrices

$$\begin{pmatrix} 5 & -2 & 0 \\ -4 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 3 & -2 \\ 4 & -7 & 2 \end{pmatrix}$$

exists since $m=2$ and $n=3$ for each matrix, and equals

$$\begin{pmatrix} 5+2 & -2+3 & 0-2 \\ -4+4 & 1-7 & 1+2 \end{pmatrix} = \begin{pmatrix} 7 & 1 & -2 \\ 0 & -6 & 3 \end{pmatrix}.$$

The difference $\begin{pmatrix} 5 & -2 & 0 \\ -4 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 & -2 \\ 4 & -7 & 2 \end{pmatrix}$ is equal to $\begin{pmatrix} 3 & -5 & 2 \\ -8 & 8 & -1 \end{pmatrix}$.

Example 4

If

$$A = \begin{pmatrix} 2-3i & 4-i \\ 1+i & 3-4i \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2+3i & -4+i \\ 2-i & 1+4i \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 2-3i+2+3i & 4-i-4+i \\ 1+i+2-i & 3-4i+1+4i \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix},$$

a real matrix.

Also

$$A - B = \begin{pmatrix} -6i & 8-2i \\ -1+2i & 2-8i \end{pmatrix}.$$

The sum of several matrices is obtained by repeated addition; since the elements are in general complex numbers, and matrix addition is merely the addition of corresponding elements, the laws of matrix addition and subtraction are the same as those for complex numbers. To be explicit, if A, B, C are three $(m \times n)$ matrices, then

$$A + B = B + A \qquad \text{(commutative law)}$$

and

$$(A + B) + C = A + (B + C) \qquad \text{(associative law).}$$

In other words, it does not matter in which order we add several $(m \times n)$ matrices.

The product of a complex number λ and a matrix A is found by multiplying every element of A by λ; thus $\lambda(a_{\rho\sigma})=(\lambda a_{\rho\sigma})$. This rule is the same as that for multiplying a scalar into a vector.

Example 5

If

$$A = \begin{pmatrix} \frac{1}{2}i & 3+i & i-1 \\ \frac{1}{4} & \frac{1}{2}+2i & -3i \end{pmatrix}$$

and $\lambda=2-i$, then

$$\lambda A = \begin{pmatrix} (2-i)\frac{1}{2}i & (2-i)(3+i) & (2-i)(i-1) \\ (2-i)\frac{1}{4} & (2-i)(\frac{1}{2}+2i) & (2-i)(-3i) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+i & 7-i & -1+3i \\ \frac{1}{2}-\frac{1}{4}i & 3-\frac{7}{2}i & -3-6i \end{pmatrix}.$$

If A and B are two $(m \times n)$ matrices, then the matrix $\lambda(A+B)$ has its (ρ, σ) element equal to $\lambda(a_{\rho\sigma}+b_{\rho\sigma})=\lambda a_{\rho\sigma}+\lambda b_{\rho\sigma}$. So

$$\lambda(A+B) = \lambda A + \lambda B.$$

It is equally obvious that the other distributive law

$$(\lambda+\mu)A = \lambda A + \mu A$$

holds, where λ and μ are complex numbers. These two properties mean that we can 'multiply out' a product of a sum of complex numbers into a sum of matrices.

EXERCISE 12.1

1. A, B, C and D are the (2×4) matrices

$$A = \begin{pmatrix} 3 & -2 & 5 & 1 \\ -4 & 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 6 & 4 & -1 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 1+i & 2-3i & 4i & 2 \\ 5i & 3-i & 4+i & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 4i & -2 & 3i & 2+2i \\ 2+4i & i & 0 & -2+i \end{pmatrix}.$$

Find $A+B$, $A-C$, $B+2D$, $3C-2D$, $2A+C-4B$, $2A-4B+4C-2D$, $A+7B-C$.

2. If A and B are given by

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -3 \\ 2 & 2 \end{pmatrix},$$

show that real numbers λ and μ can be found such that $\lambda \boldsymbol{A}+\mu \boldsymbol{B}=\boldsymbol{C}$, where

$$\boldsymbol{C}=\begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix}.$$

3. Two square matrices $\boldsymbol{A}$, $\boldsymbol{B}$ have elements

$$a_{\rho\sigma} = \exp\,(i\alpha(\rho+\sigma)) \quad \text{and} \quad b_{\rho\sigma} = \exp\,(-i\alpha(\rho+\sigma))$$

which are dependent on the real parameter α. If $\boldsymbol{C}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{B})$, find the values of α for which

(i) the diagonal elements of $\boldsymbol{C}$ are all zero,
(ii) the diagonal elements of $\boldsymbol{C}$ are all equal to 1,
(iii) all the elements of $\boldsymbol{C}$ are equal to 1.

§ 2.2. Matrix multiplication defined

For vectors, we have defined not only the product $\lambda \boldsymbol{v}$ of a scalar λ and a vector $\boldsymbol{v}$, but also products of two vectors. In order to define the product $\boldsymbol{AB}$ of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we generalise the rule for forming the scalar or inner product of two vectors. If $\boldsymbol{A}$ is the $(m\times n)$ matrix $(a_{\rho\sigma})$, we can treat the rows of the matrix as though they were vectors in component form; that is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

can be regarded as the m 'vectors' $(a_{11}, a_{12},\ldots a_{1n}),\ldots(a_{m1}, a_{m2},\ldots, a_{mn})$ written one below another. To multiply the matrix into a second matrix $\boldsymbol{B}$, we form a series of inner products from these vectors. Now we can only form an inner product of two vectors if they have the same number of components, so that the vectors of the matrix $\boldsymbol{B}$ must also have n components. It seems therefore that we must insist on $\boldsymbol{B}$ having the same number (n) of columns as $\boldsymbol{A}$ so that we can form inner products of rows of $\boldsymbol{A}$ with the rows of $\boldsymbol{B}$. This method of defining matrix products does not prove to be quite suitable, as it does not lead to a systematic algebra. It turns out that it is best to treat the *rows* of $\boldsymbol{A}$ as vectors, and the *columns* of $\boldsymbol{B}$ as vectors, so that the *columns* of $\boldsymbol{B}$ must contain n elements. We can state this as a general rule:

The product $\boldsymbol{AB}$ *of two matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *exists only if*

number of elements in a row of $\boldsymbol{A}$ = *number of elements in a column of* $\boldsymbol{B}$,

which is the same as saying that

number of columns of $\boldsymbol{A}$ = *number of rows of* $\boldsymbol{B}$.

We therefore assume that $\boldsymbol{B}$ is an $(n \times p)$ matrix, and use the inner product rule to form products of the rows of $\boldsymbol{A}$ and the columns of $\boldsymbol{B}$ in the matrix product

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \dots & a_{1\sigma} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2\sigma} & \dots & a_{2n} \\ \hline \multicolumn{6}{c}{\cdots\cdots\cdots\cdots} \\ \hline a_{\rho 1} & a_{\rho 2} & \dots & a_{\rho\sigma} & \dots & a_{\rho n} \\ \hline \multicolumn{6}{c}{\cdots\cdots\cdots\cdots} \\ \hline a_{n1} & a_{n2} & \dots & a_{n\sigma} & \dots & a_{nn} \\ \hline \end{array}\right] \left[\begin{array}{c|c|c|c|c|c} b_{11} & b_{12} & \cdot & b_{1\tau} & \cdot & b_{1p} \\ b_{21} & b_{22} & \cdot & b_{2\tau} & \cdot & b_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{\sigma 1} & b_{\sigma 2} & \cdot & b_{\sigma\tau} & \cdot & b_{\sigma p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & b_{n\tau} & \cdot & b_{np} \end{array}\right].$$

Here we have drawn horizontal and vertical lines to divide the matrices into vectors. The first row of $\boldsymbol{A}$ and the first column of $\boldsymbol{B}$ give the inner product

$$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1\sigma}b_{\sigma 1} + \dots + a_{1n}b_{n1} = \sum_{\sigma=1}^{n} a_{1\sigma}b_{\sigma 2}.$$

The first row of $\boldsymbol{A}$ and the second column of $\boldsymbol{B}$ give

$$a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1\sigma}b_{\sigma 2} + \dots + a_{1n}b_{n2} = \sum_{\sigma=1}^{n} a_{1\sigma}b_{\sigma 2}.$$

We can see that in general the ρth row of $\boldsymbol{A}$ and the τth column of $\boldsymbol{B}$, shown explicitly in the matrices above, give the inner product

$$a_{\rho 1}b_{1\tau} + a_{\rho 2}b_{2\tau} + \dots + a_{\rho\sigma}b_{\sigma\tau} + \dots + a_{\rho n}b_{n\tau} = \sum_{\sigma=1}^{n} a_{\rho\sigma}b_{\sigma\tau},$$

with the first and last suffixes in each term denoting the row of $\boldsymbol{A}$ and the column of $\boldsymbol{B}$; the middle suffixes are equated and are summed over to form the inner product. In all, we have mp inner products, since the m rows of $\boldsymbol{A}$ are each multiplied into the p columns of $\boldsymbol{B}$. These mp products are written naturally as the $(m \times p)$ matrix

$$
AB = \begin{bmatrix}
\sum_\sigma a_{1\sigma}b_{\sigma 1} & \sum_\sigma a_{1\sigma}b_{\sigma 2} & \cdots & \sum_\sigma a_{1\sigma}b_{\sigma\tau} & \cdots & \sum_\sigma a_{1\sigma}b_{\sigma p} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sum_\sigma a_{\rho\sigma}b_{\sigma 1} & \sum_\sigma a_{\rho\sigma}b_{\sigma 2} & \cdots & \sum_\sigma a_{\rho\sigma}b_{\sigma\tau} & \cdots & \sum_\sigma a_{\rho\sigma}b_{\sigma p} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\sum_\sigma a_{m\sigma}b_{\sigma 1} & \sum_\sigma a_{m\sigma}b_{\sigma 2} & \cdots & \sum_\sigma a_{m\sigma}b_{\sigma\tau} & \cdots & \sum_\sigma a_{m\sigma}b_{\sigma p}
\end{bmatrix}, \tag{12.3}
$$

which is defined to be the product of $\boldsymbol{A}$ into $\boldsymbol{B}$. In the product, the ρth row of $\boldsymbol{A}$ contributes to the ρth row of $\boldsymbol{AB}$, while the τth column of $\boldsymbol{B}$ contributes to the τth column of $\boldsymbol{AB}$. In fact, the rule for forming a matrix product leads to a natural and sensible labelling of the rows and columns of $\boldsymbol{AB}$, the (ρ, τ) element being $\sum_\sigma a_{\rho\sigma}b_{\sigma\tau}$.

The products $\boldsymbol{A0}$ and $\boldsymbol{0A}$ of any matrix $\boldsymbol{A}$ with a suitable zero matrix are all clearly zero. It is important to note however, that the product of two non-zero matrices can equal a zero matrix. For instance,

$$
\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

Example 6

The product $\boldsymbol{AB}$ of the matrices

$$
\boldsymbol{A} = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ -1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 3 & 0 \\ 2 & 4 \\ 2 & -1 \end{pmatrix}
$$

exists since the number of columns of $\boldsymbol{A}$ is 3, equal to the number of rows of $\boldsymbol{B}$. Using (12.3) we find

$$
\boldsymbol{AB} = \begin{pmatrix}
2\cdot 3 + 1\cdot 2 + (-3)\cdot 2 & 2\cdot 0 + 1\cdot 4 + (-3)\cdot(-1) \\
0\cdot 3 + 2\cdot 2 + (-2)\cdot 2 & 0\cdot 0 + 2\cdot 4 + (-2)\cdot(-1) \\
(-1)\cdot 3 + (-1)\cdot 2 + 3\cdot 2 & (-1)\cdot 0 + (-1)\cdot 4 + 3\cdot(-1) \\
2\cdot 3 + 0\cdot 2 + 1\cdot 2 & 2\cdot 0 + 0\cdot 4 + 1\cdot(-1)
\end{pmatrix}
$$

or

$$
\boldsymbol{AB} = \begin{pmatrix} 2 & 7 \\ 0 & 10 \\ 1 & -7 \\ 8 & -1 \end{pmatrix}.
$$

Note that the product $\boldsymbol{BA}$ does not exist, since $\boldsymbol{B}$ has two columns, while $\boldsymbol{A}$ has four rows.

§ 2.3. Failure of the commutative law

If two matrices A and B are given, then the matrix product BA is in general quite different from the product AB, since in BA the inner products are formed from the rows (not columns) of B and the columns (not rows) of A. In fact, although (12.3) defines the product AB of an $(m \times n)$ matrix A into an $(n \times p)$ matrix B, we have no reason to suppose that BA will exist, since this would require $p=m$, which is not true in general. Even when the condition $p=m$ is satisfied, so that BA does exist, the inner products formed in calculating AB and BA are generally quite different, so that $AB \neq BA$.

Example 7

If

$$A = \begin{pmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -4 \\ 3 & -2 \\ -1 & 1 \end{pmatrix}$$

then both AB and BA exist; but AB is the (2×2) matrix

$$\begin{pmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ 2 & 9 \end{pmatrix},$$

while BA is the (3×3) matrix

$$\begin{pmatrix} 1 & -4 \\ 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -9 & 0 \\ 12 & -7 & 10 \\ -5 & 3 & -3 \end{pmatrix}.$$

The products AB and BA of two $(n \times n)$ square matrices both exist, and are themselves $(n \times n)$ square matrices. But in general $AB \neq BA$.

Example 8

If A and B are the (2×2) matrices

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 22 & 6 \\ -5 & 5 \end{pmatrix}, \quad \text{but} \quad BA = \begin{pmatrix} 9 & 11 \\ 2 & 13 \end{pmatrix}.$$

§ 2.4. Associative and distributive laws

Thus matrix multiplication does not satisfy the commutative law $AB=BA$, meaning that it matters a great deal in which order we

write matrices in matrix products. However, this is the only fundamental law of ordinary algebra (the algebra of complex numbers) which is violated by matrices. We have already seen that the associative and distributive laws of addition and scalar multiplication are satisfied. The associative law

$$\lambda(AB) = (\lambda A)B = A(\lambda B)$$

is clearly seen to be true, since the (ρ, τ) elements in these products are

$$\lambda \sum_{\sigma} a_{\rho\sigma}b_{\sigma\tau}, \qquad \sum_{\sigma} (\lambda a_{\rho\sigma})b_{\sigma\tau} \quad \text{and} \quad \sum_{\sigma} a_{\rho\sigma}(\lambda b_{\sigma\tau}),$$

which are all equal. Further, the associative and distributive laws

$$(AB)C = A(BC) \tag{12.4}$$

and

$$A(B + C) = AB + AC \tag{12.5}$$

are true for any matrices A, B and C for which the products exist; these laws mean that matrix multiplications may be done in any order. and that products of sums of matrices may be multiplied out.

★ Let us assume that one product in (12.4), say $(AB)C$, exists. Then we can show that $A(BC)$ exists and equals $(AB)C$. For we know that

(i) number of columns of A = number of rows of $B = n$, say,

(ii) number of columns of AB = numbers of rows of $C = p$, say.

Now the (ρ, τ) element of AB is $\sum_{\sigma=1}^{n} a_{\rho\sigma}b_{\sigma\tau}$, and B, like AB, has p columns, which is equal to the number of rows of C by (ii). Hence BC exists, and the (σ, ω) element of BC is $\sum_{\tau=1}^{p} b_{\sigma\tau}c_{\tau\omega}$. The number of rows of BC, equal to the number of rows of B, is n; so by (i), the product $A(BC)$ exists, and its (ρ, ω) element is $\sum_{\sigma=1}^{n} a_{\rho\sigma}(\sum_{\tau=1}^{p} b_{\sigma\tau}c_{\tau\omega})$. But the (ρ, ω) element of $(AB)C$ is $\sum_{\tau=1}^{p}(\sum_{\sigma=1}^{n} a_{\rho\sigma}b_{\sigma\tau})c_{\tau\omega}$. These repeated sums are both equal to the double sum

$$\sum_{\sigma=1}^{n} \sum_{\tau=1}^{p} a_{\rho\sigma}b_{\sigma\tau}c_{\tau\omega}, \tag{12.6}$$

establishing (12.4). ★

The distributive law (12.5) is even easier to prove, and we leave the proof as an exercise to the reader. The outcome of this discussion of fundamental laws is that matrices can be added and multiplied as freely as ordinary numbers, except that we must never change the order of matrices in any product.

Example 9

If

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 4 & 6 \\ -1 & 3 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} -3 & -1 \\ 4 & 0 \end{pmatrix},$$

then

$$\boldsymbol{AB} = \begin{pmatrix} 7 & 15 \\ -11 & -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{BC} = \begin{pmatrix} 12 & -4 \\ 15 & 1 \end{pmatrix}.$$

Thus

$$(\boldsymbol{AB})\boldsymbol{C} = \begin{pmatrix} 39 & -7 \\ 21 & 11 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{BC}) = \begin{pmatrix} 39 & -7 \\ 21 & 11 \end{pmatrix},$$

verifying the associative law (12.4).

The rule (12.6) for calculating elements in the product of three matrices can be generalised very simply to products of any finite number of matrices. If $\boldsymbol{A}, \boldsymbol{B}, \ldots \boldsymbol{G}, \boldsymbol{H}$ are respectively $(m \times n)$, $(n \times p), \ldots (r \times s)$, $(s \times t)$ matrices, then the product $\boldsymbol{AB} \cdots \boldsymbol{GH}$ exists, and is the $(m \times t)$ matrix with general element

$$\sum_{\sigma=1}^{n} \sum_{\tau=1}^{p} \cdots \sum_{\mu=1}^{r} \sum_{\nu=1}^{s} a_{\rho\sigma} b_{\sigma\tau} \cdots g_{\mu\nu} h_{\nu\omega}. \tag{12.7}$$

§ 2.5. Suffix notation

The suffixes ρ and ω label the rows and columns of the product matrix defined by (12.7) and are known as *free* suffixes, since they may take any value we choose to give them. The suffices $\sigma, \tau, \ldots \mu, \nu$ are summed over their full ranges, and hence cannot take any particular value; they are known as *dummy suffixes*, since they have no special significance in the product. We notice that dummy suffixes always occur twice in matrix products; in fact, if a suffix is repeated we are very often summing over its full range, and the summation signs such as those written in (12.6) and (12.7) are superfluous. It is customary to dispense with these summation signs, using instead the *summation convention* by which it is understood that a suffix which occurs twice or is *repeated* is automatically treated as a dummy suffix and is summed over its full range. For example, the result (12.6) can be written

$$\boldsymbol{ABC} = (a_{\rho\sigma} b_{\sigma\tau} c_{\tau\omega}). \tag{12.8}$$

Here the repeated suffixes σ and τ are automatically summed over their complete ranges, and the free suffixes ρ and ω label the rows and columns

of the product matrix. There are occasions when we do not wish to sum over a repeated suffix; we must then be careful to say explicitly that we are not summing.

Example 10

If $A=(a_{\rho\sigma})$ is a square matrix, then the element on the leading diagonal which is in the τth row and τth column is $a_{\tau\tau}$; the suffix here is repeated, but is *not summed* over all values of τ. If we were to sum over all values of τ we would obtain a quantity known as the *trace* of the matrix A, denoted by Tr A. Thus

$$a_{\tau\tau} \quad \text{(not summed)} = \tau\text{th diagonal element,}$$

while

$$a_{\tau\tau} = \operatorname{Tr} A = \text{sum of all elements on the leading diagonal.}$$

Example 11

A square $(n \times n)$ matrix A can be multiplied into itself, giving the square AA or A^2, also an $(n \times n)$ matrix. Hence the cube $A^3=A(A^2)=(A^2)A$ exists, and in general the kth powers A^k of A is the $(n \times n)$ matrix formed by repeated multiplication.

If A and B are two $(n \times n)$ matrices, the kth power of their sum $(A+B)^k$, exists, but care must be taken in writing down the binomial theorem; for instance,

$$(A + B)^2 = (A + B)(A + B) = A^2 + (AB + BA) + B^2$$

and

$$(A+B)^3=(A+B)(A+B)^2=A^3+(A^2B+ABA+BA^2)+(AB^2+BAB+B^2A)+B^3.$$

If we define $S(A^i, B^j)$ as the sum of all different terms formed by multiplying i matrices A and j matrices B together in all possible orders, then

$$(A + B)^k = \sum_{i=0}^{k} S(A^i, B^{k-i}).$$

§ 2.6. Transposed, symmetric and antisymmetric matrices

The *transposed* of an $(m \times n)$ matrix A is found by writing the rows of A as columns, forming an $(n \times m)$ matrix $\bar{A}$. For example, the transposed of

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{is} \quad \bar{A} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}.$$

So the (ρ, σ) element of $\bar{A}$ is $a_{\sigma\rho}$, the interchange of rows and columns being reflected in the interchange of suffixes. Denoting the (ρ, σ) element of $\bar{A}$ by $\bar{a}_{\rho\sigma}$, we thus have $\bar{a}_{\rho\sigma}=a_{\sigma\rho}$.

Suppose that a matrix $\boldsymbol{M}$ is the matrix product

$$\boldsymbol{M} = \boldsymbol{ABC}\cdots\boldsymbol{GH}.$$

The general element of $\boldsymbol{M}$ is given by (12.7), with ρ and ω labelling the rows and columns. So the (ω, ρ) element of the transposed matrix $\overline{\boldsymbol{M}}$ is given by (12.7). Writing $a_{\rho\sigma}=\bar{a}_{\sigma\rho}$, and so on, we therefore have

$$\bar{m}_{\omega\rho} = \sum_{\sigma=1}^{n} \sum_{\tau=1}^{p} \cdots \sum_{\mu=1}^{r} \sum_{\nu=1}^{s} \bar{h}_{\omega\nu}\bar{g}_{\nu\mu}\cdots\bar{b}_{\tau\sigma}\bar{a}_{\sigma\rho}$$

or

$$\overline{\boldsymbol{M}} = \overline{\boldsymbol{H}}\overline{\boldsymbol{G}}\cdots\overline{\boldsymbol{C}}\overline{\boldsymbol{B}}\overline{\boldsymbol{A}}.$$

So *the transposed of a matrix product is the product of the transposed matrices written in reverse order.*

A *symmetric matrix* is a real square matrix which is its own transposed, so that $\overline{\boldsymbol{A}}=\boldsymbol{A}$ or $a_{\rho\sigma}=a_{\sigma\rho}$ for all ρ and σ. For example,

$$\begin{pmatrix} 3 & -1 & 4 \\ -1 & 2 & 6 \\ 4 & 6 & -3 \end{pmatrix}$$

is a symmetric (3×3) matrix. Clearly there are only $3+2+1=6$ independent elements in this matrix; an $(n\times n)$ symmetric matrix has $n+(n-1)+(n-2)+\ldots+2+1=\frac{1}{2}n(n+1)$ independent elements.

An *antisymmetric matrix* is a real square matrix with $\overline{\boldsymbol{A}}=-\boldsymbol{A}$ or $a_{\rho\sigma}=-a_{\sigma\rho}$. It follows at once that all diagonal elements are zero. An example of a (3×3) antisymmetric matrix is

$$\begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix}$$

which has three independent elements. An $(n\times n)$ antisymmetric matrix has $(n-1)+(n-2)+\ldots+2+1=\frac{1}{2}n(n-1)$ independent elements.

If any matrix $\boldsymbol{A}=(a_{\rho\sigma})$ is given, we can find a symmetric matrix $\boldsymbol{A}'$ and an antisymmetric matrix $\boldsymbol{A}''$ such that $\boldsymbol{A}=\boldsymbol{A}'+\boldsymbol{A}''$. For we can write

$$a_{\rho\sigma} = \tfrac{1}{2}(a_{\rho\sigma} + a_{\sigma\rho}) + \tfrac{1}{2}(a_{\rho\sigma} - a_{\sigma\rho});$$

so if $\boldsymbol{A}'$ and $\boldsymbol{A}''$ have elements $a'_{\rho\sigma}=\frac{1}{2}(a_{\rho\sigma}+a_{\sigma\rho})$ and $a''_{\rho\sigma}=\frac{1}{2}(a_{\rho\sigma}-a_{\sigma\rho})$ respec-

tively, then $A = A' + A''$, with $a'_{\rho\sigma} = a'_{\sigma\rho}$ and $a''_{\rho\sigma} = -a''_{\sigma\rho}$, as required. The matrices A' and A'' are known respectively as the *symmetric and antisymmetric parts* of A.

Example 12

Find the symmetric and antisymmetric parts of

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 3 & -4 & 0 \\ 4 & 6 & 2 \end{pmatrix}.$$

Since $\frac{1}{2}(5+3) = 4$, $\frac{1}{2}(5-3) = 1$, $\frac{1}{2}(-2+4) = 1$, $\frac{1}{2}(-2-4) = -3$, and so on, we can write $A = A' + A''$, where

$$A' = \begin{pmatrix} 1 & 4 & 1 \\ 4 & -4 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad A'' = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -3 \\ 3 & 3 & 0 \end{pmatrix}.$$

§ 2.7. Column and row matrices

We have already remarked that an n-component vector can be written either as a $(1 \times n)$ row matrix or an $(n \times 1)$ column matrix. It is therefore natural to use a notation similar to vector notation for row and column matrices, but we must distinguish between these two types of matrix. It is customary to use the vector symbol $\boldsymbol{v}$ to denote the $(n \times 1)$ column matrix

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}. \tag{12.9}$$

The elements v_ρ of $\boldsymbol{v}$, regarded as a matrix, have only one suffix ρ, labelling the rows; the suffix labelling the columns, which takes only one value, is omitted. A $(1 \times n)$ row matrix is written as

$$\bar{\boldsymbol{v}} = (v_1\, v_2 \cdots v_n). \tag{12.10}$$

Here the suffix σ attached to the elements v_σ labels the columns of the matrix $\bar{\boldsymbol{v}}$, the trivial row suffix being omitted. The matrix (12.10) is

derived by (12.9) by writing the column as a row, so that $\bar{\boldsymbol{v}}$ is the transposed of $\boldsymbol{v}$, in accordance with the notation introduced in § 2.6.

If $\bar{\boldsymbol{v}}$ and $\boldsymbol{w}$ are n-component row and column matrices, then the matrix product

$$\bar{\boldsymbol{v}}\boldsymbol{w} = (v_1w_1 + v_2w_2 + \ldots + v_nw_n) \tag{12.11}$$

is a matrix consisting of one element equal to the scalar product of the *vectors* $\boldsymbol{v}$ and $\boldsymbol{w}$. In particular, the length v of a vector $\boldsymbol{v}$ is given in matrix form by

$$\bar{\boldsymbol{v}}\boldsymbol{v} = (v_1^2 + v_2^2 + \ldots + v_n^2) = (v^2). \tag{12.12}$$

Example 13

The relations found in Exer. 10.1 No. 1 between the components (v_1, v_2) of a vector along one set of rectangular axes in a plane and the components (v_1', v_2') along a second set of axes inclined at an angle θ to the first can be expressed as the matrix equation

$$\boldsymbol{v}' = \boldsymbol{U}\boldsymbol{v}$$

where

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{v}' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

In terms of the row matrices $\bar{\boldsymbol{v}} = (v_1v_2)$ and $\bar{\boldsymbol{v}}' = (v_1'v_2')$, the relations can be written, transposing, as

$$\bar{\boldsymbol{v}}' = \boldsymbol{v}\bar{\boldsymbol{U}};$$

$$\bar{\boldsymbol{U}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is formed by interchanging rows and columns in $\boldsymbol{U}$, and is the transposed of $\boldsymbol{U}$.

§ 2.8. Block multiplication

Sometimes the elements in a set of square $(n \times n)$ matrices fall into four well-defined groups or *blocks* so that a typical matrix can be written naturally as

$$\boldsymbol{A} = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1l} & a_{11}' & \cdots & a_{1k}' \\ a_{21} & \cdots & a_{2l} & a_{21}' & \cdots & a_{2k}' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} & a_{l1}' & \cdots & a_{lk}' \\ \hline a_{11}'' & \cdots & a_{1l}'' & a_{11}''' & \cdots & a_{1k}''' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1}'' & \cdots & a_{kl}'' & a_{k1}''' & \cdots & a_{kk}''' \end{array}\right] \tag{12.13}$$

where $l+k=n$, the number of rows and columns. The solid lines in (12.13) mark out the four blocks. We can write the matrix $\boldsymbol{A}$ more conveniently in the form

$$\boldsymbol{A} = \begin{pmatrix} \mathscr{A} & \mathscr{A}' \\ \mathscr{A}'' & \mathscr{A}''' \end{pmatrix}, \tag{12.14}$$

where $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{A}''$, $\mathscr{A}'''$ are the arrays

$$\mathscr{A} = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ a_{21} & \cdots & a_{2l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix}, \qquad \mathscr{A}' = \begin{pmatrix} a'_{11} & \cdots & a'_{1k} \\ a'_{21} & \cdots & a'_{2k} \\ \cdots & \cdots & \cdots \\ a'_{l1} & \cdots & a'_{lk} \end{pmatrix},$$

and so on. Suppose that $\boldsymbol{C}$ is another $(n\times n)$ matrix written in the form (12.13), and let us examine some typical elements of the product $\boldsymbol{AC}$. The (1, 1) element is

$$\sum_{\sigma=1}^{l} a_{1\sigma}c_{\sigma 1} + \sum_{\sigma=1}^{k} a'_{1\sigma}c''_{\sigma 1};$$

more generally, if the (ρ, τ) element of $\boldsymbol{AC}$ lies in the top left block, it has the form

$$\sum_{\sigma=1}^{l} a_{\rho\sigma}c_{\sigma\tau} + \sum_{\sigma=1}^{k} a'_{\rho\sigma}c''_{\sigma\tau}.$$

The top left block of $\boldsymbol{AC}$ can therefore be written in matrix form as the $(l\times l)$ matrix $\mathscr{A}\mathscr{C}+\mathscr{A}'\mathscr{C}''$, these products being formed from $\mathscr{A}$, $\mathscr{C}$,... by ordinary matrix multiplication. We notice that this is just the result we would obtain if we wrote $\boldsymbol{A}$ and $\boldsymbol{C}$ in the form (12.14) and multiplied them out using ordinary matrix rules:

$$\begin{pmatrix} \mathscr{A} & \mathscr{A}' \\ \mathscr{A}'' & \mathscr{A}''' \end{pmatrix} \begin{pmatrix} \mathscr{C} & \mathscr{C}' \\ \mathscr{C}'' & \mathscr{C}''' \end{pmatrix}$$

$$= \begin{pmatrix} \mathscr{A}\mathscr{C} + \mathscr{A}'\mathscr{C}'' & \mathscr{A}\mathscr{C}' + \mathscr{A}'\mathscr{C}''' \\ \mathscr{A}''\mathscr{C} + \mathscr{A}'''\mathscr{C}'' & \mathscr{A}''\mathscr{C}' + \mathscr{A}'''\mathscr{C}''' \end{pmatrix}. \tag{12.15}$$

It is easy to check that the elements in the other three blocks of $\boldsymbol{AC}$ are correctly given by (12.15), so that the blocks of elements in (12.14) can be treated formally as though they were single elements. We must of course be careful to preserve the order of matrices in $\mathscr{A}\mathscr{C}$, $\mathscr{A}'\mathscr{C}''$, and so on, in the blocks of (12.15), since these products are themselves matrix products.

EXERCISE 12.2

1. Find all products which exist of pairs of the matrices

$$A=\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix},\quad B=\begin{pmatrix}2 & 3 & 4\\ 1 & 0 & -1\\ -1 & 3 & -2\end{pmatrix},\quad C=\begin{pmatrix}6 & 2\\ -1 & 4\\ 2 & 1\end{pmatrix},$$

$$D=\begin{pmatrix}2 & 3 & 2\\ -1 & -1 & 0\end{pmatrix},\quad E=\begin{pmatrix}6 & 1 & 0 & -3\\ -2 & 1 & -1 & 2\end{pmatrix},\quad F=\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 1\\ 2 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}.$$

2. If the elements of a matrix $\boldsymbol{A}$ are $a_{\rho\sigma}=u_\rho u_\sigma$, show that $\boldsymbol{A}^n$ ($n=2, 3, 4,\ldots$) are scalar multiples of $\boldsymbol{A}$.

3. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices, under what conditions does the rule

$$\boldsymbol{A}^2-\boldsymbol{B}^2=(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B})$$

hold?

4. If a square matrix $\boldsymbol{A}$ can be written in block form as

$$\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_2 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_3\end{pmatrix},$$

show that if $p(x)$ is a polynomial in x, then

$$p(\boldsymbol{A})=\begin{pmatrix}p(\boldsymbol{A}_1) & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & p(\boldsymbol{A}_2) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & p(\boldsymbol{A}_3)\end{pmatrix}.$$

5. Find the symmetric and antisymmetric parts of

$$\begin{pmatrix}1 & -2 & 6 & 4\\ 2 & -2 & 3 & -5\\ 2 & 3 & -1 & 6\\ 0 & 1 & -6 & 4\end{pmatrix}.$$

6. Show that if $\boldsymbol{A}$ is any real matrix, then $\bar{\boldsymbol{A}}\boldsymbol{A}$ is symmetric.

7. If $\boldsymbol{A}$ is an antisymmetric $(n\times n)$ matrix and $\boldsymbol{v}$ is an n-component column vector, show that $\bar{\boldsymbol{v}}\boldsymbol{A}\boldsymbol{v}=0$.

8. If $\boldsymbol{A}$ is an antisymmetric matrix, show that for any positive integer l, $\boldsymbol{A}^{2l}$ is a symmetric matrix and $\boldsymbol{A}^{2l+1}$ is antisymmetric.

§ 3. Diagonal matrices and the unit matrix

§ 3.1. Diagonal matrices and the Kronecker delta

A *diagonal matrix* is a square matrix whose elements are all zero, except for those on the leading diagonal. The general $(n\times n)$ diagonal matrix $\boldsymbol{D}$ has the form

$$\boldsymbol{D}=\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}. \tag{12.16}$$

If $\boldsymbol{D}'$ is a second $(n\times n)$ diagonal matrix with diagonal elements $d_1', d_2',\dots, d_n'$, then

$$\boldsymbol{D}\boldsymbol{D}'=\begin{pmatrix} d_1d_1' & 0 & 0 & \dots & 0 \\ 0 & d_2d_2' & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_nd_n' \end{pmatrix};$$

this matrix is also diagonal, the ρth diagonal element being the product of the ρth diagonal elements d_ρ and d_ρ'. Moreover, it is obvious by symmetry that $\boldsymbol{D}\boldsymbol{D}'=\boldsymbol{D}'\boldsymbol{D}$, so that the multiplication of diagonal matrices is commutative. In general, if any two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy $\boldsymbol{AB}=\boldsymbol{BA}$, we say that they *commute*.

In order to express diagonal matrices in algebraic form, we introduce a symbol $\delta_{\rho\sigma}$, known as the *Kronecker delta*. This is defined as

$$\left.\begin{aligned} \delta_{\rho\sigma} &= 1 \quad \text{when} \quad \rho=\sigma \\ \delta_{\rho\sigma} &= 0 \quad \text{when} \quad \rho\neq\sigma \end{aligned}\right\}, \tag{12.17}$$

it being assumed that the ranges of ρ and σ are prescribed. In terms of this symbol, the diagonal matrix (12.16) can be written $\boldsymbol{D}=(d_\rho\delta_{\rho\sigma})$, where we are *not summing over* ρ, but are using it to label the rows; elements not on the leading diagonal have $\rho\neq\sigma$ and so are zero by (12.17), while the ρth diagonal element is $d_\rho\delta_{\rho\rho}=d_\rho$ (no summation), which is correct.

§ 3.2. The unit matrix

The $(n\times n)$ *unit matrix*, denoted by $\boldsymbol{I}$, is the diagonal matrix whose diagonal elements are all equal to unity. The (ρ, σ) element of $\boldsymbol{I}$ is simply

$\delta_{\rho\sigma}$, defined by (12.17). It is an extremely important matrix, holding the same place in matrix algebra as the unit 1 does in ordinary algebra. In fact, if $\boldsymbol{A}$ is any $(m \times n)$ matrix, then by actual multiplication it is easy to check that

(i) $\boldsymbol{IA} = \boldsymbol{A}$, $\boldsymbol{I}$ being the $(m \times m)$ unit matrix,

(ii) $\boldsymbol{AI} = \boldsymbol{A}$, $\boldsymbol{I}$ being the $(n \times n)$ unit matrix.

These properties can also be established algebraically: since $\boldsymbol{I} = (\delta_{\rho\sigma})$, the (ρ, τ) element of $\boldsymbol{IA}$ is $\delta_{\rho\sigma}a_{\sigma\tau}$; the summation $\sum_{\sigma=1}^{m}$ is understood of course, and by (12.17) all terms in the sum are zero except that with $\sigma = \rho$. So the (ρ, τ) element is

$$\delta_{\rho\sigma}a_{\sigma\tau} = a_{\rho\tau}, \tag{12.18}$$

proving that $\boldsymbol{IA} = \boldsymbol{A}$. Likewise,

$$\boldsymbol{AI} = (a_{\rho\sigma}\delta_{\sigma\tau}) = (a_{\rho\tau}) = \boldsymbol{A}. \tag{12.19}$$

Equations (12.18) and (12.19) express the most important property of the Kronecker delta, known as the *substitution property* of suffixes.

EXERCISE 12.3

1. Write down a formula for the general element of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hence show that for $n = 2, 3, 4, \ldots$,

$$\boldsymbol{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

2. If $\boldsymbol{A}$ is any $(m \times n)$ matrix and $\boldsymbol{D}$ is the $(m \times m)$ diagonal matrix with diagonal elements $d_1, d_2, \ldots d_m$, show that $\boldsymbol{DA}$ is the $(m \times n)$ matrix whose pth row consists of the elements of the pth row of $\boldsymbol{A}$ multiplied by d_p. What is the corresponding rule for a product of the form $\boldsymbol{AD}$?

3. If

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

use block notation to find a general formula for $\boldsymbol{A}^n$.

4. If

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

find two other (2×2) matrices σ_2, σ_3 such that

$$\sigma_r\sigma_s + \sigma_s\sigma_r = 2I\delta_{rs} \qquad (r, s = 1, 2, 3).$$

§ 4. The determinant of a square matrix

In applications of matrix algebra, square matrices are often of great importance, and the *determinant* is a very important quantity associated with a square matrix. The determinant of an $(n\times n)$ square matrix $A=(a_{\rho\sigma})$ is denoted by $|A|$ or $|a_{\rho\sigma}|$, and is written in full as

$$|A| \equiv |a_{\rho\sigma}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$

the array being bordered by vertical straight lines, instead of by the curved brackets of the matrix. The determinant $|A|$ is defined to be a *single number* whose value depends on the elements $a_{\rho\sigma}$. Before we can define the determinant as a particular function of the $a_{\rho\sigma}$, it is necessary to discuss some elementary properties of permutations.

§ 4.1. Properties of permutations

Suppose that the ordered set of numbers 1, 2, 3,... n can be permuted to give the set $i, j,\dots k$, each of the numbers 1, 2,... n occurring once and only once in the permutation; there are in all $n!$ possible permutations. Any permutation can be formed from the set 1, 2,... n by simple interchanges of pairs of numbers; for example, the permutation 3421 can be formed from 1234 in the three stages

$$\bar{1}\ 2\ 3\ \bar{4} \to 4\ \bar{2}\ \bar{3}\ 1 \to \bar{4}\ \bar{3}\ 2\ 1 \to 3\ 4\ 2\ 1,$$

the bars indicating which pair of numbers are to be interchanged at each stage. A particular permutation can be formed in many different ways by simple interchanges, but the number of simple interchanges required is either always odd or always even; for example, the permutation 3421 can only be formed from 1234 by an odd number of simple

interchanges. So permutations of $1, 2, \ldots n$ fall into two classes, *even permutations* and *odd permutations*, obtained by making even and odd numbers of simple interchanges respectively. For example, the even and odd permutations of 123 are

$$\left.\begin{array}{llll} \text{even:} & 123 & 231 & 312 \\ \text{odd:} & 213 & 132 & 321 \end{array}\right\}. \tag{12.20}$$

There are always equal numbers of even and odd permutations, since for each even permutation there is a unique odd permutation formed by interchanging the numbers 1 and 2, and vice-versa. We shall denote a permutation $i, j, \ldots k$ of $1, 2, \ldots n$ by $P(ij\ldots k)$, to remind us that it contains each of the numbers $1, 2, \ldots n$ once and once only.

§ 4.2. Definition of a determinant

Consider now a product of n elements of the $(n \times n)$ matrix $\boldsymbol{A}$ of the form

$$a_{1i}a_{2j}\cdots a_{nk} \tag{12.21}$$

for a particular permutation $P(ij\ldots k)$ of the second suffixes. In the product (12.21), one and only one element is chosen from each row (a_{1i} from the first row, a_{2j} from the second row, and so on) and one and only one from each column (a_{1i} from the ith column, a_{2j} from the jth, and so on). If we attach a $+$ sign to products (12.21) with even permutations $P(ij\ldots k)$, and a $-$ sign to products with odd permutations, and then sum over all $n!$ possible permutations, we obtain the determinant $|\boldsymbol{A}|$. Thus by definition,

$$|\boldsymbol{A}| = \sum_{P(ij\ldots k)} \pm a_{1i}a_{2j}\cdots a_{nk}. \tag{12.22}$$

So the determinant is formed by (i) taking the $n!$ products of n elements such that there is one element from each row and one from each column (ii) attaching a $+$ sign to a product if the column suffixes are an even permutation of the row suffixes, and a $-$ sign otherwise, and (iii) summing over the $n!$ products with these signs. Stated in this way, it is clear that the definition of a determinant is symmetrical between the rows and columns of $\boldsymbol{A}$; therefore any theorem about the determinant $|\boldsymbol{A}|$ of a matrix $\boldsymbol{A}$ which involves the rows of $\boldsymbol{A}$ is also true for the columns of $\boldsymbol{A}$, and vice-versa.

Example 14

The determinant of a (2×2) matrix $\boldsymbol{A}$ is, by (12.22),

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Thus

$$\begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} = 6\cdot 2 - 5\cdot 3 = 12 - 15 = -3,$$

and

$$\begin{vmatrix} 4+i & 2i \\ -2+3i & 2-i \end{vmatrix} = (4+i)(2-i) - 2i(-2+3i) = 15 + 2i.$$

The determinant of a (3×3) matrix $\boldsymbol{A}$ consists of six terms, corresponding to the six permutations (12.20). Thus

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned} \tag{12.23}$$

The triple scalar product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ was defined in Ch. 9 § 6.1; if (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) are rectangular components of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, then the triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \equiv [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Example 15

Using (12.23) to evaluate a particular determinant,

$$\begin{vmatrix} 1 & 4 & 2 \\ 3 & -1 & 1 \\ -2 & 3 & 2 \end{vmatrix}$$

$$\begin{aligned} &= (1\cdot -1\cdot 2) + (4\cdot 1\cdot -2) + (2\cdot 3\cdot 3) - (1\cdot 1\cdot 3) - (4\cdot 3\cdot 2) - (2\cdot -1\cdot -2) \\ &= -2 - 8 + 18 - 3 - 24 - 4 = -23. \end{aligned}$$

§ 4.3. Expansions by rows and by columns

The basic symmetry property of determinants is fairly obvious. If a matrix $\boldsymbol{A}$ is given, and a second matrix $\boldsymbol{A}_1$ is formed from it by interchanging the position of two complete columns, then every term of type (12.21) occurring in $|\boldsymbol{A}|$ will also occur in $|\boldsymbol{A}_1|$; however, the interchange of two columns of $\boldsymbol{A}$ is equivalent to interchange of two column

suffixes in the permutation $P(ij...k)$; so even permutation terms in $|\boldsymbol{A}|$ become odd permutation terms in $|\boldsymbol{A}_1|$, and vice-versa. Hence all the signs in (12.22) are changed, and $|\boldsymbol{A}_1| = -|\boldsymbol{A}|$. The argument applies equally well for an interchange of two complete rows of the matrix $\boldsymbol{A}$, which also causes a change in the sign of the determinant.

Associated with each element $a_{\rho\sigma}$ of a square matrix $\boldsymbol{A}$ is its *minor* $M_{\rho\sigma}$. The minor of $a_{\rho\sigma}$ is defined by omitting the ρth row and σth column (the row and column containing $a_{\rho\sigma}$ itself), and by forming the determinant of the resulting matrix, which has $n-1$ rows and columns. For example, in the general (3×3) matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the minor of a_{11}, formed by omitting the first row and the first column, is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32};$$

the minor of a_{12} is

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31},$$

and so on.

The determinant of an $(n\times n)$ matrix $\boldsymbol{A}$ is said to be 'of order n'; so minors of elements of $\boldsymbol{A}$ are determinants of order $n-1$. We shall now establish a formula expressing $\boldsymbol{A}$ in terms of the minors of elements of $\boldsymbol{A}$. Consider first the terms in the sum (12.22) which contain a_{11}; in these terms $i=1$ and so $P(j...k)$ is a permutation of $2,\dots n$. Further, if $P(1j...k)$ is an even (or odd) permutation of $1, 2,\dots n$, it means that $P(j...k)$ is even (or odd). So the terms containing a_{11} sum to give

$$a_{11} \sum_{P(j...k)} \pm a_{2j}\cdots a_{nk}$$

which is simply $a_{11}M_{11}$, where M_{11} is the minor of a_{11}. Next consider the terms in (12.22) which contain a particular element $a_{\rho\sigma}$. If we interchange the whole of the ρth row with the one above it, $|\boldsymbol{A}|$ changes sign; if we move the row up in this way $\rho-1$ times, it will have moved up into the place of the first row, the order of the other rows being unchanged; this process changes the determinant by a factor $(-1)^{\rho-1}$.

Likewise we can shift the σth column to the extreme left of the matrix without changing the ordering of the other columns, but changing $|A|$ by a further factor $(-1)^{\sigma-1}$; the element $a_{\rho\sigma}$ will then be in the top left corner of the matrix, in place of a_{11}; so in the determinant of the rearranged matrix, the terms containing $a_{\rho\sigma}$ sum, as above, to give $a_{\rho\sigma}M_{\rho\sigma}$. But the determinant of the rearranged matrix is

$$(-1)^{\rho+\sigma-2}|A| = (-1)^{\rho+\sigma}|A|;$$

therefore the terms in $|A|$ containing $a_{\rho\sigma}$ sum to $(-1)^{\rho+\sigma}a_{\rho\sigma}M_{\rho\sigma}$.

Now consider the expression (12.22) for $|A|$,

$$\sum_{P(ij\ldots k)} \pm a_{1i}a_{2j}\cdots a_{nk}.$$

Every term in this sum contains one and only one element from the ρth row; we have just shown that those containing $a_{\rho 1}$ sum to $(-1)^{\rho+1}a_{\rho 1}M_{\rho 1}$, those containing $a_{\rho 2}$ sum to $(-1)^{\rho+2}a_{\rho 2}M_{\rho 2}$, and so on. Therefore (12.22) can be written as

$$|A| = (-1)^{\rho+1}a_{\rho 1}M_{\rho 1} + (-1)^{\rho+2}a_{\rho 2}M_{\rho 2} + \ldots + (-1)^{\rho+n}a_{\rho n}M_{\rho n}$$

or

$$|A| = \sum_{\sigma=1}^{n} (-1)^{\rho+\sigma}a_{\rho\sigma}M_{\rho\sigma}. \tag{12.24}$$

This formula is known as 'the expansion of $|A|$ by the ρth row'; in (12.24) we have written the summation over σ explicitly, since we sum over one repeated suffix but not the other. The analogous 'expansion by the σth column' gives

$$|A| = \sum_{\rho=1}^{n} (-1)^{\rho+\sigma}a_{\rho\sigma}M_{\rho\sigma}, \tag{12.25}$$

with no summation over σ.

Example 16

The determinant of the (3×3) matrix A defined by (12.23) can be expanded by the second row, using (12.24) with $\rho=2$:

$$\begin{aligned}|A| &= (-1)^{2+1}a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33}\end{vmatrix} + (-1)^{2+2}a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix} + (-1)^{2+3}a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix} \\ &= -a_{21}(a_{12}a_{33}-a_{13}a_{32}) + a_{22}(a_{11}a_{33}-a_{13}a_{31}) - a_{23}(a_{11}a_{32}-a_{12}a_{31}).\end{aligned}$$

This is clearly the same as (12.23).

§ 4.4. Evaluation of determinants

If all the elements in the ρth row of A contain a common factor K, so that $a_{\rho\sigma}=K\alpha_{\rho\sigma}$ for all σ, with ρ fixed, then from (12.24),

$$|A| = K \sum_{\sigma=1}^{n} (-1)^{\rho+\sigma}\alpha_{\rho\sigma}M_{\rho\sigma}.$$

This is by (12.22) just K times the determinant with the row

$$(a_{\rho 1}, a_{\rho 2},\dots a_{\rho n}) \quad \text{replaced by} \quad (\alpha_{\rho 1}, \alpha_{\rho 2},\dots \alpha_{\rho n}).$$

So *when all elements of a row have a common factor, we can take this common factor outside the determinant.* Likewise, using (12.25), we see that *a common factor of all elements of a column can be taken outside the determinant.*

If *all* elements of an $(n\times n)$ matrix A have a common factor λ, so that $A=\lambda C$, then λ is a common factor of the elements in each of the n rows. Hence we can take n factors λ outside the determinant $|A|$, at the same time dividing every element of A by λ. Thus

$$|A| = \lambda^n |C|.$$

If is important to note that $A=\lambda C$ does *not* imply

$$|A| = \lambda |C|.$$

We also note here that the symmetry between the rows and columns in the definition of a determinant implies that $|A|=|\bar{A}|$, where $\bar{A}$ is the transposed of A. It follows at once that the determinant of an anti-symmetric matrix of odd order is zero.

Example 17

$$\begin{vmatrix} 4 & -7 & -4 \\ 6 & 42 & 9 \\ 4 & 7 & 2 \end{vmatrix}$$

(taking factor 3 out of second row)

$$= 3\begin{vmatrix} 4 & -7 & -4 \\ 2 & 14 & 3 \\ 4 & 7 & 2 \end{vmatrix}$$

(taking factor 2 from first column, factor 7 from second)

$$= 3\cdot 2\cdot 7\begin{vmatrix} 2 & -1 & -4 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

(expanding by first column)

$$= 42\left[2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & -4 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & -4 \\ 2 & 3 \end{vmatrix}\right]$$

$$= 42(2\cdot 1 - 2 + 2\cdot 5) = 420.$$

In practice, it does not help greatly to use the expansions (12.24) and (12.25) directly, since (2×2) and (3×3) determinants are easily evaluated directly, while higher order determinants are simplified first by a technique which we shall now develop. We first note two simple facts about determinants:

(i) If two rows (or, equally, two columns) in a matrix A are identical, then $|A|=0$. For we know that the interchange of two rows (or columns) changes the sign of $|A|$; so if A is thereby unaltered, we have $|A|=-|A|$ or $|A|=0$. Now we have shown above that a common factor of every element in a row (or column) of A can be taken outside the determinant; so *if two rows (or columns) have their corresponding elements in some fixed ratio, then* $|A|=0$.

(ii) Suppose that A is the $(n\times n)$ matrix $(a_{\rho\sigma})$ and A' is a matrix differing from A only in the ρth row, which has elements $a'_{\rho\sigma}$ $(\sigma=1, 2,\ldots n)$, then $|A|+|A'|=|A''|$, where A'' has elements $a''_{\rho\sigma}=a_{\rho\sigma}+a'_{\rho\sigma}$ in the ρth row and is otherwise identical with A. This follows at once if we expand $|A|$ and $|A'|$ by the ρth row, using (12.24), since the minors $M_{\rho\sigma}$ $(\sigma=1, 2,\ldots n)$ are the same in the two determinants. This rule applied equally to matrices with one column different.

Suppose that we choose $a'_{\rho\sigma}$ in (ii) to be $\alpha a_{\kappa\sigma}$, where α is a constant and $a_{\kappa\sigma}$ $(\sigma=1, 2,\ldots n)$ are the elements of the κth row $(\kappa\neq\rho)$. Then the κth and ρth rows of A' (elements $\alpha a_{\rho\sigma}$ and $a_{\rho\sigma}$ respectively) have their elements in the fixed ratio α, so that $|A'|=0$ by (i). Thus by (ii), $|A|=|A''|$. Now A'' is formed from A by adding to the ρth row a fixed multiple α of the κth row, where $\kappa\neq\rho$; so we conclude that $|A|$ *is unchanged if we add to any row of* A *a constant multiple of any other row; likewise adding to one column of* A *a constant multiple of another column leaves* $|A|$ *unchanged.* In the following examples we use this rule in conjunction with the expansions (12.24) and (12.25) to evaluate the (3×3) determinant of Example 17, and a (4×4) determinant.

Example 18

$$\begin{vmatrix} 4 & -7 & -4 \\ 6 & 42 & 9 \\ 4 & 7 & 2 \end{vmatrix}$$

(as in Example 17)

$$= 42\begin{vmatrix} 2 & -1 & -4 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

(adding twice the first row to the second)

$$= 42\begin{vmatrix} 2 & -1 & -4 \\ 5 & 0 & -5 \\ 2 & 1 & 2 \end{vmatrix}$$

(adding first row to third, taking factor 5 from second)

$$= 42\cdot 5\begin{vmatrix} 2 & -1 & -4 \\ 1 & 0 & -1 \\ 4 & 0 & -2 \end{vmatrix}$$

(expanding by second column)

$$= -210\cdot(-1)\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} = 210\cdot 2 = 420.$$

The essential simplification in Example 18 is the addition of multiples of one row (the first) to the other rows in order to introduce zeros in a particular column. This process is called *pivotal condensation,* and in effect reduces the order of a determinant by one. In the first stage of the next example, we shall add multiples of the second row to the other rows to introduce zeros, these multiples being indicated in brackets at the side of the determinant; the three additions can be performed simultaneously.

Example 19

$$\begin{vmatrix} 3 & 7 & 1 & 2 \\ 6 & 11 & -1 & 1 \\ 8 & 18 & 2 & 3 \\ -2 & -7 & 4 & -2 \end{vmatrix} \begin{array}{l} \text{(add second row)} \\ \\ \text{(add } 2\times \text{ second row)} \\ \text{(add } 4\times \text{ second row)} \end{array}$$

(pivotal condensation)

$$= \begin{vmatrix} 9 & 18 & 0 & 3 \\ 6 & 11 & -1 & 1 \\ 20 & 40 & 0 & 5 \\ 22 & 37 & 0 & 2 \end{vmatrix}$$

(expand by third column)

$$= -(-1)\begin{vmatrix} 9 & 18 & 3 \\ 20 & 40 & 5 \\ 22 & 37 & 2 \end{vmatrix} + \text{zero terms}$$

(take factors from first and second rows)

$$= 15\begin{vmatrix} 3 & 6 & 1 \\ 4 & 8 & 1 \\ 22 & 37 & 2 \end{vmatrix} \quad \begin{matrix} \\ \text{(add } -1 \text{ times first row)} \\ \text{(add } -2 \text{ times first row)} \end{matrix}$$

(pivotal condensation)

$$= 15\begin{vmatrix} 3 & 6 & 1 \\ 1 & 2 & 0 \\ 16 & 25 & 0 \end{vmatrix}$$

(expand by third column)

$$= 15\begin{vmatrix} 1 & 2 \\ 16 & 25 \end{vmatrix} = 15(25 - 32) = -105.$$

Certain algebraic determinants can be evaluated most simply by using the remainder theorem, together with the property that a determinant vanishes when two rows or two columns are proportional.

Example 20

Evaluate

$$\Delta = \begin{vmatrix} 1 & \alpha & \alpha^3 \\ 1 & \beta & \beta^3 \\ 1 & \gamma & \gamma^3 \end{vmatrix}.$$

When $\alpha = \beta$, the first two rows are identical, so the determinant Δ vanishes. Hence Δ contains a factor $\alpha - \beta$, by the remainder theorem. Likewise Δ contains factors $\beta - \gamma$ and $\gamma - \alpha$. Now on expansion, it is clear that every term in Δ is of fourth order in α, β, γ. So the symmetry between α, β, γ ensures that Δ must be a multiple of

$$(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma).$$

Since the coefficients of $\beta\gamma^3$ in Δ and in this expression are equal, this product must equal Δ. Applying this result when, for example, $\alpha = 1$, $\beta = 2$, $\gamma = 3$, we find

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 8 \\ 1 & 3 & 27 \end{vmatrix} = (1 - 2)(2 - 3)(3 - 1)(1 + 2 + 3) = 12.$$

§ 4.5. Determinant of a matrix product

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $(n \times n)$ square matrices, then the determinant of the product $\boldsymbol{AB}$ is given by

$$|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|.$$

We shall establish this for two (3×3) matrices $\boldsymbol{A} = (a_{\rho\sigma})$ and $\boldsymbol{B} = (b_{\rho\sigma})$;

it will be clear that a similar proof applies to matrices of any order.

Consider the six-rowed determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ -1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & -1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & -1 & b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

It is clear that the only non-zero terms in the expansion (12.22) of Δ arise when the factors from the first three rows are of the form $a_{\rho\sigma}$, and the factors from the last three rows are of the form $b_{\rho\sigma}$. Thus we must include all terms which contain three $a_{\rho\sigma}$ and three $b_{\rho\sigma}$; these terms are just those occurring in the product

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$

so that $\Delta = |\boldsymbol{A}||\boldsymbol{B}|$.

Now let us alter the fourth, fifth and sixth columns of Δ by adding certain multiples of the first three columns to them. In detail, we replace the columns by

$$\begin{aligned} &\text{column } 4 + b_{11}(\text{column } 1) + b_{21}(\text{column } 2) + b_{31}(\text{column } 3) \\ &\text{column } 5 + b_{12}(\text{column } 1) + b_{22}(\text{column } 2) + b_{32}(\text{column } 3) \\ &\text{column } 6 + b_{13}(\text{column } 1) + b_{23}(\text{column } 2) + b_{33}(\text{column } 3). \end{aligned}$$

This process leaves Δ unchanged, so that

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{1\sigma}b_{\sigma 1} & a_{1\sigma}b_{\sigma 2} & a_{1\sigma}b_{\sigma 3} \\ a_{21} & a_{22} & a_{23} & a_{2\sigma}b_{\sigma 1} & a_{2\sigma}b_{\sigma 2} & a_{2\sigma}b_{\sigma 3} \\ a_{31} & a_{32} & a_{33} & a_{3\sigma}b_{\sigma 1} & a_{3\sigma}b_{\sigma 2} & a_{3\sigma}b_{\sigma 3} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}$$

where we have used summation convention with σ taking values 1, 2, 3. In the expansion (12.22) of this determinant, it is clear that the only non-zero terms arise by choosing the three elements (-1) from the last three rows, so that the other three factors must each be of the form $a_{\rho\sigma}b_{\sigma\tau}$.

In fact, the surviving terms sum to give

$$(-1)^{1+4}(-1)^{2+5}(-1)^{3+6}(-1)^{3}\begin{vmatrix} a_{1\sigma}b_{\sigma 1} & a_{1\sigma}b_{\sigma 2} & a_{1\sigma}b_{\sigma 3} \\ a_{2\sigma}b_{\sigma 1} & a_{2\sigma}b_{\sigma 2} & a_{2\sigma}b_{\sigma 3} \\ a_{3\sigma}b_{\sigma 1} & a_{3\sigma}b_{\sigma 2} & a_{3\sigma}b_{\sigma 3} \end{vmatrix},$$

which is just $|\boldsymbol{AB}|$.

The theorem can obviously be extended inductively to a product of any number of matrices $\boldsymbol{A}, \boldsymbol{B}, \ldots \boldsymbol{G}, \boldsymbol{H}$. Thus

$$|\boldsymbol{AB}\cdots\boldsymbol{GH}| = |\boldsymbol{A}||\boldsymbol{B}|\cdots|\boldsymbol{G}||\boldsymbol{H}|. \tag{12.26}$$

EXERCISE 12.4

1. Evaluate the determinants

$$\begin{vmatrix} 2 & -3 & 5 \\ 4 & 1 & -2 \\ 7 & 12 & 6 \end{vmatrix} \quad\text{and}\quad \begin{vmatrix} 1+i & 4+3i & 6-i \\ i & -2-i & 2-3i \\ 3-2i & 4 & 3+3i \end{vmatrix}$$

(i) directly, and (ii) using pivotal condensation.

2. Evaluate the determinants

$$\begin{vmatrix} 1 & -2 & 4 & 2 \\ 3 & 0 & 5 & -2 \\ 2 & 7 & 3 & -1 \\ -4 & -3 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 7 & 13 & 10 & 6 \\ -5 & -9 & -7 & -4 \\ 8 & 12 & 11 & 7 \\ -4 & -10 & -6 & -3 \end{vmatrix}.$$

3. Evaluate the determinants

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & \gamma & \beta \\ \beta & \gamma & 0 & \alpha \\ \gamma & \beta & \alpha & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}.$$

4. Evaluate the determinants

$$\begin{vmatrix} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \\ 1 & 1 & 1 & \alpha \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha\beta & \gamma^2 \\ 1 & \beta\gamma & \alpha^2 \\ 1 & \gamma\alpha & \beta^2 \end{vmatrix},$$

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & 1 & 1 \\ -\beta & -1 & 0 & 1 \\ -\gamma & -1 & -1 & 0 \end{vmatrix}, \quad \begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}.$$

5. By considering the product of the matrices

$$\begin{pmatrix} \alpha^2 & -2\alpha & 1 & 0 \\ \beta^2 & -2\beta & 1 & 0 \\ \gamma^2 & -2\gamma & 1 & 0 \\ \delta^2 & -2\delta & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

show that

$$\begin{vmatrix} 0 & (\alpha-\beta)^2 & (\alpha-\gamma)^2 & (\alpha-\delta)^2 \\ (\beta-\alpha)^2 & 0 & (\beta-\gamma)^2 & (\beta-\delta)^2 \\ (\gamma-\alpha)^2 & (\gamma-\beta)^2 & 0 & (\gamma-\delta)^2 \\ (\delta-\alpha)^2 & (\delta-\beta)^2 & (\delta-\gamma)^2 & 0 \end{vmatrix} = 0.$$

Show that

$$\begin{vmatrix} 0 & (\alpha-\beta)^2 & (\alpha-\gamma)^2 \\ (\beta-\alpha)^2 & 0 & (\beta-\gamma)^2 \\ (\gamma-\alpha)^2 & (\gamma-\beta)^2 & 0 \end{vmatrix} = 2[(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)]^2.$$

§ 5. Linear dependence

§ 5.1. Vectors in 3-space

We shall introduce the concept of *linear dependence* by considering vectors in 3-space. Suppose that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are two non-parallel and non-zero vectors in 3-space, as shown in fig. 12.1. Then *in general* a third vector $\boldsymbol{a}_3$ will not lie in the plane of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and so will *not* be expressible as a linear combination

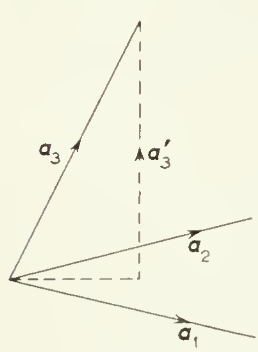

Fig. 12.1

$$\boldsymbol{a}_3 = \lambda\boldsymbol{a}_1 + \mu\boldsymbol{a}_2$$

of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. In other words, it will have a non-zero component $\boldsymbol{a}_3'$ in the direction perpendicular to this plane. But when $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ do lie in the same plane, then by Theorem 2 of Ch. 9 § 3.2, $\boldsymbol{a}_3$ is of the form $(\lambda\boldsymbol{a}_1+\mu\boldsymbol{a}_2)$; we then say that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ are *linearly dependent.* The condition for linear dependence is more symmetrically written using summation convention as

$$c_l\boldsymbol{a}_l = \boldsymbol{0}; \tag{12.27}$$

here it is assumed that at least one of the numbers c_l is non-zero, since (12.27) is otherwise trivially true. If $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ do not obey a relation of the form (12.27), they are said to be *linearly independent.* These three vectors then fill out or 'span' the 3-space, and by Theorem 3 of Ch. 9

§ 3.4, any vector in the space can be expressed as a combination of them. If a relation of type (12.27) holds, then the triple scalar product

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = 0 \tag{12.28}$$

meaning that $\boldsymbol{a}_3$ is perpendicular to the vector $\boldsymbol{a}_1 \times \boldsymbol{a}_2$. If the components of the vectors relative to given rectangular axes are

$$\begin{aligned} \boldsymbol{a}_1 &= (a_{11}, a_{21}, a_{31}), \\ \boldsymbol{a}_2 &= (a_{12}, a_{22}, a_{32}), \\ \boldsymbol{a}_3 &= (a_{13}, a_{23}, a_{33}), \end{aligned}$$

then the condition (12.28) can be written in the determinantal form established in § 4.2:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \tag{12.29}$$

The notation we have used suggests that $(a_{\rho\sigma})$ could be looked upon as the elements of a matrix $\boldsymbol{A}$, the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ being the columns of the matrix. Then the condition (12.29) is simply

$$|\boldsymbol{A}| = 0, \tag{12.30}$$

and is the condition that the columns of $\boldsymbol{A}$, regarded as vectors, are linearly dependent and obey a relation of the form (12.27). In just the same way, (12.29) is also the condition for the rows of $\boldsymbol{A}$, regarded as vectors, to be linearly dependent.

§ 5.2. Vectors in n-space

In Ch. 9 § 8, we introduced the concept of an n-component vector in n-space. The idea of linear dependence of vectors in n-space is a generalisation of linear dependence in 3-space. Consider for example the n-component unit vectors

$$\begin{aligned} \boldsymbol{u}_1 &= (1, 0, 0, \ldots 0, 0), \\ \boldsymbol{u}_2 &= (0, 1, 0, \ldots 0, 0), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{u}_n &= (0, 0, 0, \ldots 0, 1). \end{aligned}$$

If we try to find n constants satisfying $c_\sigma \boldsymbol{u}_\sigma = \boldsymbol{0}$, we see at once that $c_1 = c_2 = \ldots = c_n = 0$; in fact, there is no non-trivial relation of this type.

We therefore say that $\boldsymbol{u}_1, \boldsymbol{u}_2,\dots \boldsymbol{u}_n$ are linearly independent. If however, we consider the $n-1$ vectors $\boldsymbol{u}_1, \boldsymbol{u}_2,\dots \boldsymbol{u}_{n-1}$ and an nth vector of the form $\boldsymbol{v}=(v_1, v_2,\dots v_{n-1}, 0)$, then we see at once that

$$v_1\boldsymbol{u}_1 + v_2\boldsymbol{u}_2 + \dots + v_{n-1}\boldsymbol{u}_{n-1} - \boldsymbol{v} = 0,$$

so these n vectors are said to be linearly dependent. We notice that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2,\dots \boldsymbol{u}_{n-1}$ and $\boldsymbol{v}$ all have a zero nth component. This means that they all lie in an $(n-1)$-dimensional 'subspace', just as three vectors satisfying (12.27) lie in a plane.

Generally if $\boldsymbol{a}_1, \boldsymbol{a}_2,\dots \boldsymbol{a}_n$ are n vectors in n-space, then we say that they are linearly dependent if numbers c_σ (not all zero) exist, such that

$$c_\sigma \boldsymbol{a}_\sigma = \boldsymbol{0}. \tag{12.31}$$

If we write the vectors as

$$\boldsymbol{a}_\sigma = (a_{1\sigma}, a_{2\sigma},\dots a_{n\sigma}) \tag{12.32}$$

for $\sigma=1, 2,\dots n$, then the condition (12.31) consists of the n linear equations

$$a_{\rho\sigma}c_\sigma = 0 \qquad (\rho = 1, 2,\dots n). \tag{12.33}$$

If (12.31) or (12.33) cannot be satisfied unless every $c_\sigma=0$, the vectors $\boldsymbol{a}_\sigma$ are linearly independent. If we regard $a_{\rho\sigma}$ as the (ρ, σ) element of a matrix $\boldsymbol{A}$, then $\boldsymbol{a}_\sigma$ is the σth column of $\boldsymbol{A}$. The condition (12.31) is then the condition that the columns of $\boldsymbol{A}$ are linearly dependent.

Two results established for vectors in 3-space can be generalised to n-space:

(i) The columns $\boldsymbol{a}_\sigma$ of a matrix $\boldsymbol{A}$ are linearly dependent if and only if $|\boldsymbol{A}|=0$. This is a generalisation of the condition (12.30) for (3×3) matrices.

(ii) If $\boldsymbol{a}_1, \boldsymbol{a}_2,\dots \boldsymbol{a}_n$ are linearly independent, then any vector in n-space can be expressed as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2,\dots \boldsymbol{a}_n$. This is a generalisation to n dimensions of Theorems 2 and 3, Ch. 9 § 3.2 and § 3.4.

The proof of (ii) will be given in Ch. 13 § 2.2, since it is closely related to the solution of non-homogeneous linear equations. The result (i) will be established after we have given an example to demonstrate its meaning.

Example 21

The columns of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 4 & 0 \\ -2 & 3 & -1 & 2 \\ -4 & -1 & 5 & 2 \\ 1 & 4 & 0 & 2 \end{pmatrix}$$

are linearly dependent, since

$$2(2, -2, -4, 1) - 4(-1, 3, -1, 4) - 2(4, -1, 5, 0) + 7(0, 2, 2, 2) = (0, 0, 0, 0).$$

The determinant of $\boldsymbol{A}$ is

$$\begin{vmatrix} 2 & -1 & 4 & 0 \\ -2 & 3 & -1 & 2 \\ -4 & -1 & 5 & 2 \\ 1 & 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 4 & 0 \\ -3 & -1 & -1 & 0 \\ -5 & -5 & 5 & 0 \\ 1 & 4 & 0 & 2 \end{vmatrix}$$

$$= 2\cdot 5 \begin{vmatrix} 2 & -1 & 4 \\ -3 & -1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 10 \begin{vmatrix} 3 & 0 & 3 \\ -2 & 0 & -2 \\ -1 & -1 & 1 \end{vmatrix} = 0.$$

§ 5.3. Linear dependence of rows and columns of a matrix

In establishing the result (i) above, it is quite easy to show that $|\boldsymbol{A}|=0$ follows from (12.31) or (12.33). At least one of the constants c_σ is non-zero, so let us suppose that $c_1 \neq 0$. Now to the first column of $\boldsymbol{A}$ let us add c_2/c_1 times the second column, c_3/c_1 times the third column, and so on; this does not change the value of $|\boldsymbol{A}|$. The first column $\boldsymbol{a}_1$ is then replaced by

$$\boldsymbol{a}_1 + \frac{1}{c_1}(c_2\boldsymbol{a}_2 + c_3\boldsymbol{a}_3 + \ldots + c_n\boldsymbol{a}_n) = \frac{1}{c_1}(c_\sigma \boldsymbol{a}_\sigma),$$

which is zero by (12.31). Since every term in the sum (12.22) contains an element from the first column, and is therefore zero, it follows that $|\boldsymbol{A}|=0$.

The converse, that $|\boldsymbol{A}|=0$ implies linear dependence of the columns, is best proved by induction on n, the order of $\boldsymbol{A}$. The statement is obviously true for (1×1) matrices. Assume that it is true for matrices with $n-1$ rows and columns. Now one element of an $(n \times n)$ matrix $\boldsymbol{A}$ is non-zero, or the theorem is trivial. Assume for example that $a_{11} \neq 0$; we can then introduce $n-1$ zeros into the first row of $\boldsymbol{A}$ by the technique developed in § 4.4. Writing the columns of $\boldsymbol{A}$ in vector form, using (12.32), we have

$$\begin{aligned} |\boldsymbol{A}| &= |\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \ \ldots \ \boldsymbol{a}_\sigma \ \ldots \ \boldsymbol{a}_n| \\ &= |\boldsymbol{a}_1 \quad \boldsymbol{a}_2 - (a_{12}\boldsymbol{a}_1/a_{11}) \ \ldots \ \boldsymbol{a}_\sigma - (a_{1\sigma}\boldsymbol{a}_1/a_{11}) \ \ldots \ \boldsymbol{a}_n - (a_{1n}\boldsymbol{a}_1/a_{11})|. \end{aligned}$$

The σth component in the first row is now

$$a_{1\sigma} - (a_{1\sigma}a_{11}/a_{11}) = 0$$

for $\sigma \neq 1$. So we can write

$$|A| = \begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a'_{22} & a'_{23} & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a'_{n2} & a'_{n3} & \dots & a'_{nn} \end{vmatrix}, \text{ say.}$$

Expanding by the first column, $|A| = a_{11}M'_{11}$, where

$$M'_{11} = \begin{vmatrix} a'_{22} & a'_{23} & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots \\ a'_{n2} & a'_{n3} & \dots & a'_{nn} \end{vmatrix}.$$

Since $a_{11} \neq 0$, $|A| = 0$ implies that $M'_{11} = 0$. Now M'_{11} is of order $n-1$, so by the induction hypothesis the columns in the determinant M'_{11} are linearly dependent; therefore constants $c_2, \dots c_n$ exist such that

$$\sum_{\sigma=2}^{n} c_\sigma [\mathbf{a}_\sigma - (a_{1\sigma}\mathbf{a}_1/a_{11})] = 0.$$

If we write $c_1 = -\sum_{\sigma=2}^{n} c_\sigma a_{1\sigma}/a_{11}$, this is just the condition (12.31) for the linear dependence of the rows of A. Hence the inductive proof is complete.

This proof applies equally well to the rows of A, since $|A|$ is symmetrical between rows and columns; so we can say that the following three statements are equivalent:

(i) $|A| = 0$,

(ii) the columns of A are linearly dependent,

(iii) the rows of A are linearly dependent.

A square matrix A which satisfies these conditions is known as a *singular matrix*; other square matrices are *non-singular*.

Example 22

We have seen that the matrix A of Example 21 has zero determinant and linearly dependent columns. The rows are also linearly dependent since

$$2(2, -1, 4, 0) + 3(-2, 3, -1, 2) - (-4, -1, 5, 2) - 2(1, 4, 0, 2) = (0, 0, 0, 0).$$

LINEAR EQUATIONS. EIGENVECTORS AND EIGENVALUES

§ 1. The inverse of a square matrix

Any complex number N, except zero, has an inverse N^{-1} which satisfies $N\,N^{-1}=1$, and the existence of an inverse is fundamental to division. It is natural to ask whether 'matrix division' can be defined; that is, if $\boldsymbol{A}$ is a given matrix, can we define an inverse matrix $\boldsymbol{A}^{-1}$ such that

$$\text{either} \quad \boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I} \tag{13.1}$$

$$\text{or} \quad \boldsymbol{A}^{-1}\,\boldsymbol{A} = \boldsymbol{I}? \tag{13.2}$$

Since the order of matrices in a matrix product is important, we must consider the possibility of two different inverses existing, a *right inverse* obeying (13.1) and *left inverse* obeying (13.2). In practice we normally only wish to define the inverse of a square matrix, and no obvious unique inverse exists for other matrices. In this section, therefore, we shall tacitly assume that all matrices are square; for these a unique inverse $\boldsymbol{A}^{-1}$ generally exists and satisfies both (13.1) and (13.2); in other words, it is both a left and a right inverse.

§ 1.1. Formula for the inverse

Let us consider the sums

$$\sum_{\sigma=1}^{n} (-1)^{\rho+\sigma}\, a_{\kappa\sigma}\, M_{\rho\sigma}$$

for various values of κ and ρ, $M_{\rho\sigma}$ being the minor of the element $a_{\rho\sigma}$ of the matrix $\boldsymbol{A}$. When $\kappa=\rho$, then by (12.24) this sum is just the expansion of $|\boldsymbol{A}|$ by the ρth row. When $\kappa\neq\rho$, the sum is just the analogous expansion of $|\boldsymbol{A}'|$, $\boldsymbol{A}'$ being the matrix formed by replacing the ρth row $a_{\rho1}, a_{\rho2}, \ldots, a_{\rho n}$ by $a_{\kappa1}, a_{\kappa2}, \ldots, a_{\kappa n}$. Thus the κth and ρth rows of $\boldsymbol{A}'$ are identical, and

$|A'|=0$. Therefore we can write for all values of ρ, κ in 1, 2,..., n,

$$\sum_{\sigma=1}^{n} (-1)^{\rho+\sigma} a_{\kappa\sigma} M_{\rho\sigma} = |\boldsymbol{A}| \delta_{\kappa\rho}, \tag{13.3}$$

where $\delta_{\kappa\rho}$ is the Kronecker delta, defined by (12.17). Now unless $\boldsymbol{A}$ is singular, so that $|\boldsymbol{A}|=0$, we can divide both sides of (13.3) by $|\boldsymbol{A}|$. If we define n^2 quantities

$$b_{\sigma\rho} = |\boldsymbol{A}|^{-1}(-1)^{\rho+\sigma} M_{\rho\sigma}, \qquad (\rho, \sigma = 1, 2,\ldots, n) \tag{13.4}$$

then (13.3) can be written, using summation convention, as

$$a_{\kappa\sigma} b_{\sigma\rho} = \delta_{\kappa\rho} \qquad (\kappa, \rho = 1, 2,\ldots, n). \tag{13.5}$$

Note particularly that the suffixes on $b_{\sigma\rho}$ in (13.4) are in different order to those on $M_{\rho\sigma}$. This interchange has been made so that the left-hand member of (13.5) has the form of the (κ, ρ) element of the matrix product $\boldsymbol{AB}$, where the elements $b_{\sigma\rho}$ of $\boldsymbol{B}$ are defined by (13.4). We can therefore write (13.5) as the matrix equation

$$\boldsymbol{AB} = \boldsymbol{I}; \tag{13.6}$$

comparing with (13.1), we see that $\boldsymbol{B}$ is a right inverse of $\boldsymbol{A}$.

In a similar way, using (12.25) and expansions formed by replacing the *column* $a_{1\sigma}, a_{2\sigma},\ldots, a_{n\sigma}$ by $a_{1\kappa}, a_{2\kappa},\ldots, a_{n\kappa}$, we can establish the equations

$$b_{\sigma\rho} a_{\rho\kappa} = \delta_{\sigma\kappa} \qquad (\sigma, \kappa = 1, 2,\ldots, n) \tag{13.7}$$

which in matrix form is

$$\boldsymbol{BA} = \boldsymbol{I}, \tag{13.8}$$

showing that $\boldsymbol{B}$ is a left inverse of $\boldsymbol{A}$. So $\boldsymbol{A}^{-1} \equiv \boldsymbol{B}$ satisfies both conditions (13.1) and (13.2), and can be called simply 'the inverse of $\boldsymbol{A}$'.

§ 1.2. Existence and uniqueness

Two questions remain. First, does $\boldsymbol{A}$ have an inverse when $|\boldsymbol{A}|=0$? Second, is the inverse defined by (13.4) unique, or are there other inverses? We shall now show that

(i) if $\boldsymbol{A}$ has a finite inverse, then $|\boldsymbol{A}| \neq 0$,

(ii) $\boldsymbol{A}^{-1}$ is unique when $|\boldsymbol{A}| \neq 0$.

To prove (i), assume for example that there is a right inverse $\boldsymbol{B}$ satisfying (13.6). Taking the determinant of each side of the equation and

using the result (12.26) for the determinant of $\boldsymbol{AB}$, we have

$$|\boldsymbol{A}|\,|\boldsymbol{B}| = |\boldsymbol{AB}| = |\boldsymbol{I}| = 1.$$

This equation cannot be true if $|\boldsymbol{A}| = 0$; hence the existence of an inverse of $\boldsymbol{A}$ implies that $|\boldsymbol{A}| \neq 0$.

To establish (ii), suppose for example that there is a different left inverse $\boldsymbol{B}'$, so that $\boldsymbol{BA}=\boldsymbol{B}'\boldsymbol{A}=\boldsymbol{I}$. Then $(\boldsymbol{B}-\boldsymbol{B}')\boldsymbol{A}=\boldsymbol{0}$, where $\boldsymbol{0}$ is the zero matrix with every element zero. In terms of the elements,

$$(b_{\kappa\rho} - b'_{\kappa\rho})a_{\rho\sigma} = 0, \tag{13.9}$$

for all κ and σ. Now $\boldsymbol{B}\neq\boldsymbol{B}'$, so some elements of $\boldsymbol{B}-\boldsymbol{B}'$ are non-zero; if the μth row of $\boldsymbol{B}-\boldsymbol{B}'$ contains a non-zero element, put $b_{\mu1}-b'_{\mu1}=c_1$, $b_{\mu2}-b'_{\mu2}=c_2$, and so on. Then the equations (13.9) which have $\kappa=\mu$ can be written as

$$c_\rho a_{\rho\sigma} = 0 \qquad (\sigma = 1, 2,\ldots, n).$$

Since some of the c_ρ are non-zero, this is just the condition (12.23) for $\boldsymbol{A}$ to be singular, contrary to the hypothesis that $|\boldsymbol{A}|\neq 0$. Hence $\boldsymbol{B}'=\boldsymbol{B}$, establishing the uniqueness of the inverse.

Summarising, we have shown that *singular matrices have no inverse, while non-singular matrices have a unique inverse* $\boldsymbol{A}^{-1}=\boldsymbol{B}$, *satisfying both* (13.1) *and* (13.2), *with elements defined by* (13.4).

Example 1

Find the inverse of

$$\boldsymbol{A} = \begin{pmatrix} -8 & 9 & -3 \\ -4 & 5 & 1 \\ 2 & -2 & 3 \end{pmatrix}.$$

First, $|\boldsymbol{A}| = -4 \neq 0$, so $\boldsymbol{A}^{-1}$ exists. Now

$$M_{11} = +\begin{vmatrix} 5 & 1 \\ -2 & 3 \end{vmatrix} = 17, \text{ so by (13.4), } b_{11} = -\tfrac{17}{4}.$$

Remembering the interchange of suffixes in (13.4),

$$M_{21} = +\begin{vmatrix} 9 & -3 \\ -2 & 3 \end{vmatrix} = 21, \text{ so that } b_{12} = +\tfrac{21}{4},$$

$$M_{31} = +\begin{vmatrix} 9 & -3 \\ 5 & 1 \end{vmatrix} = 24, \text{ so that } b_{13} = -\tfrac{24}{4};$$

continuing this process, we find all elements $b_{\rho\sigma}$, and

$$A^{-1} = B = -\frac{1}{4}\begin{pmatrix} 17 & -21 & 24 \\ 14 & -18 & 20 \\ -2 & 2 & -4 \end{pmatrix},$$

the factor $-\frac{1}{4}$ being taken out of every element. It can easily be checked that $AA^{-1}=A^{-1}A=I$.

We note that if A and B are two non-singular matrices and k a complex number, then

$$(kA)^{-1} = k^{-1}A^{-1} \tag{13.10}$$

and

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{13.11}$$

Clearly $B^{-1}A^{-1}AB=B^{-1}IB=B^{-1}B=I$; so $B^{-1}A^{-1}$ is an inverse, and hence the unique inverse of AB. The proof of (13.10) is even more trivial.

EXERCISE 13.1

1. Calculate the inverse of the matrix

$$\begin{pmatrix} \alpha - 1 & 2 \\ 3 & \alpha - 2 \end{pmatrix}$$

stating for what values of α no inverse exists. Sketch the graph of the (1, 1) element of the inverse as a function of α.

2. Calculate the inverses of the matrices

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}, \quad \begin{pmatrix} 6 & -5 & 2 \\ 1 & 4 & -3 \\ 2 & -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1-i & 1-2i \\ 1+i & 1 & 1-i \\ 1+2i & 1+i & 1 \end{pmatrix}.$$

Check that equations (13.1) and (13.2) are satisfied.

3. If A is a singular matrix and B is the matrix whose (ρ, σ) element is $b_{\rho\sigma}=(-1)^{\sigma+\rho}M_{\sigma\rho}$, where $M_{\sigma\rho}$ is the minor of $a_{\sigma\rho}$, show that $AB=BA=\mathbf{0}$.

4. Calculate the inverse of the $(n \times n)$ matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

5. If $\boldsymbol{A}$ is an $(n \times n)$ matrix and $\boldsymbol{B}$ is an $(n \times n)$ non-singular matrix commuting with $\boldsymbol{A}$, show that $\boldsymbol{B}^{-1}$ commutes with $\boldsymbol{A}$.

6. Show that the inverse of $\bar{\boldsymbol{A}}$ is the transposed of $\boldsymbol{A}^{-1}$.

§ 2. Linear equations

In this section we shall discuss the solution of systems of n linear equations for n unknowns $x_1, x_2, \ldots, x_n$. The systems are of the form (12.1) with $m=n$ and can be written, using summation convention, as

$$a_{\rho\sigma}x_\sigma = k_\rho \qquad (\rho = 1, 2, \ldots, n) \tag{13.12}$$

where $(a_{\rho\sigma})$ is an $(n \times n)$ square matrix. Systems of type (13.12) fall into two main classes: *non-homogeneous systems*, in which at least one of the constants k_ρ is non-zero, and *homogeneous systems* for which *every* $k_\rho=0$. When we use system (13.12) we shall assume that it is non-homogeneous; when we deal with homogeneous systems we shall specifically use the set of equations

$$a_{\rho\sigma}x_\sigma = 0 \qquad (\rho = 1, 2, \ldots, n). \tag{13.13}$$

§ 2.1. Solubility of homogeneous systems

One basic property of a homogeneous system (13.13) is that any solution $x_\sigma = X_\sigma$ $(\sigma = 1, 2, \ldots, n)$ gives rise to an infinity of solutions $x_\sigma = \gamma X_\sigma$ $(\sigma = 1, 2, \ldots, n)$, where γ is an arbitrary constant. In other words, a homogeneous system can only tell us the ratio of the unknowns x_σ, not their absolute value. There is one obvious solution of (13.13), namely $x_\sigma=0$ for all σ; this solution is usually of no practical significance and is referred to as the *trivial solution*. If we exclude this trivial solution and insist that some of the x_σ are non-zero, then comparing with (12.33), we see that the columns of $\boldsymbol{A}$ are linearly dependent. Hence $\boldsymbol{A}$ must be singular and $|\boldsymbol{A}|=0$. This condition for a non-trivial solution is a restriction on the coefficients $a_{\rho\sigma}$; in general $|\boldsymbol{A}| \neq 0$ and (13.13) has no non-trivial solution. When we come to solve (13.13) we shall therefore assume that $\boldsymbol{A}$ is singular.

§ 2.2. Solution of non-homogeneous systems

Let us now consider the system (13.12), which can be written in matrix form as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{k}, \tag{13.14}$$

where $\boldsymbol{x}$ and $\boldsymbol{k}$ are the column matrices (x_σ) and (k_ρ). Now provided $|\boldsymbol{A}|\neq 0$, a unique inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A}$ exists. If we multiply the matrix equation (13.14) on the left by $\boldsymbol{A}^{-1}$, the left-hand member becomes $\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x}=\boldsymbol{I}\boldsymbol{x}=\boldsymbol{x}$; so we obtain

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{k}. \tag{13.15}$$

The components x_ρ of $\boldsymbol{x}$ are given by (13.15), which is thus the solution of the system of equations (13.12).

The solution (13.15) is unique. For if we suppose that two solutions $\boldsymbol{x}=\boldsymbol{x}'$ and $\boldsymbol{x}=\boldsymbol{x}''$, of (13.14) exist, then $\boldsymbol{A}\boldsymbol{x}'=\boldsymbol{k}=\boldsymbol{A}\boldsymbol{x}''$ and hence $\boldsymbol{A}(\boldsymbol{x}'-\boldsymbol{x}'')=\boldsymbol{0}$. This relation, like (13.13), is a condition for linear dependence of the columns of $\boldsymbol{A}$, unless $\boldsymbol{x}'\equiv\boldsymbol{x}''$. Since $|\boldsymbol{A}|$ is supposed non-zero, the columns are linearly independent and $\boldsymbol{x}'\equiv\boldsymbol{x}''$, establishing the uniqueness of solution.

The problem of solving (13.12) or (13.14) when $\boldsymbol{A}$ is non-singular is therefore essentially reduced to the calculation of $\boldsymbol{A}^{-1}$.

Example 2

Solve

$$\begin{aligned} -8x_1 + 9x_2 - 3x_3 &= 4 \\ -4x_1 + 5x_2 + x_3 &= 6 \\ 2x_1 - 2x_2 + 3x_3 &= -2. \end{aligned}$$

In the notation of (13.12) and (13.14),

$$\boldsymbol{A} = \begin{pmatrix} -8 & 9 & -3 \\ -4 & 5 & 1 \\ 2 & -2 & 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{k} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}.$$

The inverse $\boldsymbol{A}^{-1}$ was calculated in Example 1 of this chapter. Hence the solution (13.15) is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 17 & -21 & 24 \\ 14 & -18 & 20 \\ -2 & 2 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -106 \\ -92 \\ 12 \end{pmatrix}.$$

So $x_1=26\frac{1}{2}$, $x_2=23$, $x_3=-3$.

In solving a small system of linear equations, calculating $\boldsymbol{A}^{-1}$ and using (13.15) is generally not the simplest method. However, it is a great help in theoretical work to have a quite general, symmetrical and simply

expressed solution. In solving large systems of linear equations with the aid of electronic computers, one often aims at calculating the inverse $\boldsymbol{A}^{-1}$ of the matrix of coefficients, and then applying (13.15); there are, however, much better methods of calculating $\boldsymbol{A}^{-1}$ than by using (13.14) directly.

We can now establish property (ii) of Ch. 12 § 5.2, that an arbitrary vector $\boldsymbol{k}$ in n-space can be expressed as a linear combination $x_\sigma \boldsymbol{a}_\sigma$ of n linearly independent vectors $\boldsymbol{a}_\sigma$. For, regarding $\boldsymbol{k}$ as a column matrix and $\boldsymbol{a}_\sigma$ $(\sigma = 1, 2, \ldots n)$ as the columns of a matrix $\boldsymbol{A}$, given by (12.32), then $\boldsymbol{k} = x_\sigma \boldsymbol{a}_\sigma$ is simply $k_\rho = a_{\rho\sigma} x_\sigma$, the system (13.12). Since the columns of $\boldsymbol{A}$ are linearly independent, $\boldsymbol{A}$ is non-singular and the equations have a finite solution (13.15) for $x_1, x_2, \ldots, x_n$. Thus $\boldsymbol{k}$ is expressible in the form $x_\sigma \boldsymbol{a}_\sigma$.

§ 2.3. Failure of the solution

Let us now study equations (13.12) when $\boldsymbol{A}$ is singular. The rows of $\boldsymbol{A}$ are then linearly dependent; that is, constants c'_ρ exist such that

$$c'_\rho a_{\rho\sigma} = 0 \qquad (\sigma = 1, 2, \ldots, n). \tag{13.16}$$

Now at least one of the c'_ρ, say c'_n, is not zero. Suppose we assume that $x_1, x_2, \ldots, x_n$ satisfy the $n-1$ equations (13.12) with $\rho = 1, 2, \ldots, n-1$; then multiplying these equations by $-(c'_1/c'_n), -(c'_2/c'_n), \ldots, -(c'_{n-1}/c'_n)$ respectively and summing, we obtain (writing the summation over ρ explicitly)

$$\frac{1}{c'_n}\left[-\sum_{\rho=1}^{n-1} c'_\rho\, a_{\rho\sigma}\, x_\sigma\right] = -\frac{1}{c'_n}\sum_{\rho=1}^{n-1} c'_\rho k_\rho;$$

using the conditions (13.16) (where ρ is summed over $1, 2, \ldots, n$) this becomes

$$a_{n\sigma} x_\sigma = -\frac{1}{c'_n}\sum_{\rho=1}^{n-1} c'_\rho k_\rho. \tag{13.17}$$

Now compare this with the last of the equations (13.12),

$$a_{n\sigma} x_\sigma = k_n;$$

since the left hand members are identical, these equations are contradictory unless the right-hand members are equal, that is, unless

$$c'_\rho k_\rho = 0. \tag{13.18}$$

If (13.18) is satisfied, then the nth equation of the system is identical with (13.17), and is therefore deducible from the other $n-1$; it is there-

fore superfluous, and we really have only $n-1$ equations for n unknowns; thus one of the unknowns x_σ can be chosen arbitrarily. *So if* $\boldsymbol{A}$ *is singular, equations* (13.12) *are either inconsistent or insufficient to determine all the unknowns.*

§ 2.4. Solution of homogeneous systems

In solving the homogeneous system (13.13), we assume that $|\boldsymbol{A}|=0$; since these equations are derived from (13.12) by putting

$$k_1 = k_2 = \ldots = k_n = 0,$$

the consistency condition (13.18) is automatically satisfied. So equations (13.13) are not independent, and one of them is superfluous. We can therefore consider only $n-1$ of the equations, omitting for example the nth equation; if we divide through by one of the unknowns, say x_n, we obtain $n-1$ non-homogeneous equations for the $n-1$ ratios x_σ/x_n $(\sigma=1, 2, \ldots, n-1)$:

$$\sum_{\sigma=1}^{n-1} a_{\rho\sigma}\,(x_\sigma/x_n) = -a_{\rho n} \qquad (\rho = 1, 2, \ldots, n-1).$$

This is a system of equations of type (13.12), so that we can in general solve uniquely for the ratios of the unknowns. Since equations (13.13) cannot determine the absolute values of the variables x_ρ, this is the most complete solution that can be found.

A symmetrical form of the solution can be written down if we consider equations (13.3). Since $|\boldsymbol{A}|=0$, these give

$$\sum_{\sigma=1}^{n} (-1)^{\rho+\sigma}\, a_{\kappa\sigma}\, M_{\rho\sigma} = 0,$$

for *all* values of κ and ρ, including those with $\kappa=\rho$. If we consider the n equations of this set with a fixed value of ρ, so that $M_{\rho\sigma}$ are the minors of the elements in the ρth row of $\boldsymbol{A}$, we see that they are identical with (13.13) provided $x_\sigma=(-1)^{\rho+\sigma}M_{\rho\sigma}$ $(\sigma=1, 2, \ldots, n)$. Since only the ratios of the x_σ are determined, the general solution is therefore

$$\frac{x_1}{M_{\rho 1}} = \frac{-x_2}{M_{\rho 2}} = \frac{x_3}{M_{\rho 3}} = \ldots = \frac{(-1)^n x_n}{M_{\rho n}}\,. \tag{13.19}$$

In general, we can obtain the solution (13.19) by choosing any value of ρ in $1, 2, \ldots, n$. Since there is a unique solution for the ratios of the x_σ, the

various choices of ρ must give the same solution. This proves, incidentally, that when $|A|=0$, the minors of the elements in two different rows of A are proportional.

Example 3

For what values of α do the equations

$$\begin{aligned} x - \quad y - \quad z &= 0 \\ 2x - (\alpha+1)y - 5z &= 0 \\ -2\alpha x + \quad 3y + \quad z &= 0 \end{aligned}$$

have a non-trivial solution? Find the corresponding solutions.

A solution exists when the determinant of coefficients $|A|$ is zero; that is,

$$\begin{vmatrix} 1 & -1 & -1 \\ 2 & -(\alpha+1) & -5 \\ -2\alpha & 3 & 1 \end{vmatrix} = 0,$$

or $2\alpha^2-9\alpha+10=0$, giving $\alpha=2$ or $2\frac{1}{2}$. When $\alpha=2$,

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -3 & -5 \\ -4 & 3 & 1 \end{pmatrix}.$$

Using the minors of the third row ($\rho=3$) in (13.19) gives

$$\frac{x_1}{\begin{vmatrix} -1 & -1 \\ -3 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}},$$

or

$$\tfrac{1}{2}x_1 = \tfrac{1}{3}x_2 = -x_3.$$

Likewise, when $\alpha=2\frac{1}{2}$, using the minors of the second row,

$$\frac{x_1}{\begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ -5 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -5 & 3 \end{vmatrix}}$$

or $x_1=\frac{1}{2}x_2=-x_3$.

EXERCISE 13.2

1. Use matrix methods to solve the equations

$$\begin{aligned} 6x_1 - 5x_2 + 2x_3 &= 12 \\ x_1 + 4x_2 - 3x_3 &= 5 \\ 2x_1 - 2x_2 + x_3 &= 4. \end{aligned}$$

[Compare with Exercise 13.1, Question 2.]

2. For what values of α do the following sets of equations have solutions? Find these solutions.

(i)
$$\begin{aligned} 4x + y - z &= 0 \\ 2x + (\alpha - 2)y + 2z &= 0 \\ (\alpha - 3)x - 6y - 7z &= 0. \end{aligned}$$

(ii)
$$\begin{aligned} x - (\alpha + 1)y - \alpha z &= 0 \\ 2x + (\alpha - 3)y + (\alpha + 4)z &= 0 \\ x - 2\alpha y - (3\alpha - 2)z &= 0. \end{aligned}$$

3. Find by matrix methods the solutions for general values of α of the sets of equations

(i)
$$\begin{aligned} (\alpha + 2)x - y + z &= 10 \\ 2x + y + z &= 6 \\ x + 3y + \alpha z &= 3. \end{aligned}$$

(ii)
$$\begin{aligned} (\alpha + 2)x - y + z &= 10 \\ 2x + y + z &= 6 \\ x + 3y + \alpha z &= 2. \end{aligned}$$

For what values of α do the two sets fail to have a unique solution? Discuss the solutions for these values of α.

§ 3. Eigenvectors and eigenvalues

The concept of an *eigenvector* (known less commonly *as a characteristic vector* or *proper vector*) is one of the most important in mathematics and mathematical physics. We shall not attempt to define and discuss this concept in its full generality, limiting ourselves to the consideration of eigenvectors of finite square matrices (that is, matrices with a finite number n of rows and columns). However, not only does this limited study yield results of great importance in physics; we also find that many of the mathematical properties of this restricted theory are shared by the more general theory of eigenvectors, and this helps us to visualise the more abstract general theory in fairly simple and concrete terms.

§ 3.1. Definition for any square matrix

If $\boldsymbol{A}$ is a given $(n \times n)$ square matrix, then an eigenvector is an $(n \times 1)$ column matrix $\boldsymbol{x} \equiv (x_\rho)$ such that

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{13.20}$$

or

$$a_{\rho\sigma}x_\sigma = \lambda x_\rho \qquad (\rho = 1, 2,\ldots, n), \tag{13.21}$$

where λ is a constant. If we look upon $\boldsymbol{x}$ as a vector in an n-dimensional space with components x_ρ, then (13.20) implies that the multiplication of $\boldsymbol{x}$ by the matrix $\boldsymbol{A}$ merely changes its length by a factor λ, leaving its direction unaltered. We can write (13.20) and (13.21) in the form

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \tag{13.22}$$

or

$$(a_{\rho\sigma} - \lambda\delta_{\rho\sigma})x_\sigma = 0 \qquad (\rho = 1, 2,\ldots, n). \tag{13.23}$$

Equations (13.23) are n homogeneous equations for the n unknown components x_σ. These equations can in general be solved for the ratios of the components if the determinant of coefficients is zero:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

or in full

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{13.24}$$

Evaluating this determinant gives a polynomial in λ; and it is clear that the highest power of λ comes from the product

$$(a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda)$$

of the diagonal elements, since all other terms have fewer factors containing λ. So the determinant is a polynomial of nth degree in λ, and equation (13.24) will be satisfied by n values of λ, known as the *eigenvalues of the matrix* $\boldsymbol{A}$. We shall suppose that these n eigenvalues $\lambda_1, \lambda_2,\ldots, \lambda_n$ are all different; if some of them coincide, the discussion is complicated considerably, but the general properties of the solutions are not appreciably changed.

If we put $\lambda = \lambda_1$, in (13.23), then we can solve to find the ratios of the components of $\boldsymbol{x}$. Suppose we call this solution $\boldsymbol{x}_1$. We can likewise solve (13.23) with $\lambda = \lambda_2$, $\lambda = \lambda_3,\ldots$, $\lambda = \lambda_n$, obtaining solutions $\boldsymbol{x}_2, \boldsymbol{x}_3,\ldots, \boldsymbol{x}_n$. The solutions $\boldsymbol{x}_1, \boldsymbol{x}_2,\ldots, \boldsymbol{x}_n$ are the *eigenvectors of the matrix* $\boldsymbol{A}$ corresponding to the eigenvalues $\lambda_1, \lambda_2,\ldots, \lambda_n$, and can be looked upon as n vectors in n-space.

Example 4

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 19 & 10 \\ -30 & -16 \end{pmatrix}.$$

The eigenvalues are given by (13.24):

$$\begin{vmatrix} 19-\lambda & 10 \\ -30 & -16-\lambda \end{vmatrix} = 0$$

or $\lambda^2-3\lambda+4=0$. So the eigenvalues are $\lambda_1=4$ and $\lambda_2=-1$.

When $\lambda=\lambda_1=4$, equations (13.23) are

$$\begin{aligned} (19-\lambda_1)x_1 + \quad\quad 10x_2 &= 0 \\ -30x_1 - (16+\lambda_1)x_2 &= 0; \end{aligned}$$

both of these give $\frac{1}{2}x_1=-\frac{1}{3}x_2$, so that the eigenvector $\boldsymbol{x}_1$ corresponding to $\lambda=\lambda_1$ has components in the ratio $2:-3$; we write this as $\boldsymbol{x}_1 \propto (2, -3)$. When $\lambda=\lambda_2=-1$, $x_1=-\frac{1}{2}x_2$, so that the eigenvector $\boldsymbol{x}_2 \propto (1, -2)$.

Example 5

Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 6-2i & -1+3i \\ 9+3i & -4+3i \end{pmatrix}.$$

Equation (13.24) gives

$$(6-2i-\lambda)(-4+3i-\lambda)-(9+3i)(-1+3i)=0,$$

or

$$\lambda^2-\lambda(2+i)+2i=0.$$

So the eigenvalues are $\lambda_1=2$ and $\lambda_2=i$. When $\lambda=\lambda_1$, the eigenvectors components obey $(6-2i-2)x_1+(-1+3i)x_2=0$; thus $\boldsymbol{x}_1 \propto (1-i, 2)$. When $\lambda=\lambda_2$,

$$(6-3i)x_1+(-1+3i)x_2=0,$$

so that $\boldsymbol{x}_2 \propto (1-i, 3)$.

In Example 5, the eigenvectors necessarily have complex components; the idea of a *complex vector* with n complex components is a generalisation from real vectors. Since complex numbers obey the same rules of algebra as real numbers, the algebra of complex vectors is in many ways the same as that of real vectors.

Example 6

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -8 & 5 & -5 \\ -26 & 13 & 10 \\ -13 & 5 & -2 \end{pmatrix}.$$

The eigenvalue equation (13.24) reduces to the cubic $\lambda^3-3\lambda^2+\lambda-3$, giving eigenvalues $\lambda_1\equiv 3$, $\lambda_2\equiv i$, $\lambda_3\equiv -i$.

Putting $\lambda=\lambda_1$, the eigenvector equations are

$$\begin{aligned} -11x_1 + 5x_2 - 5x_3 &= 0 \\ -26x_1 + 10x_2 - 10x_3 &= 0, \end{aligned}$$

giving $x_1=0$, $x_2=x_3$. So $\boldsymbol{x}_1 \propto (0, 1, 1)$.

When $\lambda=\lambda_2, \lambda_3$, (12.23) become

$$\begin{aligned} (-8 \mp i)x_1 + 5x_2 - 5x_3 &= 0, \\ -26x_1 + (13 \mp i)x_2 - 10x_3 &= 0. \end{aligned}$$

These equations can be solved to give the ratios $x_1:x_2:x_3$ for the eigenvectors $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ These ratios are complex; so we have an example of a *real* matrix which has some *complex* eigenvalues and eigenvectors.

EXERCISE 13.3

1. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4+2i & 3+3i \\ 2+2i & 5+3i \end{pmatrix}.$$

2. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 2 & -2 & 1 \\ 2 & -4 & 3 \\ 2 & -6 & 5 \end{pmatrix}, \quad \begin{pmatrix} -8 & 4 & -1 \\ 3 & 1 & 3 \\ 36 & -14 & 9 \end{pmatrix}.$$

§ 3.2. Eigenvalues and eigenvectors of symmetric matrices

In physical applications we often have to find eigenvectors and eigenvalues of symmetric matrices, and these have a number of important properties. The first is the reality of all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Consider λ_1 for example: λ_1 and $\boldsymbol{x}_1$ satisfy

$$A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1. \tag{13.25}$$

If we take the complex conjugate of each of this set of linear equations, we shall obtain equations involving the complex conjugate λ_1^* and the column matrix $\boldsymbol{x}_1^*$ whose components are the conjugates x_1^* of the components of $\boldsymbol{x}_1$. Since the elements $a_{\rho\sigma}$ are real, they are unchanged by complex conjugation. So the complex conjugates of equations (13.25) are

$$A\boldsymbol{x}_1^* = \lambda_1^* \boldsymbol{x}_1^*. \tag{13.26}$$

Now let us transpose this matrix equation; we proved in Ch. 12 § 2.6 that this is done by transposing all matrices and reversing their order. Since $\boldsymbol{A}$ is symmetric, its transposed is $\bar{\boldsymbol{A}}=\boldsymbol{A}$; the transposed of the column matrix $\boldsymbol{x}_1^*$ is the row matrix $\bar{\boldsymbol{x}}_1^*$, usually written as $\boldsymbol{x}_1^\dagger$ and called the *Hermitian conjugate* of $\boldsymbol{x}_1$. Then the transposed of (13.26) is the row matrix equation

$$\boldsymbol{x}_1^\dagger \boldsymbol{A} = \lambda_1^* \boldsymbol{x}_1^\dagger; \tag{13.27}$$

we remind the reader that, regarded as a set of equations for the components of $\boldsymbol{x}_1^*$, (13.27) is *identical* with (13.26).

If we now multiply the column matrix equation (13.25) on the left by the row matrix $\boldsymbol{x}_1^\dagger$ to form inner products on each side, we find

$$\boldsymbol{x}_1^\dagger \boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1^\dagger \boldsymbol{x}_1; \tag{13.28}$$

on the other hand, multiplying (13.27) on the right by the column matrix $\boldsymbol{x}_1$ gives

$$\boldsymbol{x}_1^\dagger \boldsymbol{A}\boldsymbol{x}_1 = \lambda_1^* \boldsymbol{x}_1^\dagger \boldsymbol{x}_1. \tag{13.29}$$

Now $\boldsymbol{x}_1$ is a column vector; if we let its components be $x_{\rho 1}$ ($\rho=1, 2,\dots, n$), then the components of $\boldsymbol{x}_1^*$ are $x_{\rho 1}^*$. As in (12.11), the inner product of $\boldsymbol{x}_1^*$ and $\boldsymbol{x}_1$ is

$$\boldsymbol{x}_1^\dagger \boldsymbol{x}_1 = (x_{11}^* x_{11} + x_{21}^* x_{21} + \dots + x_{n1}^* x_{n1}),$$

which is positive since all the terms are non-negative and at least one is non-zero. Hence we can divide equations (13.28) and (13.29) by $\boldsymbol{x}_1^\dagger \boldsymbol{x}_1$ and obtain $\lambda_1=\lambda_1^*$. Thus λ_1 and likewise all other eigenvalues are real.

If λ is put equal to any eigenvalue, say λ_τ, all the coefficients $(a_{\rho\sigma}-\lambda_\tau\delta_{\rho\sigma})$ in equations (13.23) are real. Hence the ratios of the components x_σ, determined by (13.23) and (13.19), are also real. We can therefore find a real eigenvector $\boldsymbol{x}_\tau$ corresponding to the real eigenvalue λ_τ; as we have seen in Example 6, real matrices in general do not always have real eigenvalues and eigenvectors. Usually we are only interested in real eigenvectors of symmetric matrices, so we shall tacitly assume that eigenvectors $\boldsymbol{x}_\tau$ are chosen to be real; in terms of the components $x_{\sigma\tau}$ ($\sigma=1, 2,\dots, n$) of $\boldsymbol{x}_\tau$, $x_{\sigma\tau}^*=x_{\sigma\tau}$. Then the Hermitian conjugate $\boldsymbol{x}_\tau^\dagger$ is equal to the transposed $\bar{\boldsymbol{x}}_\tau$. The *magnitudes* of the vectors $\boldsymbol{x}_\tau$ are not determined by the homogeneous equations of type (13.25); if $\boldsymbol{x}_\tau$ is chosen as some convenient solution of the equations with $\lambda=\lambda_\tau$, then the length $|\boldsymbol{x}_\tau|$ of $\boldsymbol{x}_\tau$, regarded as a vector in n-space, is given by (12.12):

$$|\boldsymbol{x}_\tau| = (x_{1\tau}^2 + x_{2\tau}^2 + \dots + x_{n\tau}^2)^{\frac{1}{2}} = (\bar{\boldsymbol{x}}_\tau \boldsymbol{x}_\tau)^{\frac{1}{2}}. \tag{13.30}$$

We can define the unit eigenvector or *normalised eigenvector* $\boldsymbol{u}_\tau$ by dividing $\boldsymbol{x}_\tau$ by $|\boldsymbol{x}_\tau|$:

$$\boldsymbol{u}_\tau = \frac{\boldsymbol{x}_\tau}{|\boldsymbol{x}_\tau|}. \tag{13.31}$$

Since $\boldsymbol{x}_\tau$ is uniquely defined apart from its length, $\boldsymbol{u}_\tau$ is uniquely determined by (13.31) apart from its sign.

Example 7

The eigenvalues of the symmetric matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

are given by $(\lambda-1)(\lambda+2)-4=0$, and are $\lambda_1=-2$, $\lambda_2=3$. The eigenvector $\boldsymbol{x}_1$ has $-x_1+2x_2=0$, so we can take $\boldsymbol{x}_1=(2, 1)$; the eigenvector $\boldsymbol{x}_2$ has $4x_1+2x_2=0$, and we can take $\boldsymbol{x}_2=(-1, 2)$. The lengths of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are $|\boldsymbol{x}_1|=(4+1)^{\frac{1}{2}}=\sqrt{5}$ and $|\boldsymbol{x}_2|=\sqrt{5}$. Hence the normalised eigenvectors, given by (13.31), are

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

§ 3.3. Orthonormal sets of eigenvectors

We shall now establish the *orthogonality property* of the n eigenvectors $\boldsymbol{x}_\tau$ of a symmetric matrix. This property can be looked upon as the mutual perpendicularity of these n vectors in n-space, expressed by the vanishing of the scalar products $(\boldsymbol{x}_\tau \cdot \boldsymbol{x}_\omega)$ of any pairs of eigenvectors $\boldsymbol{x}_\tau$ and $\boldsymbol{x}_\omega$ $(\tau \neq \omega)$. In matrix form we can express this result as

$$\bar{\boldsymbol{x}}_\tau \boldsymbol{x}_\omega \equiv x_{\sigma\tau}\, x_{\sigma\omega} = 0 \qquad (\tau \neq \omega), \tag{13.32}$$

summation convention being used for the suffix σ labelling the vector components. As an example of this result, we note that in Example 7 above, $\bar{\boldsymbol{u}}_1\boldsymbol{u}_2=0$. The conditions (13.32) can be expressed very neatly in terms of the normalised eigenvectors $\boldsymbol{u}_\tau$, defined by (13.31). These vectors obey the n^2 equations

$$\bar{\boldsymbol{u}}_\tau \boldsymbol{u}_\omega = u_{\sigma\tau}\, u_{\sigma\omega} = \delta_{\tau\omega} \qquad (\tau, \omega = 1, 2, \ldots, n), \tag{13.33}$$

expressed the fact that the vectors $\boldsymbol{u}_\tau$ are both normalised and mutually perpendicular. Such a set of n vectors in n-space is known as a *complete orthonormal set.*

★ To establish (13.32) for $\tau=1$ and $\omega=2$ for instance, multiply (13.25) on the left by $\bar{\boldsymbol{x}}_2$, giving

$$\bar{\boldsymbol{x}}_2 \boldsymbol{A} \boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_2 \boldsymbol{x}_1. \tag{13.34a}$$

Likewise

$$\bar{\boldsymbol{x}}_1 \boldsymbol{A} \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1 \boldsymbol{x}_2. \tag{13.34b}$$

Now $\bar{\boldsymbol{x}}_2\boldsymbol{x}_1=\bar{\boldsymbol{x}}_1\boldsymbol{x}_2$, since each is the scalar product of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Likewise since $\bar{\boldsymbol{x}}_2\boldsymbol{A}\boldsymbol{x}_1$ is a (1×1) matrix or 'scalar', it is equal to its transposed; hence since $\bar{\boldsymbol{A}}=\boldsymbol{A}$,

$$\bar{\boldsymbol{x}}_2 \boldsymbol{A} \boldsymbol{x}_1 = (\overline{\bar{\boldsymbol{x}}_2 \boldsymbol{A} \boldsymbol{x}_1}) = \bar{\boldsymbol{x}}_1 \boldsymbol{A} \boldsymbol{x}_2.$$

So equations (13.34) give

$$(\lambda_1 - \lambda_2)\boldsymbol{x}_1\boldsymbol{x}_2 = 0;$$

since we have assumed that no two eigenvalues are equal, the scalar product $\boldsymbol{x}_1\boldsymbol{x}_2=0$. The argument holds for all pairs of eigenvectors $\boldsymbol{x}_\tau$ and $\boldsymbol{x}_\omega$ with $\lambda_\tau \neq \lambda_\omega$, so (13.32) is established. If two eigenvalues happen to be equal, it turns out that eigenvectors are to some extent arbitrary, and that this arbitrariness always allows us to *choose* eigenvectors which satisfy the orthogonality property (13.32). Thus it is always possible to find a complete orthonormal set of eigenvectors $\boldsymbol{u}_\tau$ of a symmetric matrix $\boldsymbol{A}$. ★

Example 8

Find the eigenvalues and normalised eigenvectors of the matrix

$$\frac{1}{3}\begin{pmatrix} -7 & 2 & 10 \\ 2 & 2 & -8 \\ 10 & -8 & -4 \end{pmatrix}.$$

The eigenvalue equation is

$$\begin{vmatrix} -7-3\lambda & 2 & 10 \\ 2 & 2-3\lambda & -8 \\ 10 & -8 & -4-3\lambda \end{vmatrix} = 0,$$

which reduces to $\lambda(\lambda-3)(\lambda+6)=0$. The eigenvalues are therefore $\lambda_1\equiv 0$, $\lambda_2\equiv 3$ and $\lambda_3\equiv -6$. When $\lambda=\lambda_1$, the components obey

$$-7x_1 + 2x_2 + 10x_3 = 0,$$
$$2x_1 + 2x_2 - 8x_3 = 0.$$

The unit eigenvector satisfying these equations is

$$\boldsymbol{u}_1 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}; \quad \text{likewise} \quad \boldsymbol{u}_2 = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}.$$

It is easy to check the orthogonality property, namely $\bar{\boldsymbol{u}}_1\boldsymbol{u}_2=\bar{\boldsymbol{u}}_2\boldsymbol{u}_3=\bar{\boldsymbol{u}}_3\boldsymbol{u}_1=0$, so that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ form a complete orthonormal set in 3-space.

EXERCISE 13.4

1. Find the eigenvalues and normalised eigenvectors of the matrices

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 6 \\ 6 & -4 \end{pmatrix}.$$

2. Find the eigenvalues and normalised eigenvectors of the matrices

$$\begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 3 & -2 \\ \sqrt{2} & -2 & 3 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 5a & 2a-6b & 4a-6b \\ 2a-6b & 8a-3b & -2a \\ 4a-6b & -2a & 5a+3b \end{pmatrix}.$$

Check that the eigenvectors obey the orthogonality property.

§ 4. Orthogonal transformations

§ 4.1. Change of rectangular axes

In § 3.3 of this chapter we found that a symmetric $(n\times n)$ matrix $\boldsymbol{A}$ defines a complete orthonormal set of vectors $\boldsymbol{u}_\tau$ $(\tau=1, 2,\ldots, n)$. In 3-space, for example, the normalised eigenvectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are mutually perpendicular unit vectors. This suggests that they could be used as the basis vectors of a set of rectangular axes in 3-space. So let us suppose that $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as usual are the basis vectors of a right-handed frame F, and that $(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$, obeying the orthonormality conditions (13.33), are the basis vectors of a different rectangular frame F′. We shall assume that the order of the vectors $\boldsymbol{u}_\tau$ has been chosen to make F′ a right-handed frame also, so that $\boldsymbol{u}_3=\boldsymbol{u}_1\times\boldsymbol{u}_2$. The components of $\boldsymbol{u}_\tau$ relative to the frame F are $u_{\sigma\tau}$ $(\sigma=1, 2, 3)$, so that

$$\boldsymbol{u}_\tau = u_{1\tau}\boldsymbol{i} + u_{2\tau}\boldsymbol{j} + u_{3\tau}\boldsymbol{k} \qquad (\tau = 1, 2, 3). \tag{13.35}$$

In other words, $u_{\sigma\tau}$ $(\sigma=1, 2, 3)$ are the direction cosines of the τth axis of F′ relative to F. Now by Theorem 3 of Ch. 9 § 3.4, any arbitrary vector can be expressed relative to F in the form

$$w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}, \tag{13.36}$$

or relative to F′ as

$$w_1'\,\boldsymbol{u}_1 + w_2'\,\boldsymbol{u}_2 + w_3'\,\boldsymbol{u}_3 = w_\tau'\,\boldsymbol{u}_\tau. \qquad (13.37$$

Substituting equations (13.35) in (13.37) we find that the vector is

$$(u_{1\tau}\, w'_\tau)\boldsymbol{i} + (u_{2\tau}\, w'_\tau)\boldsymbol{j} + (u_{3\tau}\, w'_\tau)\boldsymbol{k}.$$

Since the expression in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is unique, we can equate coefficients in (13.36) and in this expression, giving the relations between the components in the frames F and F′ as

$$w_\sigma = u_{\sigma\tau}\, w'_\tau \qquad (\sigma = 1, 2, 3). \tag{13.38}$$

Equation (13.38) has the form of a matrix equation, (w_σ) and (w'_τ) being column matrices, and suggests that we look upon the nine components $u_{\sigma\tau}$ of the three vectors $\boldsymbol{u}_\tau$ as the elements of a (3×3) matrix $\boldsymbol{U}=(u_{\sigma\tau})$. The three vectors $\boldsymbol{u}_\tau$ are in fact the columns of $\boldsymbol{U}$. Then equation (13.38) expresses in terms of the matrix $\boldsymbol{U}$ the relation between the components of a vector $\boldsymbol{w}$ in the two frames. We say that 'the matrix $\boldsymbol{U}$ transforms the vector from F′ to the frame F'.

We note that the most general transformation from one set of rectangular coordinates to another in a plane is simply a rotation of axes through some angle θ. The matrix form of this transformation is given in Ch. 12 § 2.7, Example 13.

Example 9

Two of the basis vectors of a frame F′, relative to a frame F, are in the directions of the vectors (1, 2, 2) and (4, 3, −5). The components of a vector $\boldsymbol{w}$ relative to F′ are $(\sqrt{2}, 3, 4)$; find the components of $\boldsymbol{w}$ relative to F.

Using vector notation, $\boldsymbol{u}_1 \propto (1, 2, 2)$ and $\boldsymbol{u}_2 \propto (4, 3, -5)$, so that $\boldsymbol{u}_1 = \frac{1}{3}(1, 2, 2)$ and $\boldsymbol{u}_2 = \frac{1}{10}\sqrt{2}(4, 3, -5)$. First we note that $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$, which is essential. The third unit basis vector of the right-handed frame F′ is

$$\boldsymbol{u}_3 = \boldsymbol{u}_1 \times \boldsymbol{u}_2 = \tfrac{1}{30}\sqrt{2}(-16, 13, -5).$$

$\boldsymbol{U}$ is the matrix with columns $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$:

$$\boldsymbol{U} = \tfrac{1}{30}\sqrt{2}\begin{pmatrix} 5\sqrt{2} & 12 & -16 \\ 10\sqrt{2} & 9 & 13 \\ 10\sqrt{2} & -15 & -5 \end{pmatrix}.$$

Using (13.38) with $(w'_1, w'_2, w'_3) = (\sqrt{2}, 3, 4)$, we find

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \tfrac{1}{30}\sqrt{2}\begin{pmatrix} -18 \\ 99 \\ -45 \end{pmatrix} = \tfrac{3}{10}\sqrt{2}\begin{pmatrix} -2 \\ 11 \\ -5 \end{pmatrix}.$$

§ 4.2. ORTHOGONAL MATRICES

A matrix $\boldsymbol{U}$ whose columns consist of an orthonormal set of vectors is known as an *orthogonal matrix*; the orthonormality property (13.33) can be expressed as a matrix equation if we remember that the transposed $\boldsymbol{U}$ has elements $\bar{u}_{\tau\sigma}=u_{\sigma\tau}$; then (13.33) can be written

$$\bar{u}_{\tau\sigma}\, u_{\sigma\omega} = \delta_{\tau\omega} \qquad (\tau,\, \omega = 1,\, 2,\, 3). \tag{13.39}$$

or

$$\overline{\boldsymbol{U}}\boldsymbol{U} = \boldsymbol{I}. \tag{13.40}$$

Now as in Ch. 12 § 4.2, the determinant $|\boldsymbol{U}|$ is the triple scalar product $[\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$, and is thus unity. So the inverse $\boldsymbol{U}^{-1}$ of $\boldsymbol{U}$ exists. If we multiply (13.40) on the right by $\boldsymbol{U}^{-1}$, we find

$$\overline{\boldsymbol{U}} = \boldsymbol{U}^{-1}, \tag{13.41}$$

so that the *inverse of an orthogonal matrix is its transposed*. Multiplying (13.41) by $\boldsymbol{U}$ on the left gives

$$\boldsymbol{U}\overline{\boldsymbol{U}} = \boldsymbol{I}.$$

If we denote by $\boldsymbol{w}$ and $\boldsymbol{w}'$ the column matrices with components w_1, w_2, w_3 and w_1', w_2', w_3' respectively, defined by the frames F and F′, then (13.38) can be written

$$\boldsymbol{w} = \boldsymbol{U}\boldsymbol{w}' \tag{13.42}$$

so that

$$\boldsymbol{w}' = \boldsymbol{U}^{-1}\boldsymbol{w} = \overline{\boldsymbol{U}}\boldsymbol{w}. \tag{13.43}$$

Just as $\boldsymbol{U}$ transforms column vectors $\boldsymbol{w}'$ from the frame F′ to F, so $\boldsymbol{U}^{-1}$ or $\overline{\boldsymbol{U}}$ transforms $\boldsymbol{w}$ from F to F′. And just as the columns of $\boldsymbol{U}$ are the directions cosines of the basis vectors of F′ relative to F, so the columns of $\overline{\boldsymbol{U}}$ are the direction cosines of the basis vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of F relative to the frame F′. The columns of $\overline{\boldsymbol{U}}$ are of course the rows of $\boldsymbol{U}$, so these three rows give the direction cosines of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ relative to F′.

Example 10

In Example 9, we found a particular orthogonal matrix $\boldsymbol{U}$. We can easily verify (13.40) for this matrix. Further, relative to the frame F′, the basis vectors of F are given by the rows of $\boldsymbol{U}$; thus

$$\boldsymbol{i} = \left(\frac{1}{3}, \frac{2\sqrt{2}}{5}, -\frac{8\sqrt{2}}{15}\right), \ \boldsymbol{j} = \left(\frac{2}{3}, \frac{3}{5\sqrt{2}}, \frac{13}{15\sqrt{2}}\right), \ \boldsymbol{k} = \left(\frac{2}{3}, -\frac{1}{\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right).$$

It is easy to check that $\boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=1$ and $\boldsymbol{i}\cdot\boldsymbol{j}=\boldsymbol{j}\cdot\boldsymbol{k}=\boldsymbol{k}\cdot\boldsymbol{i}=0$ in this frame.

In this section we have confined ourselves to the discussion of changes of rectangular axes in 3-space. Orthogonal matrices used in 3-dimensional geometry and in classical physics are usually (3×3) matrices. However, it is clear that the properties of orthogonal matrices depend only on the orthonormality condition (13.39) and simply by allowing the suffices $\sigma, \tau, \ldots$ to take the values $1, 2, \ldots, n$, the results of this section can be generalised to $(n \times n)$ orthogonal matrices, representing transformations between different rectangular frames in n-space.

§ 4.3. Hermitian and unitary matrices

The results we have established in §§ 3.2, 3.3, 4.1 and 4.2 apply to real matrices; in particular, all symmetric and orthogonal matrices are by definition real. In some branches of mathematics and physics, analogous results for complex matrices are of great importance. In this subsection we shall summarise these results, and shall give an example to illustrate them. The proofs of the results are simple generalisations of the proofs already given for real matrices, and we leave it to the reader to make the necessary modifications.

★ We have already defined the Hermitian conjugate of a column matrix $\boldsymbol{x}$ to be $\boldsymbol{x}^\dagger = \bar{\boldsymbol{x}}^*$, the conjugate of the transposed of $\boldsymbol{x}$. For any $(m \times n)$ matrix A, the *Hermitian conjugate* is likewise defined as the $(n \times m)$ matrix $A^\dagger = \bar{A}^*$; the (ρ, σ) element of $A^\dagger$ is thus $a^\dagger_{\rho\sigma} = a^*_{\sigma\rho}$. A square matrix is said to be *Hermitian* if it is its own Hermitian conjugate, so that $A = A^\dagger$ or $a_{\rho\sigma} = a^*_{\sigma\rho}$ for all ρ, σ; this implies that the diagonal elements $a_{\rho\rho}$ are real. For example

$$A = \begin{pmatrix} 2 & 4+i & 2i \\ 4-i & 0 & 1+3i \\ -2i & 1-3i & -3 \end{pmatrix}$$

is a (3×3) Hermitian matrix.

As for a symmetric matrix, it can be shown that

(i) the eigenvalues of an $(n \times n)$ Hermitian matrix are all real;

(ii) the eigenvectors $\boldsymbol{x}_\tau$ $(\tau = 1, 2, \ldots, n)$ obey the 'orthogonality relations' $\boldsymbol{x}^\dagger_\tau \boldsymbol{x}_\omega = 0$ $(\tau \neq \omega)$, or in vector notation $\boldsymbol{x}^*_\tau \cdot \boldsymbol{x}_\omega = 0$. We note that the eigenvectors are not necessarily real, and that 'scalar products' appearing in the orthogonality relations are between one eigenvector $\boldsymbol{x}_\omega$ and the complex conjugate $\boldsymbol{x}^*_\tau$ of another eigenvector. If we adopt this definition of the scalar product of two complex vectors, then the *length x of a*

complex vector $\boldsymbol{x}$ is naturally taken to be given by

$$x^2 = \boldsymbol{x}^* \cdot \boldsymbol{x} = x_1^* x_1 + x_2^* x_2 + \ldots + x_n^* x_n; \tag{13.44}$$

x^2 is equal to the one component of the matrix $\boldsymbol{x}^\dagger \boldsymbol{x}$, and as in § 3.2, is real and non-negative. Thus we have a natural definition of the length of a complex vector.

If we define n complex vectors $\boldsymbol{u}_\tau$ by

$$\boldsymbol{u}_\tau = \boldsymbol{x}_\tau / x_\tau \qquad (\tau = 1, 2, \ldots, n), \tag{13.45}$$

then each of these vectors is of unit length; this fact, together with the 'orthogonality' of these vectors, can be expressed neatly in the form

$$\boldsymbol{u}_\tau^\dagger \boldsymbol{u}_\omega = \delta_{\tau\omega} \qquad (\tau, \omega = 1, 2, \ldots, n),$$

or if the components of $\boldsymbol{u}_\tau$ are $u_{\sigma\tau}$ ($\sigma = 1, 2, \ldots, n$),

$$u_{\sigma\tau}^* u_{\sigma\omega} = \delta_{\tau\omega} \qquad (\tau, \omega = 1, 2, \ldots, n). \tag{13.46}$$

If we look upon $\boldsymbol{U} = (u_{\sigma\tau})$ as a matrix, then (13.46) can be written as the matrix equation

$$\boldsymbol{U}^\dagger \boldsymbol{U} = \boldsymbol{I}. \tag{13.47}$$

A matrix $\boldsymbol{U}$ satisfying this condition is a *unitary matrix*, the complex generalisation of an orthogonal matrix. From (13.47) we can deduce that $\boldsymbol{U}^{-1} = \boldsymbol{U}^\dagger$ and $\boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}$.

We can summarise these results thus

(iii) if $\boldsymbol{u}_\tau$ are the normalised eigenvectors defined by (13.44) and (13.45), they form an orthonormal set in the sense that (13.46) is satisfied;

(iv) the matrices $\boldsymbol{U}$, whose columns are $\boldsymbol{u}_\tau$, is a unitary matrix; its inverse is equal to its Hermitian conjugate. We shall illustrate the properties (i),..., (iv) by a simple example.

Example 11

The matrix

$$\frac{1}{5}\begin{pmatrix} 7 & 6i \\ -6i & -2 \end{pmatrix}$$

is a Hermitian (2×2) matrix. The eigenvalues are given by

$$(7 - 5\lambda)(-2 - 5\lambda) - 36 = 0,$$

and are therefore $\lambda_1 \equiv 2$ and $\lambda_2 \equiv -1$, exemplifying property (i). When $\lambda = \lambda_1$, the eigenvector components satisfy $-3x_1 + 6ix_2 = 0$, so we can take $\boldsymbol{x}_1 = (2i, 1)$. When $\lambda = \lambda_2$, $12x_1 + 6ix_2 = 0$, we can take $\boldsymbol{x}_2 = (1, 2i)$. These satisfy $\boldsymbol{x}_1^* \cdot \boldsymbol{x}_2 = 0$, demon-

strating property (ii). Now the lengths x_1 and x_2 defined by (13.44) are each $\sqrt{5}$, so

$$\boldsymbol{u}_1 = \left(\frac{2i}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \quad \text{and} \quad \boldsymbol{u}_2 = \left(\frac{1}{\sqrt{5}}, \frac{2i}{\sqrt{5}}\right),$$

obeying (13.46). The matrix whose columns are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is

$$\boldsymbol{U} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2i & 1 \\ 1 & 2i \end{pmatrix}.$$

It is easy to check that its inverse is

$$\boldsymbol{U}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2i & 1 \\ 1 & -2i \end{pmatrix},$$

equal to its Hermitian conjugate. ★

§ 5. The diagonal form of a symmetric matrix

§ 5.1. Reduction to diagonal form

The vectors $\boldsymbol{u}_\tau$ forming the orthonormal set were originally introduced in § 3.2 as eigenvectors of a symmetric matrix $\boldsymbol{A}$. The matrix $\boldsymbol{U}$ whose columns are the eigenvectors $\boldsymbol{u}_\tau$ has a close connection with the matrix $\boldsymbol{A}$. The original eigenvalue equation (13.20) or (13.21) is satisfied by n eigenvalues λ_τ $(\tau = 1, 2,\ldots, n)$, which we still suppose to be distinct, and the corresponding eigenvectors $\boldsymbol{x}_\tau$ can be taken as the unit vectors $\boldsymbol{u}_\tau = (u_{1\tau}, u_{2\tau},\ldots, u_{n\tau})$, which are the columns of the matrix $\boldsymbol{U}$. These n solutions, written in the form (13.21) are

$$a_{\rho\sigma}\, u_{\sigma\tau} = \lambda_\tau u_{\rho\tau} = u_{\rho\sigma}\, \delta_{\sigma\tau}\, \lambda_\tau. \tag{13.48}$$

This set of equations, with ρ and τ both ranging over the values $1, 2,\ldots, n$, can be looked upon as a matrix equation involving the orthogonal matrix $\boldsymbol{U} \equiv (u_{\rho\sigma})$; if we define the matrix

$$\boldsymbol{\Lambda} = (\delta_{\sigma\tau}\, \lambda_\tau), \tag{13.49}$$

then (13.48) can be written as

$$\boldsymbol{AU} = \boldsymbol{U\Lambda}. \tag{13.50}$$

If we multiply this equation on the left by $\boldsymbol{U}^{-1} = \overline{\boldsymbol{U}}$, we find

$$\overline{\boldsymbol{U}}\boldsymbol{AU} = \boldsymbol{U}^{-1}\boldsymbol{AU} = \boldsymbol{\Lambda}. \tag{13.51}$$

From the definition (13.49) we see that *the matrix $\boldsymbol{\Lambda}$ is the diagonal matrix whose diagonal elements are the eigenvalues $\lambda_1, \lambda_2,\ldots, \lambda_n$ of the matrix $\boldsymbol{A}$*. The result (13.51) is referred to as 'the reduction of the sym-

metric matrix $\boldsymbol{A}$ to diagonal form', and is a theorem which has many important applications in geometry and throughout mathematical physics. ★ The corresponding theorem on the reduction of a Hermitian matrix to diagonal form can be proved in exactly the same way. The matrix $\boldsymbol{U}$ is then the unitary matrix whose columns are the eigenvectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, and the theorem states that

$$\boldsymbol{U}^\dagger \boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is the real diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$. ★

Example 12

Reduce the symmetric matrix

$$\frac{1}{5}\begin{pmatrix} 6 & 12 \\ 12 & -1 \end{pmatrix}$$

to diagonal form, and find the orthogonal matrix which effects this reduction.

The eigenvalue equation is

$$(6 - 5\lambda)(-1 - 5\lambda) - 144 = 0$$

or

$$\lambda^2 - \lambda - 6 = 0,$$

giving eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$.
$\lambda = 3$: eigenvector equation is $-9x_1 + 12x_2 = 0$, giving normalised eigenvector $\boldsymbol{u}_1 = (\frac{4}{5}, \frac{3}{5})$.
$\lambda = -2$: eigenvector equation $16x_1 + 12x_2 = 0$ gives $\boldsymbol{u}_2 = (-\frac{3}{5}, \frac{4}{5})$.
So the matrix $\boldsymbol{A}$ can be reduced to the diagonal form

$$\boldsymbol{\Lambda} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

by the orthogonal matrix $\boldsymbol{U}$ whose columns are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$:

$$\boldsymbol{U} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}.$$

It is easy to check that equation (13.51) is satisfied.

Example 13

The eigenvalues and normalised eigenvectors of the symmetric matrix

$$\boldsymbol{A} = \frac{1}{3}\begin{pmatrix} -7 & 2 & 10 \\ 2 & 2 & -8 \\ 10 & -8 & -4 \end{pmatrix}$$

were found in Example 8 to be $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = -6$, and

$$\boldsymbol{u}_1 = (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}), \qquad \boldsymbol{u}_2 = (\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3}), \qquad \boldsymbol{u}_3 = (\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}).$$

Hence if $\boldsymbol{U}$ is the orthogonal matrix

$$\boldsymbol{U} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & 1 \\ 1 & 2 & -2 \end{pmatrix},$$

then by (13.51), $\bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda}$, where

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

It is easy to check by actual multiplication that this is true.

§ 5.2. Application to quadric surfaces

One important use of the theorem (13.51) is in the study of quadric surfaces, which are the analogues in 3-space of conic sections in 2-space. In a rectangular frame F with basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, a quadric surface is represented by the general second degree equation in the coordinates (y_1, y_2, y_3); it can easily be shown that the first-order terms can be made to vanish by a simple change of origin, reducing the equation to the form

$$a_{11}y_1^2 + a_{22}y_2^2 + a_{33}y_3^2 + 2a_{12}y_1y_2 + 2a_{13}y_1y_3 + 2a_{23}y_2y_3 = 1. \tag{13.52}$$

We have chosen a notation for the six coefficients of the quadratic terms which allows us to write equation (13.52) as a matrix equation. If $\boldsymbol{A}$ is defined as the *symmetric* matrix $(a_{\rho\sigma})$, and $\bar{\boldsymbol{y}}$ and $\boldsymbol{y}$ are the row and column matrices with components y_1, y_2, y_3, then (13.52) can be written as

$$\bar{\boldsymbol{y}}\boldsymbol{A}\boldsymbol{y} = 1. \tag{13.53}$$

Suppose that we now take a new rectangular frame of reference F′ with basis vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$; then the new coordinates y_1', y_2', y_3' are related to the old coordinates by the transformation (13.38), or in matrix form (13.42),

$$\boldsymbol{y} = \boldsymbol{U}\boldsymbol{y}'. \tag{13.54}$$

The transposed of (13.54) is

$$\bar{\boldsymbol{y}} = \boldsymbol{y}'\bar{\boldsymbol{U}}; \tag{13.55}$$

using (13.55) and (13.54) the equation of the quadric surface (13.53) in

the frame F′ is found to be

$$\bar{\boldsymbol{y}}'\bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}\boldsymbol{y}' = 1. \tag{13.56}$$

So far we have not specified the new frame F′. If we now define the basis vectors $\boldsymbol{u}_\tau$ to be the orthonormal set of eigenvectors of $\boldsymbol{A}$, then the matrix product $\bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}$ occurring in (13.56) becomes the diagonal matrix $\boldsymbol{\Lambda}$, by (13.51), and equation (13.56) is

$$(y_1'\ y_2'\ y_3')\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = 1\ ,$$

or

$$\lambda_1 y_1'^2 + \lambda_2 y_2'^2 + \lambda_3 y_3'^2 = 1. \tag{13.57}$$

Thus we have shown that by choosing the eigenvectors of $\boldsymbol{A}$ as coordinate axes, the equation of the quadric surface is reduced to the simple form (13.57). These axes are called *principal axes* of the quadric; the surface is symmetrical about each of these axes, since if (y_1', y_2', y_3') lies on the surface (13.57), then all of the eight points $(\pm y_1', \pm y_2', \pm y_3')$ do. The cross-section of the quadric surface by a plane $y_1' = \alpha$ perpendicular to the first principal axis is the conic $\lambda_2 y_2'^2 + \lambda_3 y_3'^2 = 1 - \lambda_1 \alpha^2$; for all values of the constant α this conic has principal axes parallel to the other two principal axes of the quadric surface. This property is shared of course by plane sections of the quadric perpendicular to the other two principal axes. The first principal axis meets the quadric where $y_2' = y_3' = 0$, so that $y_1' = \pm\lambda_1^{-\frac{1}{2}}$. Hence the distance along this axis from the origin to the points where it meets the quadric is $\lambda_1^{-\frac{1}{2}}$; this is known as the *length of the semi-axis*. Likewise the lengths of the other semi-axes are $\lambda_2^{-\frac{1}{2}}$ and $\lambda_3^{-\frac{1}{2}}$.

The choice of the principal axes of a quadric as coordinate axes allows us to assume the form (13.57) for the equation of any quadric surface. Many analytic properties of quadrics are best investigated by using this simple form of the equation.

Example 14

Find the directions of the principal axes of the quadric

$$11y_1^2 + 5y_2^2 + 2y_3^2 + 16y_1y_2 + 20y_2y_3 - 4y_1y_3 = 9,$$

and find the equation of the quadric when the coordinate axes coincide with the principal axes.

The symmetric matrix A defined by comparing with (13.52) is

$$A = \frac{1}{9}\begin{pmatrix} 11 & 8 & -2 \\ 8 & 5 & 10 \\ -2 & 10 & 2 \end{pmatrix}.$$

The eigenvalue equation for this matrix is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$, giving eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$.

When $\lambda = \lambda_1$, the eigenvector components obey

$$\begin{aligned} -7x_1 + 8x_2 - 2x_3 &= 0 \\ 8x_1 - 13x_2 + 10x_3 &= 0. \end{aligned}$$

So the normalised eigenvector is $\boldsymbol{u}_1 = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$. When $\lambda = \lambda_2$, the normalised eigenvector is found to be $\boldsymbol{u}_2 = (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$. For $\lambda = \lambda_3$, the normalised eigenvector is $\boldsymbol{u}_3 = \boldsymbol{u}_1 \times \boldsymbol{u}_2 = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$.

If U has columns $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, then (13.51) tells us that $\bar{U}AU = \Lambda$ where Λ is the diagonal matrix whose diagonal elements are 2, 1, -1, in that order. In the frame with coordinate axes along $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, the equation of the quadric is given by (13.57), and is

$$2y_1'^2 + y_2'^2 - y_3'^2 = 1.$$

EXERCISE 13.5

1. Find the eigenvalues and normalised eigenvectors of the matrix

$$\frac{1}{3}\begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & -2 \\ 6 & -2 & 1 \end{pmatrix}.$$

Hence find the orthogonal matrix which reduces the matrix to diagonal form.

Find the equation of the quadric

$$2y_1^2 + 6y_2^2 + y_3^2 + 8y_1y_2 + 12y_1y_3 - 4y_2y_3 = 3$$

when it is referred to its principal axes.

2. Find the equation of the quadric

$$y_1^2 + 3y_2^2 + 3y_3^2 + 2\sqrt{2}y_1y_2 + 2\sqrt{2}y_1y_3 - 4y_2y_3 = 1$$

when it is referred to its principal axes. Find also the direction cosines of the original axes relative to the frame defined by the principal axes.

3. Show that the quadrics

$$35y_1^2 + 77y_2^2 - 14y_3^2 - 36y_1y_3 - 108y_2y_3 = 49$$

and

$$3y_1^2 + 16y_2^2 + 9y_3^2 + 12y_1y_2 - 12y_2y_3 = 7$$

have the same principal axes. Find the equations of the quadrics referred to these axes.

4. Apply the theorem on the reduction of symmetric matrices to simplify the equation of a conic in a plane, assuming the centre of the conic is at the origin of coordinates. Show that if the eigenvalues of the matrix associated with the conic are equal, then the conic is a circle.

5. If $\boldsymbol{A}$ is a symmetric matrix and $\boldsymbol{U}$ is an orthogonal matrix, show that

(i) $\boldsymbol{A}' \equiv \bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}$ is symmetric;
(ii) $\operatorname{Tr} \boldsymbol{A} = \operatorname{Tr} \boldsymbol{A}'$, the 'trace' $\operatorname{Tr} \boldsymbol{A}$ being the sum of the diagonal elements of $\boldsymbol{A}$;
(iii) $\operatorname{Tr} \boldsymbol{A}$ is equal to the sum $\sum_r \lambda_r$ of the eigenvalues of $\boldsymbol{A}$;
(iv) the determinants $|\boldsymbol{A}|$ and $|\boldsymbol{A}'|$ are both equal to the product $\prod_r \lambda_r$ of the eigenvalues of $\boldsymbol{A}$.

State the corresponding results for a Hermitian matrix $\boldsymbol{A}$.

6. Reduce the Hermitian matrices

$$\frac{1}{9}\begin{pmatrix} 10 & -2(1+i) \\ -2(1-i) & 17 \end{pmatrix} \quad \text{and} \quad \frac{1}{4}\begin{pmatrix} 1 & -3i & 5\sqrt{2} \\ 3i & 1 & -5i\sqrt{2} \\ 5\sqrt{2} & 5i\sqrt{2} & -2 \end{pmatrix}$$

to diagonal form, and find the unitary matrices which effect these reductions.

§ 6. Euler's angles

We have discussed in general terms the properties of orthogonal matrices representing rotations. In physics we often need a concrete expression

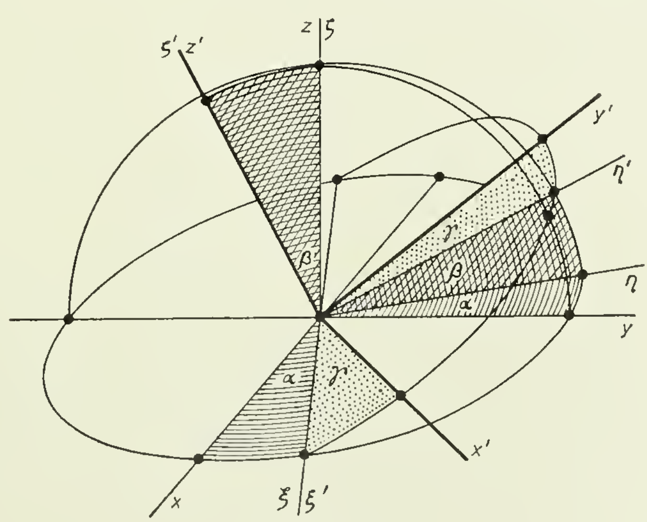

Fig. 13.1

for a rotation in 3-space, in terms of angles which can be specified. We can regard any rotation in 3-space as a combination of three rotations, performed successively, about three different directions in space; the

most useful description of this kind is in terms of *Euler's angles* α, β,(γ), which we now define.

In fig. 13.1, the rectangular axes x, y, z are the axes before any rotation has been performed. Consider a set of axes (ξ, η, ζ), originally coinciding with (x, y, z), subjected to a rotation α about the z-axis; the ξ- and η-axes will take up the position shown in fig. 13.1, the ζ-axis remaining coincident with the z-axis. The direction cosines of the (ξ, η, ζ)-axes relative to the (x, y, z)-axes are ($\cos\alpha$, $\sin\alpha$, 0), ($-\sin\alpha$, $\cos\alpha$, 0) and (0, 0, 1). Hence, by (13.43), the component $\boldsymbol{\chi} \equiv (\chi_1, \chi_2, \chi_3)$ of any vector in thə (ξ, η, ζ)-frame are related to its components $\boldsymbol{w} \equiv (w_1, w_2, w_3)$ in the (x, y, z)-frame by the transformation

$$\boldsymbol{\chi} = \boldsymbol{U}_1\boldsymbol{w}, \tag{13.58}$$

where $\boldsymbol{U}_1$ is the unitary matrix whose *rows* are these direction cosines:

$$\boldsymbol{U}_1 = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.59}$$

Now consider a second rotation, through an angle β about the ξ-axis, to the frame (ξ', η', ζ') shown. The components $\boldsymbol{\chi}' = (\chi_1', \chi_2', \chi_3')$ in this new frame are given, as with the first rotation, by

$$\boldsymbol{\chi}' = \boldsymbol{U}_2\boldsymbol{\chi} \tag{13.60}$$

where

$$\boldsymbol{U}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}. \tag{13.61}$$

The third rotation, through an angle γ about the ζ'-axis is indicated in the figure; the final position of the axes after the three rotations is along (x', y', z'). The components $\boldsymbol{w}' = (w_1', w_2', w_3')$ in this final frame are related to $\boldsymbol{\chi}'$ by

$$\boldsymbol{w}' = \boldsymbol{U}_3\boldsymbol{\chi}' \tag{13.62}$$

where

$$\boldsymbol{U}_3 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.63}$$

Equations (13.58)–(13.64) tell us that the change of axes from the (x, y, z)-

frame to the (x', y', z')-frame relates the components $\boldsymbol{w}$ and $\boldsymbol{w}'$ by

$$\boldsymbol{w}' = \boldsymbol{U}_3\boldsymbol{U}_2\boldsymbol{U}_1\boldsymbol{w} \equiv \boldsymbol{V}\boldsymbol{w}, \tag{13.64}$$

say. The transformation matrix for the over-all rotation of axes is thus

$$\boldsymbol{V} = \boldsymbol{U}_3\boldsymbol{U}_2\boldsymbol{U}_1 =$$

$$\begin{pmatrix} \cos\gamma\cos\alpha - \sin\gamma\cos\beta\sin\alpha & \cos\gamma\sin\alpha + \sin\gamma\cos\beta\cos\alpha & \sin\gamma\sin\beta \\ -\sin\gamma\cos\alpha + \cos\gamma\cos\beta\cos\alpha & -\sin\gamma\sin\alpha + \cos\gamma\cos\beta\cos\alpha & \cos\gamma\sin\beta \\ \sin\beta\sin\alpha & -\sin\beta\cos\alpha & \cos\beta \end{pmatrix}. \tag{13.65}$$

It is not difficult to check that V *is* an orthogonal matrix, by actually showing that $\boldsymbol{V}\bar{\boldsymbol{V}} = \boldsymbol{I}$. The inverse transformation, from the (x', y', z')-frame to the (x, y, z)-frame, is of course represented by the transposed matrix $\bar{V}$.

It is worth while noting that different authors define Euler's angles in slightly different ways. We have adopted the definition used in many textbooks on classical mechanics; we have used the notation (α, β, γ) for the angles to avoid confusion with the spherical polar coordinates θ and φ.

§ 7. Tensors

We have seen that the components w_σ of a vector are transformed by eqs. (13.38), (13.42) or (13.43) under a change of orthogonal axes. This transformation is a simple example of a *tensor transformation*, and vectors can also be called *tensors of the first order*. Once again limiting ourselves to quantities defined in 3-space, we define a *tensor of the second order* or more correctly, a *second order Cartesian tensor*, as a set of nine quantities $T_{\rho\sigma}$ $(\rho, \sigma = 1, 2, 3)$ which are transformed under a change of axes to the set $T'_{\mu\nu}$ $(\mu, \nu = 1, 2, 3)$, where

$$T_{\rho\sigma} = u_{\rho\mu}u_{\sigma\nu}T'_{\mu\nu}, \tag{13.66}$$

$\boldsymbol{U} \equiv (u_{\rho\mu})$ being the orthogonal matrix corresponding to the change of axes; summation convention applies to both the μ and the ν suffixes in eq. (13.66). We see that each of the two suffixes on $T_{\rho\sigma}$ obeys the same transformation law as the suffix σ in eq. (13.38), and can therefore be looked upon as a 'vector suffix'. In fact, one simple but important type of second order tensor $T_{\rho\sigma}$ can be formed from the components v_ρ $(\rho = 1, 2, 3)$ and w_σ $(\sigma = 1, 2, 3)$ of any two vectors thus:

$$T_{\rho\sigma} = v_\rho w_\sigma. \tag{13.67}$$

Since v_ρ and w_σ each obey the first order tensor transformation law (13.38), it is obvious that $T_{\rho\sigma}$ obeys the second law (13.66). The components $T_{\rho\sigma}$ defined by (13.67) are those of the *outer product* or *dyad* of the vectors v_ρ and w_σ.

It is natural to write the components $T_{\rho\sigma}$ of a second order tensor as a matrix $\boldsymbol{T}=(T_{\rho\sigma})$, the 'vector suffixes' ρ and σ labelling the rows and columns of the matrix. Then using the transposed matrix $\bar{\boldsymbol{U}}$ whose elements are $\bar{u}_{\nu\sigma}=u_{\sigma\nu}$, the law (13.66) can be written as the matrix equation

$$T_{\rho\sigma} = u_{\rho\mu}T'_{\mu\nu}\bar{u}_{\nu\sigma}, \tag{13.68}$$

or

$$\boldsymbol{T} = \boldsymbol{U}\boldsymbol{T}'\bar{\boldsymbol{U}}. \tag{13.69}$$

Using (13.69) and (13.41), the components $T'_{\mu\nu}$ are expressible in terms of $T_{\rho\sigma}$ by

$$\boldsymbol{T}' = \bar{\boldsymbol{U}}\boldsymbol{T}\boldsymbol{U}, \tag{13.70}$$

which in suffix notation is

$$T'_{\mu\nu} = u_{\rho\mu}u_{\sigma\nu}T_{\rho\sigma}. \tag{13.71}$$

Example 15

Let x_σ $(\sigma=1, 2, 3)$ be the coordinates of a point in 3-space, and suppose that $\varphi(x_1, x_2, x_3)$ is any scalar function of position. Then

$$w_\sigma = \frac{\partial\varphi}{\partial x_\sigma} \qquad (\sigma = 1, 2, 3)$$

s a vector or tensor of the first order. Under the change of orthogonal axes definedi by the orthogonal matrix $\boldsymbol{U}=(u_{\sigma\tau})$, the new coordinates x'_τ of the point will be given by eq. (13.38):

$$x_\sigma = u_{\sigma\tau}x'_\tau \quad \text{or conversely} \quad x'_\tau = u_{\sigma\tau}x_\sigma.$$

Supposing $\varphi(x_1, x_2, x_3)$ to be expressed in terms of the transformed coordinates x'_τ, the transformation for w_σ will be

$$w'_\tau = \frac{\partial\varphi}{\partial x'_\tau} = \frac{\partial\varphi}{\partial x_\sigma}\,\frac{\partial x_\sigma}{\partial x'_\tau} = u_{\sigma\tau}w_\sigma,$$

or conversely

$$w_\sigma = \frac{\partial\varphi}{\partial x_\sigma} = \frac{\partial\varphi}{\partial x'_\tau}\,\frac{\partial x'_\tau}{\partial x_\sigma} = u_{\sigma\tau}w'_\tau,$$

and so $w_\sigma=\partial\varphi/\partial x_\sigma$ is a tensor of the first order. This tensor is the 'gradient' of the scalar function $\varphi(x_1, x_2, x_3)$.

Example 16

If w_ρ $(\rho=1, 2, 3)$ is a first order tensor which is a function of position x_σ, then

$$T_{\rho\sigma} = \frac{\partial w_\rho}{\partial x_\sigma}, \tag{13.72}$$

are the components of a second order tensor. Under a change of orthogonal axes, we have

$$T_{\rho\sigma} = \frac{\partial w_\rho}{\partial x_\sigma} = \frac{\partial w_\rho}{\partial x'_\nu}\,\frac{\partial x'_\nu}{\partial x_\sigma}.$$

But $x'_\nu = u_{\sigma\nu} x_\sigma$ and $w_\rho = u_{\rho\mu} w'_\mu$ so that

$$T_{\rho\sigma} = u_{\sigma\nu}\frac{\partial}{\partial x'_\nu}(u_{\rho\mu} w'_\mu) = u_{\sigma\nu} u_{\rho\mu}\frac{\partial w'_\mu}{\partial x'_\nu} = u_{\sigma\nu} u_{\rho\mu} T'_{\mu\nu},$$

which is the transformation (13.66). Thus $\partial w_\rho/\partial x_\sigma$ $(\rho, \sigma=1, 2, 3)$ is a second order tensor.

Example 17

A special case of the second order tensor in eq. (13.72) is when $w_\rho = \partial\varphi/\partial x_\rho$ as in Example 15. Then

$$T_{\rho\sigma} = \frac{\partial}{\partial x_\sigma}\left(\frac{\partial\varphi}{\partial x_\rho}\right) = \frac{\partial^2\varphi}{\partial x_\rho \partial x_\sigma} \qquad (\rho, \sigma = 1, 2, 3)$$

is a second order tensor.

Example 18

Another special case of the tensor in eq. (13.72) is when the first order tensor w_ρ is the vector x_ρ itself. This tensor has the components

$$\delta_{\rho\sigma} = \frac{\partial x_\rho}{\partial x_\sigma},$$

and since x_1, x_2, x_3 are independent coordinates, this tensor does have the components

$$\delta_{\rho\sigma} = 1 \quad \text{when} \quad \rho = \sigma,$$
$$\delta_{\rho\sigma} = 0 \quad \text{when} \quad \rho \neq \sigma,$$

which are those of the Kronecker delta. The reader should verify that the transformation law

$$\delta_{\rho\sigma} = u_{\rho\mu}\, u_{\sigma\nu}\, \delta'_{\mu\nu},$$

is satisfied, using the equation (13.33). In fact, the Kronecker delta is transformed into itself, i.e. it has the same components referred to both sets of axes.

Further we see that if w_ρ is a tensor of the first order

$$w_\rho \delta_{\rho\sigma} = w_1\delta_{1\sigma} + w_2\delta_{2\sigma} + w_3\delta_{3\sigma} = w_\sigma,$$

for $\sigma=1, 2, 3$ and w_σ is also a tensor of the first order. We see therefore that by multiplying w_ρ by $\delta_{\rho\sigma}$, the suffix ρ has been replaced by the suffix σ. For this reason the tensor $\delta_{\rho\sigma}$ is also called the *substitution tensor*.

§ 7.1. Symmetric tensors

The form (13.70) of the transformation law enables us to discover a remarkable and important property of *symmetric tensors*, for which $T_{\rho\sigma}=T_{\sigma\rho}$. The matrix $\boldsymbol{T}$ in eq. (13.70) is then symmetric, and the theorem (13.51) on reduction to diagonal form can be applied: thus, if we are given the components $T_{\rho\sigma}$ in one frame F, we can find a frame F′ (related to F by the orthogonal transformation $\boldsymbol{U}$) such that $T'_{\mu\nu}$ is a diagonal matrix. We shall give two examples of the use of this property of symmetric tensors.

Our first example is simply a re-statement of the results of § 5.2 in tensor language. The coefficients $a_{\rho\sigma}$ of the quadratic form (13.52) were written there as a symmetric matrix $\boldsymbol{A}$; we saw that when the coordinates undergo the transformation (13.38) or (13.54), the matrix $\boldsymbol{A}$ transforms to the form $\boldsymbol{\bar{U}AU}$; that is, $\boldsymbol{A}$ is a *symmetric tensor*. So there is a set of rectangular axes, the *principal axes* of the quadric, with respect to which $\boldsymbol{A}$ is a diagonal matrix; relative to these axes, the equation of the quadric takes the simple form (13.57).

§ 7.2. The strain tensor

Our second example of a symmetrical second order tensor arises in the analysis of the strain of an elastic solid. When an elastic solid is acted on by constant forces it changes its shape to some extent, and we say that the body is 'strained'. Let us consider the strain near a fixed point of the solid, which we choose to be the origin of coordinates. Suppose that a neighbouring point of the solid has coordinates x_ρ ($\rho=1, 2, 3$) when the body is unstrained. When the constant forces are applied, this point of the body will move, say, to the position $x_\rho+\delta x_\rho$. Since the origin remains fixed in the body, the displacement δx_ρ is a measure of the strain; this displacement is a function of the coordinates themselves, so that we can write

$$\delta x_\rho = v_\rho(x_\sigma), \tag{13.73}$$

say. Provided that the distortion of the solid is not discontinuous, we can expand the functions $v_\rho(x_\sigma)$ by the Taylor expansion (eq. 6.35) to the first order in x_σ; provided the x_σ are sufficiently small, eq. (13.71) becomes approximately

$$\delta x_\rho = v_\rho(0) + \frac{\partial v_\rho}{\partial x_\sigma}\, x_\sigma.$$

However, the displacement $v_\rho(0)$ of the origin is zero by definition, so we have

$$\delta x_\rho = \frac{\partial v_\rho}{\partial x_\sigma} x_\sigma. \tag{13.74}$$

Thus the strain at any point x_σ is defined by the nine quantities $\partial v_\rho/\partial x_\sigma$, which we can write as a (3×3) matrix. It is important, however, to distinguish between the symmetric and anti-symmetric parts of this matrix, so we write eq. (13.74) in the form

$$\delta x_\rho = e_{\rho\sigma} x_\sigma + \tilde{\omega}_{\rho\sigma} x_\sigma, \tag{13.75}$$

where

$$e_{\rho\sigma} = \frac{1}{2}\left(\frac{\partial v_\rho}{\partial x_\sigma} + \frac{\partial v_\sigma}{\partial x_\rho}\right), \tag{13.76}$$

and

$$\tilde{\omega}_{\rho\sigma} = \frac{1}{2}\left(\frac{\partial v_\rho}{\partial x_\sigma} - \frac{\partial v_\sigma}{\partial x_\rho}\right). \tag{13.77}$$

First we shall show that the term $\tilde{\omega}_{\rho\sigma} x_\sigma$ in (13.75) does not in fact represent any strain of the solid, but merely a rotation in space. For since $\tilde{\omega}_{\rho\sigma}$ is antisymmetrical, we can write

$$\tilde{\omega}_{32} = -\tilde{\omega}_{23} = \theta_1, \qquad \tilde{\omega}_{13} = -\tilde{\omega}_{31} = \theta_2, \qquad \tilde{\omega}_{21} = -\tilde{\omega}_{12} = \theta_3;$$

then this second term is

$$\begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \theta_2 x_3 - \theta_3 x_2 \\ \theta_3 x_1 - \theta_1 x_3 \\ \theta_1 x_2 - \theta_2 x_1 \end{pmatrix}, \tag{13.78}$$

which is just $\boldsymbol{\theta}\times\boldsymbol{x}$ in the vector notation of Ch. 9 eq. (9.72). So the term $\tilde{\omega}_{\rho\sigma} x_\sigma$ represents a rotation of that part of the solid near the origin, of angle $\theta = |\boldsymbol{\theta}|$ about an axis in the direction of $\boldsymbol{\theta}$.

So the actual deformation of the body Δx_ρ say, is given by the first term in eq. (13.75):

$$\Delta x_\rho = e_{\rho\sigma} x_\sigma,$$

or

$$\Delta \boldsymbol{x} = \boldsymbol{E}\boldsymbol{x}, \tag{13.79}$$

where $\boldsymbol{E} \equiv (e_{\rho\sigma})$ is a symmetrical matrix. If we now choose a new set of rectangular coordinates x'_τ related to x_σ by (13.38) or (13.42), then

$\boldsymbol{x}=\boldsymbol{U}\boldsymbol{x}'$ and the displacements $\Delta\boldsymbol{x}$ and $\Delta\boldsymbol{x}'$ are similarly related:

$$\Delta\boldsymbol{x} = \boldsymbol{U}\Delta\boldsymbol{x}'.$$

Thus in the new coordinates, the equations (13.79) for the displacement vector becomes

$$\boldsymbol{U}\Delta\boldsymbol{x}' = \boldsymbol{E}\boldsymbol{U}\boldsymbol{x}',$$

or

$$\Delta\boldsymbol{x}' = (\bar{\boldsymbol{U}}\boldsymbol{E}\boldsymbol{U})\boldsymbol{x}', \tag{13.80}$$

remembering (13.41). Thus the symmetric matrix $\boldsymbol{E}$ transforms like a tensor under change of axes; it is known as the *strain tensor.*

We can now apply the theorem (13.51), choosing the coordinate system so that $\bar{\boldsymbol{U}}\boldsymbol{E}\boldsymbol{U}$ in (13.80) is a diagonal matrix. Equations (13.80) take a particularly simple form in this coordinate system, the component $\Delta x_1'$ of $\Delta\boldsymbol{x}'$ depending only on coordinate x_1' and so on. In other words, the displacements of any point lying on one of the coordinate axes is along that coordinate axis. These three mutually perpendicular coordinate axes are called the *principal axes of strain.* The strain has been analysed only in the neighbourhood of a single point of the solid; in general the strain tensor $\boldsymbol{E}$ will be a function of position in the solid, and the principal axes will usually lie in different directions at different points.

§ 7.3. Tensors of higher order; the alternating tensor

The definition in § 7 of a second order Cartesian tensor can be generalised to nth order tensors; roughly speaking, an nth order Cartesian tensor in 3-space is an array $T_{\rho\sigma\ldots\tau}$ with n 'vector suffixes'. More precisely, $T_{\rho\sigma\ldots\tau}$ ($\rho, \sigma,\ldots, \tau=1, 2, 3$) are a set of $3n$ quantities which obey the transformation law

$$T_{\rho\sigma\ldots\tau} = u_{\rho\mu}\, u_{\sigma\nu}\cdots u_{\tau\eta} T'_{\mu\nu\ldots\eta}, \tag{13.81}$$

or

$$T'_{\mu\nu\ldots\eta} = u_{\rho\mu}\, u_{\sigma\nu}\cdots u_{\tau\eta} T_{\rho\sigma\ldots\tau},$$

under a change of axes corresponding to the orthogonal matrix $\boldsymbol{U}$.

The *alternating tensor* is a particularly important tensor of the third order in 3-space. It is represented by $\varepsilon_{\rho\sigma\tau}$ defined as follows:

$\varepsilon_{\rho\sigma\tau} = 0$, when two of ρ, σ, τ are equal,

$\varepsilon_{\rho\sigma\tau} = +1$, when ρ, σ, τ are unequal and in cyclic order,

$\varepsilon_{\rho\sigma\tau} = -1$, when ρ, σ, τ are unequal and not in cyclic order.

Therefore it has 27 components, 21 of which are zero, 3 are $+1$, 3 are -1. We must first show that it is a tensor. If so, its transformation should be

$$\varepsilon'_{\mu\nu\eta} = u_{\rho\mu}\, u_{\sigma\nu}\, u_{\tau\eta}\, \varepsilon_{\rho\sigma\tau}.$$

Using the definition of $\varepsilon_{\rho\sigma\tau}$, this result can be written out in full as

$$\begin{aligned} \varepsilon'_{\mu\nu\eta} &= u_{1\mu}\, u_{2\nu}\, u_{3\eta} + u_{2\mu}\, u_{3\nu}\, u_{1\eta} + u_{3\mu}\, u_{1\nu}\, u_{2\eta} \\ &\quad - u_{1\mu}\, u_{3\nu}\, u_{2\eta} - u_{2\mu}\, u_{1\nu}\, u_{3\eta} - u_{3\mu}\, u_{2\nu}\, u_{1\eta}. \end{aligned} \tag{13.82}$$

If now $\nu=\eta$, the right hand side is clearly zero; similarly when $\eta=\mu$ or $\mu=\nu$. Thus $\varepsilon'_{\mu\nu\eta}=0$ when two of μ, ν, η are equal. When μ, ν, η are all unequal, the expression on the right hand side in eq. (13.82) is the determinant

$$\begin{vmatrix} u_{1\mu} & u_{1\nu} & u_{1\eta} \\ u_{2\mu} & u_{2\nu} & u_{2\eta} \\ u_{3\mu} & u_{3\nu} & u_{3\eta} \end{vmatrix},$$

which is equal to $+1$ when μ, ν, η are in cyclic order, and equal to -1 when they are not in cyclic order. Hence $\varepsilon_{\rho\sigma\tau}$ is transformed into itself by the rule for transforming third order tensors.

Tensors whose components are unaltered by transformation of the axes are called *isotropic* tensors. It can be shown that the only isotropic tensors of the second and third orders are scalar multiples of $\delta_{\rho\sigma}$ and $\varepsilon_{\rho\sigma\tau}$.

§ 7.4. Contraction of tensors

Consider a second order tensor $T_{\rho\sigma}$ for which the transformation is

$$T_{\rho\sigma} = u_{\rho\mu}\, u_{\sigma\nu} T'_{\mu\nu}.$$

We can form a tensor of lower order by putting $\sigma=\rho$ in this result and summing for the repeated suffix ρ according to the summation convention, that is

$$T_{\rho\rho} = u_{\rho\mu}\, u_{\rho\nu} T'_{\mu\nu} = \delta_{\mu\nu} T'_{\mu\nu} = T'_{\mu\mu}.$$

This establishes the equality of the *traces* of $\boldsymbol{T}$ and $\boldsymbol{T}'$ (see Ex. 13.5, no. 5),

$$T_{11} + T_{22} + T_{33} = T'_{11} + T'_{22} + T'_{33};$$

so $T_{\rho\rho}$ is a scalar which is transformed into itself. Thus this operation has changed a second order tensor into a scalar.

The operation of putting two suffixes in a tensor equal, and summing accordingly, is known as *contraction*: in general it yields a new tensor whose order is less by 2 than the order of the original tensor.

Example 19

If we contract the second order tensor $T_{\rho\sigma}=v_\rho w_\sigma$, we obtain

$$T_{\rho\rho} = v_\rho w_\rho = v_1w_1 + v_2w_2 + v_3w_3,$$

which is the scalar product $\boldsymbol{v}\cdot\boldsymbol{w}$ of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Example 20

The second order tensor $\partial w_\rho/\partial x_\sigma$ gives, on contraction, the scalar

$$\frac{\partial w_\rho}{\partial x_\rho} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3},$$

which is called the 'divergence' of $\boldsymbol{w}$.

Example 21

Consider the fifth order tensor $\varepsilon_{\rho\sigma\tau}v_\mu w_\nu$ contracted twice by putting $\mu=\sigma$ and $\nu=\tau$ and summing accordingly. The result should be a first order tensor or vector, which we will call θ_ρ ($\rho=1, 2, 3$). Using the results for $\varepsilon_{\rho\sigma\tau}$, we have for $\rho=1$

$$\theta_1 = \varepsilon_{1\sigma\tau}v_\sigma w_\tau = \varepsilon_{123}v_2w_3 + \varepsilon_{132}v_3w_2 = v_2w_3 - v_3w_2,$$

whilst similarly

$$\theta_2 = v_3w_1 - v_1w_3,$$

and

$$\theta_3 = v_1w_2 - v_2w_1.$$

We see that these are the components of the vector product $\boldsymbol{v}\times\boldsymbol{w}$. Thus this vector product can be written in the tensor notation $\varepsilon_{\rho\sigma\tau}v_\sigma w_\tau$.

Example 22

By the same method as in Example 21 the tensor $\varepsilon_{\rho\sigma\tau}(\partial w_\sigma/\partial x_\tau)$ is a vector of components

$$\theta_1 = \frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_1}, \qquad \theta_2 = \frac{\partial w_3}{\partial x_1} - \frac{\partial w_1}{\partial x_3}, \qquad \theta_3 = \frac{\partial w_1}{\partial x_2} - \frac{\partial w_2}{\partial x_1},$$

and is called the 'curl' of the vector $\boldsymbol{w}$. Thus curl $\boldsymbol{w}$ is a first order tensor or vector which can be written in the tensor notation $\varepsilon_{\rho\sigma\tau}(\partial w_\sigma/\partial x_\tau)$.

CURVILINEAR COORDINATES AND MULTIPLE INTEGRALS

§ 1. Scalar and vector fields

In the study of the dynamics of particles or rigid bodies, we are faced with the problem of predicting how a small number of variables (for instance, the three coordinates of a particle) will vary with time. In physics however we often have to deal with the properties of substances spread throughout a continuous region, such as liquids and gases, and we cannot describe these properties in detail by a reasonably small number of variables. Gases for instance contain about 10^{23} or 10^{24} molecules per litre at S.T.P. Clearly it is impossible to solve the equations of motion for the molecules in detail, even if one knew what initial conditions to impose on the solution. One way to surmount this difficulty is to use the statistical method; this consists of calculating the average properties of molecules in the substance, thereby predicting such properties as the pressure and the specific heats. Another method, with which we are now concerned, is to treat the substance as a 'continuous medium', whose properties are smoothly varying functions of the position vector $\boldsymbol{r}$. For example, the density ρ of a gas may vary throughout the volume it occupies, and we must discuss this density as a function of position $\rho(\boldsymbol{r})$. We can say loosely that the density is given by an infinite number of variables, one for each point within the volume. Quantities such as the density or the velocity of a fluid, the gravitational potential or the strength of an electric field, which are defined throughout a continuous region of space, are known as *field quantities*, or simply as *fields*. We shall deal only with fields in 3-space; the results we obtain can usually be generalised to fields in higher dimensional spaces; fields in the 4-space occur naturally in relativistic mechanics.

Many important field quantities in physics consist of a single quantity $\psi(\boldsymbol{r})$ defined at each point $\boldsymbol{r}$ of space, the value at any particular point being independent of any coordinate system used. Such fields are called *scalar fields*; examples are the density and the pressure within a fluid,

the height h above sea level on the earth's surface, the gravitational potential (which is just gh locally on the earth's surface), and the potential in an electric field. Other field quantities have vector properties, being described by both a magnitude and a direction at each point within a continuous region; these are called *vector fields*, and it is important to remember that both the magnitude and the direction may vary from point to point throughout the region. Examples of vector fields are the velocity in a fluid, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ in space, and the intensity of magnetisation $\boldsymbol{I}$ within magnetic material. If we use a system of rectangular coordinates (x, y, z) then any vector field $\boldsymbol{w}(\boldsymbol{r})$ has three components $w_x(\boldsymbol{r})$, $w_y(\boldsymbol{r})$ and $w_z(\boldsymbol{r})$; each component is a function of the coordinates, and if we change our system of coordinates the quantities (w_x, w_y, w_z) obey the vector law of transformation, expressed in matrix form by (13.38), at each point $\boldsymbol{r}$ in space. We shall frequently write scalar and vector fields simply as ψ and $\boldsymbol{w}$, their dependence on $\boldsymbol{r}$ being understood. Fields may of course depend on time, but for the present we are concerned only with the analysis of fields in one given configuration.

Example 1

If a vector field $\boldsymbol{w}$ is of the form

$$\boldsymbol{w} = (w_x(x, y), w_y(x, y), 0)$$

it is called a *two-dimensional field*. The relation between the rectangular coordinates (x, y) and another pair (x', y') in the same plane are as in Ch. 12 § 2.7, Example 13,

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta.$$

The components (w_x', w_y') of the vector $\boldsymbol{w}$ in the (x', y') frame are given by the inverse transformation

$$w_x' = w_x \cos \theta + w_y \sin \theta$$

$$w_y' = -w_x \sin \theta + w_y \cos \theta.$$

We must remember that w_x, w_y are defined as functions of x, y; so the expressions for w_x' in terms of x', y' is, in full,

$$w_x' = w_x(x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta)\cos \theta$$
$$+ w_y(x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta)\sin \theta.$$

The expression for w_y' is similar.

Example 2

A two-dimensional field $\boldsymbol{w}$ is given by

$$\boldsymbol{w} = (x^2 + y^2 + \lambda(x + y), x^2 + y^2 + \lambda(x - y), 0).$$

If (x', y') are coordinates along axes at an angle $\frac{1}{4}\pi$ to the (x, y) axes, then as in Example 1,

$$x' = (x + y)/\sqrt{2}$$
$$y' = (-x + y)/\sqrt{2}$$

and $x'^2+y'^2=x^2+y^2$. Therefore

$$(w_x, w_y) = (x'^2 + y'^2 + \lambda\sqrt{2}\,x', x'^2 + y'^2 - \lambda\sqrt{2}\,y').$$

Hence, as in Example 1,

$$w'_x = (w_x + w_y)/\sqrt{2} = \lambda(x' - y') + (x'^2 + y'^2)\sqrt{2}$$
$$w'_y = (-w_x + w_y)/\sqrt{2} = -\lambda(x' + y').$$

Example 3

Suppose that the components w_ρ of a vector field in a frame F are linear functions of the coordinates x_ρ, so that we can write in matrix form

$$w_\rho = a_{\rho\sigma}x_\sigma$$

or

$$\boldsymbol{w} = \boldsymbol{A}\boldsymbol{x}.$$

If F′ is a frame related to F by the orthogonal matrix $\boldsymbol{U}$, then by (13.38),

$$\boldsymbol{w} = \boldsymbol{U}\boldsymbol{w}',$$

where $\boldsymbol{w}'$ stands for the components relative to the frame F′. In particular, $\boldsymbol{x}=\boldsymbol{U}\boldsymbol{x}'$. Therefore

$$\boldsymbol{w}' = \boldsymbol{U}^{-1}\boldsymbol{w} = \bar{\boldsymbol{U}}\boldsymbol{w} = \bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{x} = \bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}\boldsymbol{x}'.$$

If the matrix $\boldsymbol{A}$ is symmetric, we can take the eigenvectors of $\boldsymbol{A}$ as the coordinate axes in F′. Then $\bar{\boldsymbol{U}}\boldsymbol{A}\boldsymbol{U}=\boldsymbol{\Lambda}$, a diagonal matrix. Thus the components w'_ρ in frame F′ have the simple form $w'_1=\lambda_1x'_1$, $w'_2=\lambda_2x'_2$, $w'_3=\lambda_3x'_3$.

§ 1.1. Level surfaces of a scalar field

The use of contour lines to indicate the height of land above sea level is a familiar feature of Ordnance Survey maps. If we were to specify the position on a map by two rectangular coordinates (x, y) referred to chosen axes, then the height $h(x, y)$ at any point (x, y) would be a scalar function of position; in other words, $h(x, y)$ would be a scalar field in the 2-space of the map. Contour lines pass through points at the same height; if h_0 is the common height along a contour line, then

$$h(x, y) = h_0 \tag{14.1}$$

at all points on the line. Equation (14.1) is therefore the equation of the contour line, and if we vary h_0 we obtain the equations of all contour lines. So we may regard (14.1) as defining the family of curves on a map known as the contour lines, h_0 being the parameter of the family. We note that normally no two contour lines on a map intersect, since one point on the map cannot have two different heights unless the land has some exceptional feature such as a vertical cliff.

This picture can be generalised to scalar fields in 3-space. Suppose that $\psi(\boldsymbol{r})$ is a given scalar field and (x, y, z) are a set of rectangular coordinates; then the points at which ψ has a given value ψ_0 satisfy

$$\psi(x, y, z) = \psi_0. \tag{14.2}$$

For fixed ψ_0, (14.2) defines a surface in 3-space analogous to the contour line (14.1) in 2-space. This surface is called a *level surface* of the scalar field ψ. If we allow the parameter ψ_0 in (14.2) to vary, we obtain a family of level surfaces in 3-space, ψ_0 being the parameter of the family. We assume that a physical field $\psi(\boldsymbol{r})$ is a single valued function of $\boldsymbol{r}$; it follows that there will be one and only one level surface of ψ passing through any point (x, y, z), namely, the surface (14.2) with the appropriate constant ψ_0. We have in fact tacitly assumed that equation (14.2) defines a surface and not a set of points scattered in some peculiar way through space. Physical fields are generally 'well-behaved' in a mathematical sense; that is to say, they and any partial derivatives that we encounter exist and are continuous at most points in space. However, there may well be isolated points or particular lines or surfaces where a physical field is not well-behaved, and these must be treated with care; at present however we are discussing general properties and we shall ignore these difficulties. We shall not therefore always state precise conditions to be satisfied by field functions in our analysis; when we do mention precise conditions, we shall try to explain the physical importance of these conditions.

Example 4

The electrostatic potential at $\boldsymbol{r}$ produced by a point charge e placed at $\boldsymbol{r}'$ is $\psi(\boldsymbol{r}) = e/|\boldsymbol{r}-\boldsymbol{r}'|$. The level surfaces $\psi = \psi_0$ (constant) are spheres with centres at $\boldsymbol{r}'$, with equations

$$|\boldsymbol{r} - \boldsymbol{r}'| = e/\psi_0 = a, \text{ say.}$$

The common potential on a sphere of radius a with centre at $\boldsymbol{r}'$ is $\psi_0 \equiv e/a$.

§ 2. Curvilinear coordinates

The position vector $\boldsymbol{r}$ in 3-space can be specified by the coordinates (x, y, z) relative to a given set of rectangular axes. However, as we have seen in Ch. 10 § 2, there are other ways of specifying $\boldsymbol{r}$, and many problems can best be tackled by using non-rectangular coordinates. For example, we know that it is helpful to use polar coordinates (r, θ) in certain types of two-dimensional problems in dynamics. The rectangular coordinates (x, y, z) can in principle be determined if the values u_1, u_2, u_3 of three independent functions f_1, f_2, f_3 of x, y, z are given; that is,

$$\begin{aligned} f_1(x, y, z) &= u_1 \\ f_2(x, y, z) &= u_2 \\ f_3(x, y, z) &= u_3 \end{aligned} \tag{14.3}$$

in principle determine x, y, z. For example, equations (10.7) determine x, y, z when r, θ, φ are given; x, y, z are in fact of the form (10.6). If u_1, u_2, u_3 are changed, then different values x, y, z will be obtained. So points in 3-space can be specified by giving the quantities u_k $(k=1, 2, 3)$, which are known as *curvilinear coordinates.*

§ 2.1. Ranges of coordinates

We shall assume that equations (14.3) can be solved, determining x, y, z, as functions X, Y, Z of the parameters u_k:

$$\begin{aligned} x &= X(u_1, u_2, u_3), \\ y &= Y(u_1, u_2, u_3), \\ z &= Z(u_1, u_2, u_3). \end{aligned} \tag{14.4}$$

Generally the parameters u_k do not need to take all values in the range $(-\infty, \infty)$ in order that x, y, z shall take all values in $(-\infty, \infty)$ independently. For example, consider the relations in the form (14.4) between rectangular coordinates (x, y) and polar coordinates (r, θ) in a plane; these relations are

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta, \end{aligned} \tag{14.5}$$

r and θ taking the place of u_1 and u_2. If we make r vary from 0 to ∞ while keeping θ constant, then from equations (14.5) we see that the

point $P(x, y)$ will trace out the line OP_∞ shown in fig. 14.1. If further we make θ vary from 0 to 2π, this line will rotate from OA_∞ ($\theta=0$) in an anti-clockwise sense, sweeping over the whole plane until it returns to its original position OA_∞ when $\theta=2\pi$. Thus the whole plane is covered by letting r and θ vary in the ranges $0 \leqslant r < \infty$ and $0 \leqslant \theta < 2\pi$. Alternatively, if r varies from $-\infty$ to ∞, the line $P_{-\infty}P_\infty$ is traced out, and θ need only vary from 0 to π. The ranges of θ can be altered by letting the lines OP_∞ and $P_{-\infty}P_\infty$ begin sweeping out the plane from the position $\theta=\alpha$, for any angle α. Then the ranges are $0 \leqslant r < \infty$, $\alpha \leqslant \theta < \alpha + 2\pi$, or alternatively

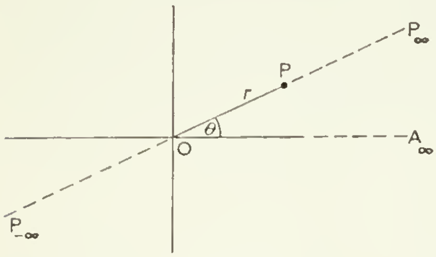

Fig. 14.1

$$-\infty < r < \infty, \quad \alpha \leqslant \theta < \alpha + \pi.$$

In practice therefore it is important to specify clearly the ranges of the coordinates u_k. We shall assume at present that u_k in their specified ranges determine x, y, z uniquely through (14.3) or (14.4); for example, we shall assume that the relations are not of the type $x^2=F(u_k)$ with x unrestricted, which would give two values of x for given values of u_k.

Example 5

Cylindrical polar coordinates (ρ, φ, z) are defined by

$$\rho = (x^2 + y^2)^{\frac{1}{2}}$$
$$\varphi = \tan^{-1}(y/x),$$

the rectangular coordinate z being retained. These coordinates are simply polar coordinates ρ, φ in the xy-plane with the z-coordinate added. From the discussion above of the ranges of polar coordinates, it is clear that the whole of 3-space corresponds to ranges $\rho \geqslant 0$, $0 \leqslant \varphi < 2\pi$, $-\infty < z < \infty$.

Example 6

Spherical polar coordinates (r, θ, φ) obey the relations

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta,$$

and the coordinates r, θ, φ are shown in fig. 14.2. Points with a given value of φ lie on a plane containing the z-axis (shaded in the figure); the whole plane corresponds to the ranges $r \geqslant 0$, $0 \leqslant \theta < 2\pi$. If φ varies, this plane rotates about the z-axis, and

sweeps out the whole 3-space as φ varies throughout $0 \leqslant \varphi < \pi$. In practice the range of θ is usually halved and that of φ doubled, the whole 3-space corresponding to the ranges $r \geqslant 0$, $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$.

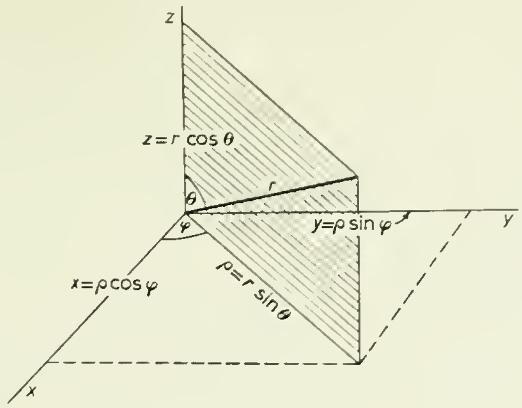

Fig. 14.2

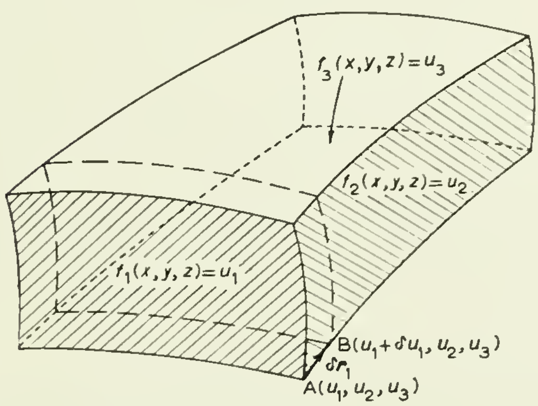

Fig. 14.3

§ 2.2. Displacements due to variations of coordinates

For fixed u_k, each of the equations (14.3) can be looked upon as the equation of a level surface in 3-space; the solution (x, y, z), given by (14.4), is the point of intersection of the three surfaces, and is the point labelled $A(u_1, u_2, u_3)$ in fig. 14.3. If u_1 is allowed to increase to $u_1+\delta u_1$, then the level surface $f_1(x, y, z)=u_1$ is replaced by a different level surface $f_1(x, y, z)=u_1+\delta u_1$ of the same family. If δu_1 is small, this surface, indicated by broken lines in fig. 14.3, will be near to the original surface $f_1=u_1$. The point of intersection of $f_1=u_1+\delta u_1$ with $f_2=u_2$ and $f_3=u_3$ is $B(u_1+\delta u_1, u_2, u_3)$, as shown; it lies on the line of intersection

of $f_2 = u_2$ and $f_3 = u_3$, along which only the parameter u_1 can vary. The displacement vector AB, which we denote by $\delta \boldsymbol{r}_1$, can be found from equations (14.4) by making the variation $u_1 \to u_1 + \delta u_1$. Using the Taylor expansion of Ch. 6 § 3, for the variable $u_1 + \delta u_1$, the corresponding small change in x is

$$\delta x_1 = X(u_1 + \delta u_1, u_2, u_3) - X(u_1, u_2, u_3) \approx \frac{\partial X(u_1, u_2, u_3)}{\partial u_1} \delta u_1,$$

to first order in δu_1. Similar expressions hold for δy_1 and δz_1, so that to first order in δu_1,

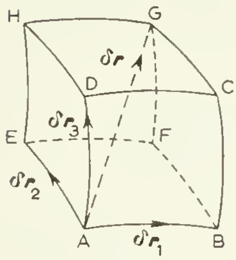

Fig. 14.4

$$\delta \boldsymbol{r}_1 \equiv \mathsf{AB} = (\delta x_1, \delta y_1, \delta z_1) \approx \left(\frac{\partial X}{\partial u_1}, \frac{\partial Y}{\partial u_1}, \frac{\partial Z}{\partial u_1} \right) \delta u_1. \tag{14.6}$$

Small variations δu_2 and δu_3 of the parameters u_2 and u_3 will likewise cause small displacements $\delta \boldsymbol{r}_2$ and $\delta \boldsymbol{r}_3$ along the other lines of intersection of pairs of level surfaces, as shown in fig. 14.4. Equation (14.6) can be immediately generalised to give

$$\delta \boldsymbol{r}_k = (\delta x_k, \delta y_k, \delta z_k) \approx \left(\frac{\partial X}{\partial u_k}, \frac{\partial Y}{\partial u_k}, \frac{\partial Z}{\partial u_k} \right) \delta u_k, \tag{14.7}$$

for $k = 1, 2, 3$.

§ 2.3. Orthogonal curvilinear coordinates

The most important type of curvilinear coordinates are *orthogonal curvilinear coordinates*, or simply *orthogonal coordinates*. For such systems of coordinates, the three surfaces (14.3) are mutually perpendicular, whatever the values of u_k; this is the same as saying that as $\delta u_k \to 0$, the three small displacements $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$, $\delta \boldsymbol{r}_3$ become mutually perpendicular at every point in 3-space, so that in this limit

$$(\delta \boldsymbol{r}_k \cdot \delta \boldsymbol{r}_l) = 0 \qquad (k \neq l); \tag{14.8}$$

the small solid in fig. 14.4, with edges along $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$, $\delta \boldsymbol{r}_3$ is then a rectangular parallelepiped. Using equations (14.7), conditions (14.8) can be expressed as

$$\frac{\partial X}{\partial u_k} \frac{\partial X}{\partial u_l} + \frac{\partial Y}{\partial u_k} \frac{\partial Y}{\partial u_l} + \frac{\partial Z}{\partial u_k} \frac{\partial Z}{\partial u_l} = 0 \qquad (k \neq l). \tag{14.9}$$

If variations δu_1, δu_2, δu_3 are made in the coordinates u_k simultaneously, the corresponding displacement will be the vector sum

$$\delta \boldsymbol{r} = \sum_{k=1}^{3} \delta \boldsymbol{r}_k \tag{14.10}$$

represented by AG in fig. 14.4. The magnitude of the individual displacements $\delta \boldsymbol{r}_k$ are, from (14.7)

$$|\delta \boldsymbol{r}_k| \approx h_k \delta u_k \tag{14.11}$$

where

$$h_k = \left[\left(\frac{\partial X}{\partial u_k}\right)^2 + \left(\frac{\partial Y}{\partial u_k}\right)^2 + \left(\frac{\partial Z}{\partial u_k}\right)^2\right]^{\frac{1}{2}}. \tag{14.12}$$

For orthogonal coordinate systems, since the small solid in fig. 14.4 is rectangular, with $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$, $\delta \boldsymbol{r}_3$ perpendicular to each other, we can write (14.10) in the form

$$\delta \boldsymbol{r} \approx (h_1 \delta u_1,\ h_2 \delta u_2,\ h_3 \delta u_3),$$

these components being in the three orthogonal directions defined by the vectors $\delta \boldsymbol{r}_k$. It is important to realise that these three directions are in general different at different points in space. The three quantities h_k defined by (14.12) are in general functions of all the coordinates u_k; the evaluation of the functions h_k is an essential preliminary to using any particular orthogonal system of coordinates.

Example 7

Cylindrical polar coordinates (ρ, φ, z) have, in equations (14.4),

$$X = \rho \cos \varphi$$
$$Y = \rho \sin \varphi$$
$$Z = z.$$

Small changes $\delta\rho$, $\delta\varphi$ and δz produce spatial displacements

$$\delta \boldsymbol{r}_1 = (\cos \varphi,\ \sin \varphi,\ 0)\,\delta\rho$$
$$\delta \boldsymbol{r}_2 = (-\rho \sin \varphi,\ \rho \cos \varphi,\ 0)\,\delta\varphi$$
$$\delta \boldsymbol{r}_3 = (0,\ 0,\ 1)\,\delta z.$$

These three displacements obviously satisfy (14.8) for all values of ρ, φ, z, so the coordinates are orthogonal. Further, $|\delta \boldsymbol{r}_1| = \delta\rho$, $|\delta \boldsymbol{r}_2| = \rho\delta\varphi$, $|\delta \boldsymbol{r}_3| = \delta z$, so that the general small displacement referred to the orthogonal system is $(\delta\rho,\ \rho\delta\varphi,\ \delta z)$; the quantities h_k defined by (14.11) or (14.12) are with h_1 written as h_ρ and so on,

$$h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1.$$

The three sets of orthogonal surfaces are

(i) ρ constant, the circular cylinders $x^2+y^2=\rho^2$ shown in fig. 14.5;
(ii) φ constant, the planes $y=x \tan \varphi$ through the z-axis;
(iii) z constant, the planes parallel to the xy-plane.

The curves of intersection of pairs of orthogonal surfaces are

(iv) the normal circular cross sections of the cylinders (i) by the planes (iii);
(v) the radii of the circles (iv);
(vi) the straight lines parallel to the z-axis on the surfaces of the cylinders (i).

For a typical point P these curves are indicated by thick lines in fig. 14.5. They are clearly mutually perpendicular and the three families of surfaces are mutually orthogonal families.

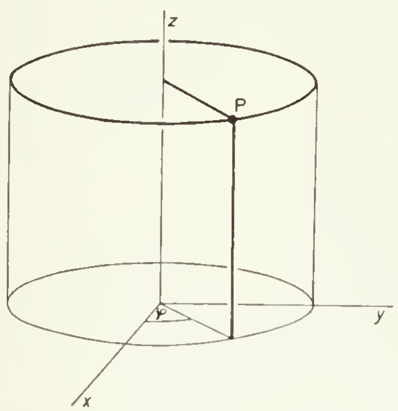

Fig. 14.5

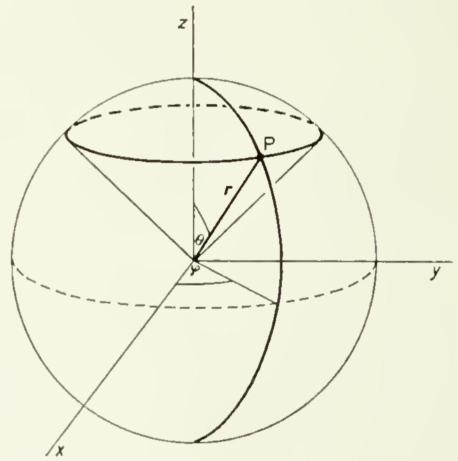

Fig. 14.6

Example 8

Spherical polar coordinates (r, θ, φ) have

$$X = r \sin \theta \cos \varphi$$
$$Y = r \sin \theta \sin \varphi$$
$$Z = r \cos \theta.$$

Small changes δr, $\delta\theta$, $\delta\varphi$ produce displacements

$$\delta \mathbf{r}_1 = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\, \delta r,$$
$$\delta \mathbf{r}_2 = (r \cos \theta \cos \varphi, r \cos \theta \sin \varphi, -r \sin \theta)\, \delta\theta,$$
$$\delta \mathbf{r}_3 = (-r \sin \theta \sin \varphi, r \sin \theta \cos \varphi, 0)\, \delta\varphi.$$

These displacement vectors are clearly orthogonal for all (r, θ, φ) and the general small displacement, referred to the orthogonal axes, is $(h_r \delta r, h_\theta \delta\theta, h_\varphi \delta\varphi)$, where

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r \sin \theta.$$

The three families of orthogonal surfaces are

(i) r constant, the spheres $x^2+y^2+z^2=r^2$ with centres at the origin, as shown in fig. 14.6;

(ii) θ constant, the right circular cones $x^2+y^2=z^2 \tan^2 \theta$ with axes along the z-axis;
(iii) φ constant, the planes $y=x \tan \varphi$ through the z-axis.
The curves of intersection of pairs of these surfaces are
(iv) the radii of the spheres (i), which are the intersections of surfaces (ii) and (iii);
(v) the circles of intersection of the spheres (i) and the cones (ii), corresponding to the circles of latitude on the surface of the earth.
(vi) the circles of intersection of the spheres (i) and planes (iii), corresponding to the circles of longitude on the surface of the earth.

For a typical point P these curves are shown in fig. 14.6 as thick lines; these curves and the surfaces are clearly mutually orthogonal.

Example 9

Oblate spheroidal coordinates (ξ, η, φ) are related to rectangular coordinates (x, y, z) by

$$\begin{aligned} x &= c \cosh \xi \cos \eta \cos \varphi \\ y &= c \cosh \xi \cos \eta \sin \varphi \\ z &= c \sinh \xi \sin \eta. \end{aligned}$$

The coordinates ξ, η are related to the cylindrical polar coordinates ρ, z of Example 7 by

$$\begin{aligned} \rho &= c \cosh \xi \cos \eta \\ z &= c \sinh \xi \sin \eta, \end{aligned}$$

φ being a common coordinate of the two systems. If the range of φ is $0 \leqslant \varphi < 2\pi$, the range of ρ is $\rho \geqslant 0$, while z can take all real values. The ranges of ρ and z are covered independently if we let ξ and η vary in the ranges $-\infty < \xi < \infty$, $0 \leqslant \eta < \frac{1}{2}\pi$. We can, if we wish, halve the range of φ and double the range of η, so that 3-space corresponds to $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \eta < \pi$, ξ taking all real values.

Incremental changes $\delta\xi$, $\delta\eta$ and $\delta\varphi$ produce first order displacements

$$\begin{aligned} \delta \boldsymbol{r}_1 &= c(\sinh \xi \cos \eta \cos \varphi,\ \sinh \xi \cos \eta \sin \varphi,\ \cosh \xi \sin \eta)\,\delta\xi \\ \delta \boldsymbol{r}_2 &= c(-\cosh \xi \sin \eta \cos \varphi,\ -\cosh \xi \sin \eta \sin \varphi,\ \sinh \xi \cos \eta)\,\delta\eta \\ \delta \boldsymbol{r}_3 &= c(-\cosh \xi \cos \eta \sin \varphi,\ \cosh \xi \cos \eta \cos \varphi,\ 0)\,\delta\varphi; \end{aligned}$$

these displacements are clearly mutually orthogonal, and if their lengths are $h_\xi \delta\xi$, $h_\eta \delta\eta$, $h_\varphi \delta\varphi$, then it is not difficult to show that

$$h_\xi = h_\eta = c(\cosh^2 \xi - \cos^2 \eta)^{\frac{1}{2}}$$

while $h_\varphi = c \cosh \xi \cos \eta = \rho$, as we know already. One set of orthogonal surfaces are given by taking φ constant, and as in Example 7 are planes through the z-axis. The other orthogonal surfaces are best pictured by considering their intersections with one of these planes; ρ, z are rectangular coordinates in such a plane, as in fig. 14.7, and from the relations between (ρ, z) and (ξ, η) we find that the curves with ξ constant are

$$\frac{\rho^2}{c^2 \cosh^2 \xi} + \frac{z^2}{c^2 \sinh^2 \xi} = 1.$$

This curve is an ellipse which intersects the ρ-axis at $\rho = \pm c \cosh \xi$ and the z-axis at $\rho = \pm c \sinh \xi$, as shown in fig. 14.7. The surface in 3-space with ξ constant is given by rotating this ellipse round the z-axis, giving an oblate spheroid. A curve in the (ρ, z) plane with η constant is

$$\frac{\rho^2}{c^2 \cos^2 \eta} - \frac{z^2}{c^2 \sin^2 \eta} = 1,$$

a hyperbola which intersects the ρ-axis when $\rho = \pm c \cos \eta$. The surface in 3-space with η constant is the hyperboloid of one sheet formed by rotating this hyperbola about the z-axis.

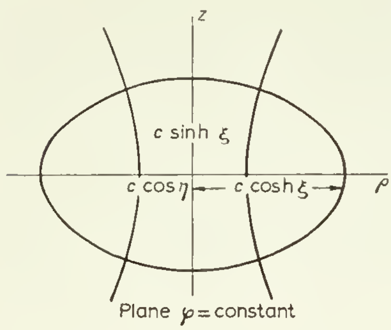

Fig. 14.7

EXERCISE 14.1

1. Show that the following pairs of relations define orthogonal coordinates in a plane:

(i) $x = \xi^2 - \eta^2, \qquad y = 2\xi\eta;$

(ii) $x = e^{\xi}(\xi \cos \eta - \eta \sin \eta), \qquad y = e^{\xi}(\eta \cos \eta + \xi \sin \eta);$

(iii) $x = \dfrac{c \sin v}{\cosh u - \cos v}, \qquad y = \dfrac{c \sinh u}{\cosh u - \cos v}.$

Find the ranges of the orthogonal coordinates corresponding to the whole xy-plane, and the incremental displacements in the orthogonal directions. Discuss the nature of the orthogonal sets of curves defined by the curvilinear coordinates.

2. Show that the following relations define sets of orthogonal coordinates:

(i) $x = \operatorname{sech} \lambda \sec \mu \cos \varphi$
$y = \operatorname{sech} \lambda \sec \mu \sin \varphi$
$z = \tanh \lambda \tan \mu;$

(ii) $x = c \sinh \xi \sin \eta \cos \varphi$
$y = c \sinh \xi \sin \eta \sin \varphi$
$z = c \cosh \xi \cos \eta.$

Find the ranges of the coordinates corresponding to the whole of 3-space, and the incremental displacements in the orthogonal sets of surfaces defined by the two coordinate systems.

§ 3. Multiple integrals

Curvilinear and orthogonal systems of coordinates are of great value when we wish to define and evaluate integrals over areas in a plane and over

volumes in 3-space. We shall see that particular integrals can be simplified by an appropriate choice of coordinate system (a) by simplification of the integrand and (b) by simplifying the definition of the range or 'region' of integration. We are now going to discuss the specification of areas in a plane, prior to the definition of integration over plane regions; we must bear in mind that while it is often useful to use coordinates which simplify the task of specifying an area, we must be prepared to use coordinates which are not particularly well adapted to this purpose.

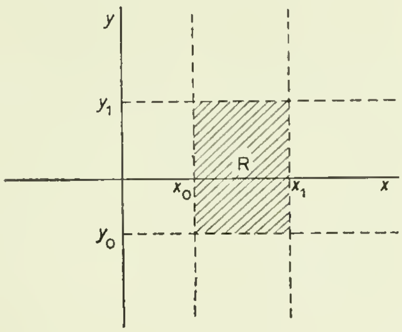

Fig. 14.8

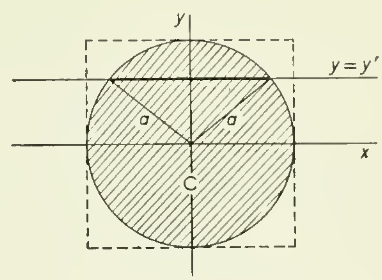

Fig. 14.9

§ 3.1. The specification of areas in a plane

If x and y are rectangular coordinates in a plane, then any continuous region in the plane corresponds to a certain range of values of the pair of variables (x, y). For example, the shaded rectangle R in fig. 14.8 corresponds to the range $x_0 \leqslant x \leqslant x_1$, $y_0 \leqslant y \leqslant y_1$. The shaded area C in fig. 14.9, bounded by the circle of radius a with centre at the origin, corresponds to real values of x and y for which $x^2+y^2 \leqslant a^2$. This restriction certainly implies that $|x| \leqslant a$ and $|y| \leqslant a$; but these two restrictions are not sufficient to ensure that (x, y) is within C — they only limit (x, y) to the square circumscribing C, shown by dotted lines in fig. 14.9. Suppose we consider a fixed value y' of y such that $y' \leqslant a$; then the allowed values of x lie on the thickened portion of $y=y'$, so that the range of x is $[-(a^2-y'^2)^{\frac{1}{2}}, +(a^2-y'^2)^{\frac{1}{2}}]$. The region C therefore corresponds to the ranges $|y| \leqslant a$, $|x| \leqslant (a^2-y^2)^{\frac{1}{2}}$. Since the original restriction $x^2+y^2 \leqslant a^2$ is symmetrical between x and y, we can equally well specify C by the restrictions $|x| \leqslant a$, $|y| \leqslant (a^2-x^2)^{\frac{1}{2}}$. The essential point is that, except for rectangles such as R in fig. 14.8, regions in the xy-plane correspond to ranges of x and y which are not independent.

Example 10

The triangle bounded by the axes $x=0$, $y=0$, and the line $ax+by=1$ $(a, b>0)$, is shown in fig. 14.10. Clearly all points in the triangle have $0 \leqslant y \leqslant b^{-1}$. For any given value of y in this range, x takes values all along the thickened segment, range from $x=0$ to $x=a^{-1}(1-by)$. Thus the triangle corresponds to the ranges $0 \leqslant y \leqslant b^{-1}$, $0 \leqslant x \leqslant a^{-1}(1-by)$. Alternatively, the triangle corresponds to the ranges $0 \leqslant x \leqslant a^{-1}$, $0 \leqslant y \leqslant b^{-1}(1-ax)$.

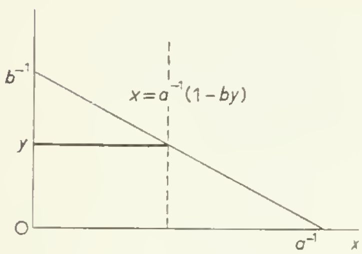

Fig. 14.10

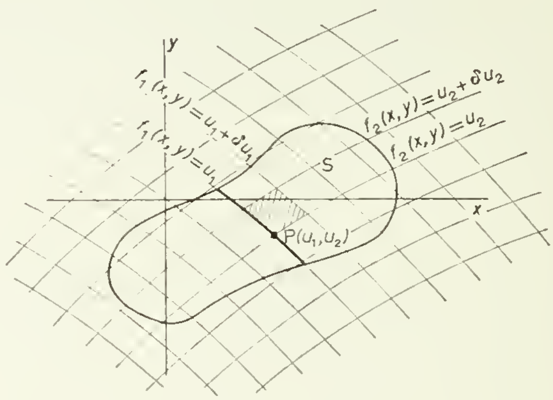

Fig. 14.11

It has probably already occurred to the reader that it would be much easier to specify the circular area C of fig. 14.9 in terms of polar coordinates (r, θ), since the ranges of these variables for the circle, namely $0 \leqslant r \leqslant a$, $0 \leqslant \theta < 2\pi$, are independent of each other. Likewise, a sector of a circle of radius a, subtending an angle α at the centre, can be specified most easily by the ranges $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \alpha$ of polar coordinates with origin at the centre of the circle.

The use of polar coordinates to specify a circular area exemplifies the use of a pair of curvilinear coordinates (u_1, u_2) to specify an area. Generally, curvilinear coordinates in a plane are defined by two relations

$$\begin{aligned} f_1(x, y) &= u_1, \\ f_2(x, y) &= u_2. \end{aligned} \tag{14.13}$$

These equations define two families of curves, with u_1 and u_2 as parameters, which form a 'mesh' covering the plane, as in fig. 14.11. Given values of u_1 and u_2 define the point of intersection $P(u_1, u_2)$ of a curve of each family, and if u_1 is fixed, the values of u_2 corresponding to points within S (along the thickened segment) will in general depend upon u_1. However, as with polar coordinates for the circle C, we can sometimes choose curvilinear coordinates for an area S whose ranges are independent.

Example 11

The shaded region in fig. 14.12 is enclosed by the axes $x=0$, $y=0$, and by one branch of the rectangular hyperbola $xy=a^2$. It is often specified in terms of the curvilinear coordinates

$$u_1 = 2xy$$
$$u_2 = x^2 - y^2.$$

The range of u_1 is clearly $(0, \frac{1}{2}a^2)$. For any given value of u_1, the point (x, y) lies on the hyperbola $xy=\frac{1}{2}u_1$, shown by the dotted line. On this hyperbola, $u_2=x^2-y^2$ varies over the complete range

$$-\infty < u_2 < \infty,$$

so that the area is specified by

$$0 < u_1 < 2a^2, \quad -\infty < u_2 < \infty.$$

The half of this region enclosed by the (thickened) lines $x=y$, $y=0$ and $xy=a^2$, corresponds to the ranges $0<u_1<2a^2$ and $0<u_2<\infty$, since on the hyperbola $xy=\frac{1}{2}u_1$, u_2 increases continuously from zero at the point of intersection with $x=y$, up to infinity.

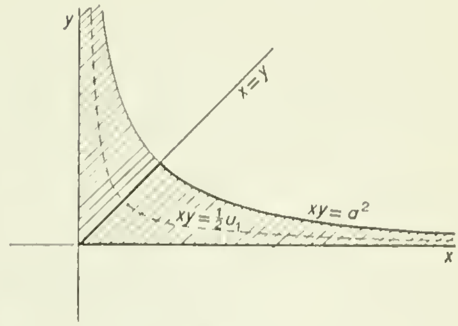

Fig. 14.12

The coordinates u_1, u_2 in Example 11 can easily be seen to be orthogonal. It is not always possible to choose orthogonal coordinate systems to specify areas in this way. For instance, if in Example 10 we use a coordinate u_1 which is constant on the line $ax+by=1$, then the other coordinate of an orthogonal system must be $u_2=bx-ay$ (or a multiple of this). On the lines $x=0$, $y=0$, both u_1 and u_2 vary, so that the limits of these orthogonal coordinates depend upon each other. In fact, the rectangular coordinates x, y are the simplest orthogonal system of coordinates we can use to specify this triangular area. Thus it is important to learn to specify areas in terms of any pair of coordinates, irrespective of whether they vary between independent limits.

§ 3.2. Integrals over two rectangular coordinates

In defining the definite integral $\int_a^b f(x)\,dx$ of a function of x, the interval (a, b) is divided into small intervals δx_r; then the integral is the limit of the sum

$$\sum_r f(\xi_r)\,\delta x_r$$

as all the intervals $\delta x_r \to 0$, ξ_r being any point in the rth interval. To

define the *double integral*

$$\iint_S f(x, y)\,dx\,dy \tag{14.14}$$

of a function of two variables x, y, over a given area S of the xy-plane, we similarly divide the area into small elements by a 'mesh' of lines parallel to the x-axis and y-axis, as in fig. 14.13. We shall assume that

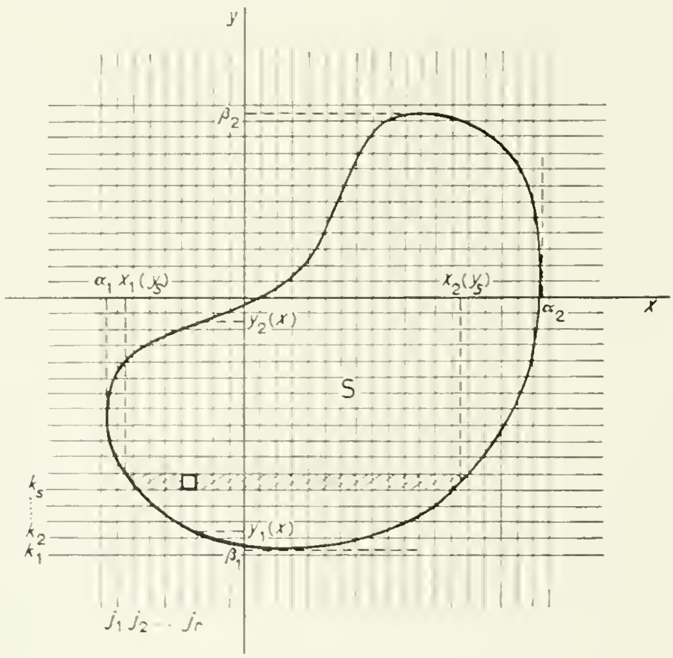

Fig. 14.13

the boundary of S is a simple closed curve which does not cross itself, and for the present that any line parallel to the x-axis or y-axis cuts the boundary at no more than two points; the second restriction will be removed later on.

Let the distances between the lines of the mesh with x constant be $j_1, j_2, \ldots, j_r, \ldots$, and those between the lines with y constant be $k_1, k_2, \ldots, k_s, \ldots$, as shown. Then the rectangle labelled by suffixes r, s, heavily outlined in fig. 14.13, has area $j_r k_s$. Now suppose that $f(x, y)$ is a continuous function of x and y, and that (x_r, y_s) is some point within the (r, s) rectangle. Then as every $j_r \to 0$ and every $k_s \to 0$, so that the mesh becomes finer and finer, the value of $f(x_r, y_s)$ will in the limit be the same for all points within the (r, s) rectangle. We define the double integral

(14.14) over the region S as

$$\lim_{j_r, k_s \to 0} \sum_{r,s} f(x_r, y_s)\, j_r\, k_s, \tag{14.15}$$

where $\sum_{r,s}$ implies summation over all rectangular elements within S. This is directly analogous to the definition of the integral of a function of one variable, since in (14.15) we weight each element of area $j_r k_s$ by multiplying by the value of $f(x, y)$ within that element, and then sum over all elements of S. In general, the boundary of S will intersect a number of elements of the mesh. For finite functions $f(x, y)$, it does not matter whether we include these elements in the sum (14.15) or not, for in the limit of a fine mesh, these elements combined give small total area close to the boundary of S which tends to zero as all j_r, $k_s \to 0$.

Let us perform the summation in (14.15) by summing first over r, keeping s fixed, and then summing over s; we therefore write (14.15) as

$$\lim_{j_r, k_s \to 0} \sum_{s} k_s \sum_{r} f(x_r, y_s)\, j_r. \tag{14.16}$$

The summation $\sum_r f(x_r, y_s)\, j_r$ is along the shaded strip in fig. 14.13 so that y_s can be assumed to be the same for all terms in the sum. In the limit $j_r \to 0$, this sum is just the integral

$$g(y_s) = \int_{x_1(y_s)}^{x_2(y_s)} f(x, y_s)\, \mathrm{d}x \tag{14.17}$$

along the shaded strip, y_s being treated as a constant parameter; the limits of integration $x_1(y_s)$ and $x_2(y_s)$ shown in fig. 14.13, are in general dependent on the value of y_s in the way we discussed in § 2.1. So the integral (14.17) depends on the value y_s both through the integrand $f(x, y_s)$ and through the limits of integration. The double sum (14.16) then becomes

$$\lim_{k_s \to 0} \sum_{s} k_s\, g(y_s) = \int_{\beta_1}^{\beta_2} g(y)\, \mathrm{d}y, \tag{14.18}$$

the limits β_1 and β_2 denoting the maximum and minimum values of y in the region S. We have then performed the summation over all horizontal strips like that shaded in fig. 14.13, thereby covering the whole region S. Substituting from (14.17) into (14.18) we find that the double integral equals

$$\int_{\beta_1}^{\beta_2} \mathrm{d}y \int_{x_1(y)}^{x_2(y)} \mathrm{d}x\, f(x, y). \tag{14.19}$$

The integral (14.19), with the order of integration specified (over x first, then over y), is called a *repeated integral.*

We can equally well integrate first over y; then the limits of the y-integration will be functions $y_1(x)$ and $y_2(x)$ of x, the limits of the x-integration will be constants α_1 and α_2; these limits are shown in fig. 14.13. The double integral (14.14) is then equal to the repeated integral

$$\int_{\alpha_1}^{\alpha_2} dx \int_{y_1(x)}^{y_2(x)} dy\, f(x, y). \tag{14.20}$$

In practice a double integral is evaluated by treating it as a repeated integral.

If the function $f(x, y)=1$ throughout the region of integration, the integral (14.14) gives the area of the region of integration.

Example 12

Evaluate

$$\iint \frac{dx\, dy}{(2-x-y)^2}$$

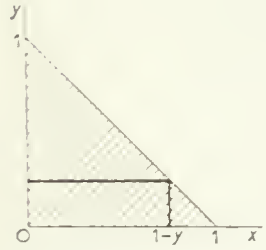

Fig. 14.14

over the triangle bounded by the axes $x=0$, $y=0$, and the line $x+y=1$; this is the shaded area in fig. 14.14.

If we integrate over x first, the limits are $(0, 1-y)$ as shown, with y lying in the range $(0, 1)$. Hence the integral equals the repeated integral

$$\int_0^1 dy \int_0^{1-y} dx\, (2-x-y)^{-2}.$$

Performing the x-integration, treating y as a fixed parameter, we have

$$\int_0^1 dy\, [(2-x-y)^{-1}]_{x=0}^{1-y} = \int_0^1 dy\left[1-\frac{1}{2-y}\right] = [y+\log(2-y)]_0^1 = 1-\log 2.$$

Example 13

Express the repeated integral

$$\int_0^a dy \int_0^{(4a^2-4ay)^{\frac{1}{2}}} dx\, f(x, y)$$

as a repeated integral over y and then x. For a given value of y in $(0, a)$, the values of x lies between the y-axis and the parabola $x^2=4a(a-y)$; so the integration is over the shaded region in fig. 14.15. Clearly x takes values over the range $(0, 2a)$;

for any x in this range, the maximum value of y lies on the parabola, and so equals $a-(x^2/4a)$; the minimum value of y is always zero. Hence the integral can be written

$$\int_0^{2a} \mathrm{d}x \int_0^{a-(x^2/4a)} \mathrm{d}y\, f(x, y).$$

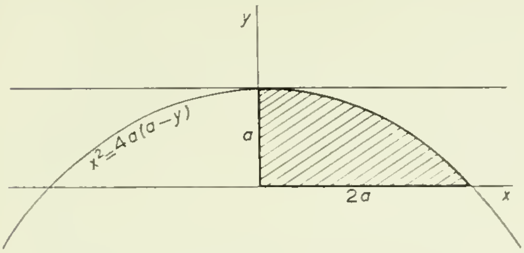

Fig. 14.15

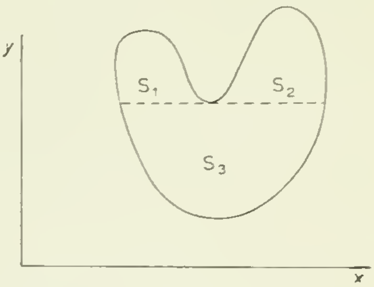

Fig. 14.16

It may happen that the region of integration S has a rather irregular shape, as in fig. 14.16. It is then best to subdivide the region into several parts, for example, the regions S_1, S_2 and S_3, each of which give rise to simple repeated integrals of the form (14.19) or (14.20). The integral over the whole region is then the sum of the integrals over all the sub-regions.

We have only discussed the integration of a continuous function $f(x, y)$. If the function is discontinuous along one or more lines in the region S, once again we subdivide S into regions throughout which $f(x, y)$ is continuous, and perform the integrations over these regions separately. When the region S can be divided into a finite number of regions in each of which $f(x, y)$ is continuous, f is said to be *piecewise continuous* in S. In this section, we shall only consider integrals of functions which are piecewise continuous.

§ 3.3. Double integrals using curvilinear coordinates

Very often the original variables of integration (x, y) are not the most convenient for evaluating an integral; for example, for an integration over a circle with centre at the origin, we have seen in § 2.1 that the limits of integration using variables (x, y) are rather complicated, whereas the limits of integration using polar coordinates (r, θ) are particularly simple. It is therefore desirable to define double integrals in terms of a general pair of curvilinear coordinates.

Suppose that curvilinear coordinates (u_1, u_2) are defined by (14.13), so that the point $\mathrm{P}(u_1, u_2)$ is defined, as in fig. 14.11, as the point of inter-

section of the curves (14.13) with particular values of u_1 and u_2. Now suppose that slightly different parameters $u_1+\delta u_1$ and $u_2+\delta u_2$ define adjacent curves of the families, as shown in fig. 14.11; then the four curves with parameters u_1, $u_1+\delta u_1$ and u_2, $u_2+\delta u_2$ enclose a small area δS (shaded in the figure) which becomes a parallelogram as $\delta u_1, \delta u_2 \to 0$. For a very fine mesh, any area S can be regarded as the aggregate of a large number of these small elements.

We can solve equations (14.13) for x, y in terms of u_1, u_2, giving

$$\begin{aligned} x &= X(u_1, u_2), \\ y &= Y(u_1, u_2), \end{aligned} \tag{14.21}$$

say. If $f(x, y)$ is a function of x and y then its value at any point can be expressed in terms of the curvilinear coordinates (u_1, u_2) by substituting for x and y from (14.21). Suppose that this substitution gives

$$f(x, y) = f[X(u_1, u_2), Y(u_1, u_2)] \equiv F(u_1, u_2). \tag{14.22}$$

Then the double integral of $f(x, y)$ over S is given in terms of the curvilinear coordinates as

$$\lim_{\delta u_1, \delta u_2 \to 0} \sum_S F(u_1, u_2)\, \delta S. \tag{14.23}$$

In (14.23), as in the definition (14.15), each element of area δS is weighted by multiplying by the value of $f(x, y) = F(u_1, u_2)$ at a point in that ele-

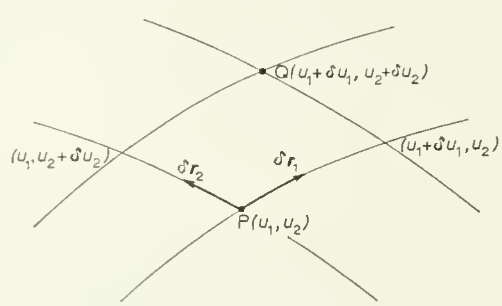

Fig. 14.17

ment; the sum is then taken over all elements in S. When the rectangular coordinates (x, y) themselves are used, the element δS is simply $j_r k_s$, and the definition (14.23) reduces at once to the form (14.15).

In general, we still need to calculate the element of area δS in terms of the curvilinear coordinates. In fig. 14.17 we have drawn the elemental parallelogram which was shaded in fig. 14.11, with a pair of opposite

corners at the points $P(u_1, u_2)$ and $Q(u_1+\delta u_1, u_2+\delta u_2)$. The displacement vectors $\delta \boldsymbol{r}_1$ and $\delta \boldsymbol{r}_2$ along the edges of the parallelogram are, as in (14.7), given to first order by

$$\delta \boldsymbol{r}_1 \approx \left(\frac{\partial X}{\partial u_1}, \frac{\partial Y}{\partial u_1}\right) \delta u_1$$

$$\delta \boldsymbol{r}_2 \approx \left(\frac{\partial X}{\partial u_2}, \frac{\partial Y}{\partial u_2}\right) \delta u_2.$$

Now as in Ch. 9 § 4.2, the area δS is approximately equal to the magnitude of the vector product $\delta \boldsymbol{r}_1 \times \delta \boldsymbol{r}_2$; so in the limit of a fine mesh

$$\delta S = \left| \frac{\partial X}{\partial u_1} \frac{\partial Y}{\partial u_2} - \frac{\partial X}{\partial u_2} \frac{\partial Y}{\partial u_1} \right| \delta u_1 \, \delta u_2. \tag{14.24}$$

The coefficient of $\delta u_1 \, \delta u_2$ in (14.24) is, apart possibly from change of sign, equal to the determinant

$$\begin{vmatrix} \dfrac{\partial X}{\partial u_1} & \dfrac{\partial Y}{\partial u_1} \\[2ex] \dfrac{\partial X}{\partial u_2} & \dfrac{\partial Y}{\partial u_2} \end{vmatrix} \equiv \frac{\partial(x, y)}{\partial(u_1, u_2)}. \tag{14.25}$$

This determinant is called the *Jacobian determinant*, or simply the Jacobian; denoting the determinant by

$$\frac{\partial(x, y)}{\partial(u_1, u_2)}$$

indicates that it arises when we change variables from (x, y) to (u_1, u_2); it is analogous to the factor $\mathrm{d}x/\mathrm{d}u$ occurring in an integral of one variable x when we change the variable to u. Using (14.24) and (14.25), the integral given by (14.23) is

$$\lim_{\delta u_1, \delta u_2 \to 0} \sum_S \pm F(u_1, u_2) \frac{\partial(x, y)}{\partial(u_1, u_2)} \delta u_1 \, \delta u_2$$

$$= \iint_S \pm F(u_1, u_2) \frac{\partial(x, y)}{\partial(u_1, u_2)} \, \mathrm{d}u_1 \, \mathrm{d}u_2. \tag{14.26}$$

The ranges of integration of u_1 and u_2 have to be chosen to cover the region S. Treated as a repeated integral, the range of the first variable

to be integrated may depend upon the second variable, as with rectangular coordinates; however, for a given region S we are often able to choose a pair of curvilinear coordinates whose limits are all constant; for example, we have already seen that the limits of polar coordinates are constant when S is a circular region. The indeterminacy of the sign in (14.26) is usually resolved by common sense methods. If the integrand is positive in a certain region, we choose the sign $\pm$ to ensure a positive integral, and likewise for a negative integrand.

The area of the region of integration is given in curvilinear coordinates by putting $F(u_1, u_2) \equiv 1$.

Example 14

Evaluate the integral

$$\iint x^2y^2 \, dx \, dy$$

over the interior of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The coordinates of a point *on* the ellipse can be written parametrically as $(x, y) = (a \cos \theta, b \sin \theta)$, the point traversing the whole ellipse as θ varies between 0 and 2π. So the coordinates of any point inside the ellipse can be written as

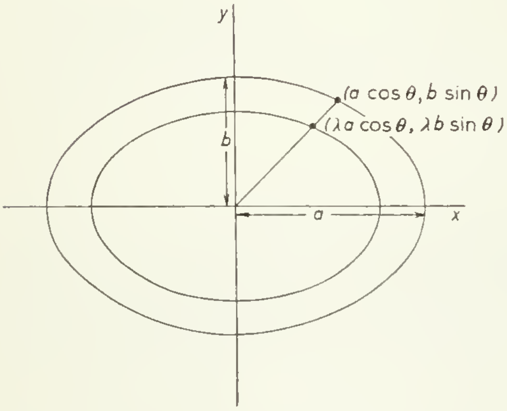

Fig. 14.18

$$x = \lambda a \cos \theta$$

$$y = \lambda b \sin \theta$$

defining the functions $X(\lambda, \theta)$, $Y(\lambda, \theta)$ of (14.21). Any point inside the ellipse corresponds to values of λ, θ with $0 \leqslant \theta < 2\pi$ and $0 \leqslant \lambda < 1$, as shown in fig. 14.18. This point is at the intersection of the straight line $ay = bx \tan \theta$ with θ constant, and the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$$

with λ constant. We shall take λ, θ as curvilinear coordinates; note that the ellipses with λ constant and the straight lines with θ constant are not orthogonal unless $a = b$.

In order to write down the integral in the form (14.26), we must evaluate the Jacobian determinant given by (14.25) with $u_1 = \lambda$, $u_2 = \theta$. Since $X(\lambda, \theta) = \lambda a \cos \theta$

and $Y(\lambda, \theta) = \lambda b \sin \theta$,

$$\frac{\partial(x, y)}{\partial(\lambda, \theta)} = \begin{vmatrix} a \cos \theta & b \sin \theta \\ -\lambda a \sin \theta & \lambda b \cos \theta \end{vmatrix} = \lambda ab.$$

The integral, in form (14.26), is therefore

$$\int_{\lambda=0}^{1} \int_{\theta=0}^{2\pi} (\lambda a \cos \theta)^2 (\lambda b \sin \theta)^2 (\lambda ab) \, d\lambda \, d\theta,$$

the plus sign being chosen since the integrand is positive. Integrating over λ and using result (2.33) twice, we have

$$\frac{2}{3} a^3 b^3 \int_0^{\frac{1}{2}\pi} \cos^2 \theta \sin^2 \theta \, d\theta = \frac{2}{3} a^3 b^3 \cdot \frac{\pi}{16} = \frac{\pi a^3 b^3}{24},$$

using the result (5.30).

Example 15

Evaluate the infinite integral

$$I = \int_0^\infty \exp(-x^2) \, dx.$$

This integral cannot be evaluated by simple means as an integral over a single variable. It can be evaluated by writing its square as a double integral and then changing variables. The integral is

$$I = \lim_{A \to \infty} \int_0^A \exp(-x^2) \, dx,$$

which can also be written as

$$\lim_{A \to \infty} \int_0^A \exp(-y^2) \, dy.$$

Thus we can write

$$I^2 = \lim_{A \to \infty} \int_0^A dx \int_0^A dy \exp[-(x^2 + y^2)],$$

the region of integration S in the xy-plane being the shaded square in fig. 14.19. The two quarter circles (dotted lines) of radii A and $A\sqrt{2}$ define, with the axes, quadrants which we call Q_1 and Q_2. The square S completely contains Q_1 and is completely contained in Q_2. So, since the integrand $\exp[-(x^2+y^2)]$ is positive everywhere,

$$\iint_{Q_1} \exp[-(x^2 + y^2)] \, dx \, dy \leqslant \iint_S \exp[-(x^2 + y^2)] \, dx \, dy \leqslant \iint_{Q_2} \exp[-(x^2 + y^2)] \, dx \, dy.$$

The integral over Q_1 can be evaluated by using polar coordinates $r=(x^2+y^2)^{\frac{1}{2}}$ and $\theta=\tan^{-1}(y/x)$, and becomes

$$\int_0^A \mathrm{d}r \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta \left[\pm \frac{\partial(x, y)}{\partial(r, \theta)} \exp[-(x^2 + y^2)]\right].$$

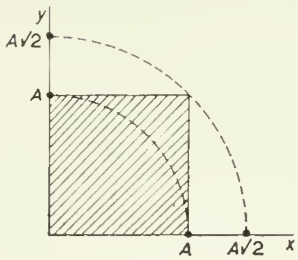

Fig. 14.19

Now from (14.25),

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -y & x \end{vmatrix} = \frac{x^2 + y^2}{r} = r.$$

Thus the Q_1-integral is

$$\int_0^A \mathrm{d}r\, r \exp(-r^2) \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta = \tfrac{1}{4}\pi[-\exp(-r^2)]_0^A = \tfrac{1}{4}\pi[1 - \exp(-A^2)].$$

The Q_2-integral is the same, with A replaced by $A\sqrt{2}$. Hence the inequalities above give

$$\tfrac{1}{4}\pi[1 - \exp(-A^2)] \leqslant \iint_S \exp(-x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y \leqslant \tfrac{1}{4}\pi[1 - \exp(-2A^2)].$$

As $A \to \infty$, the integral over S becomes I^2, $\exp(-A^2) \to 0$ and $\exp(-2A^2) \to 0$. Thus $I^2 = \frac{1}{4}\pi$, so that

$$\int_0^\infty \exp(-x^2)\,\mathrm{d}x = \tfrac{1}{2}\pi^{\frac{1}{2}}.$$

Sometimes the relations between the coordinates (x, y) and (u_1, u_2) are defined through an intermediate pair of variables, say (s, t); thus if

$$x = H(s, t), \qquad y = K(s, t), \tag{14.27}$$

while

$$s = S(u_1, u_2), \qquad t = T(u_1, u_2), \tag{14.28}$$

then

$$\left.\begin{aligned} x &= H[S(u_1, u_2), T(u_1, u_2)] \equiv X(u_1, u_2), \\ y &= K[S(u_1, u_2), T(u_1, u_2)] \equiv Y(u_1, u_2), \end{aligned}\right\} \tag{14.29}$$

comparing with (14.21). From (14.27), (14.28) and the definition (14.25),

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(u_1, u_2)} = \left(\frac{\partial H}{\partial s}\frac{\partial K}{\partial t} - \frac{\partial H}{\partial t}\frac{\partial K}{\partial s}\right)\left(\frac{\partial S}{\partial u_1}\frac{\partial T}{\partial u_2} - \frac{\partial S}{\partial u_2}\frac{\partial T}{\partial u_1}\right)$$

$$= \left(\frac{\partial H}{\partial s}\,\frac{\partial S}{\partial u_1} + \frac{\partial H}{\partial t}\,\frac{\partial T}{\partial u_1}\right)\left(\frac{\partial K}{\partial s}\,\frac{\partial S}{\partial u_2} + \frac{\partial K}{\partial t}\,\frac{\partial T}{\partial u_2}\right)$$

$$- \left(\frac{\partial H}{\partial s}\,\frac{\partial S}{\partial u_2} + \frac{\partial H}{\partial t}\,\frac{\partial T}{\partial u_2}\right)\left(\frac{\partial K}{\partial s}\,\frac{\partial S}{\partial u_1} + \frac{\partial K}{\partial t}\,\frac{\partial T}{\partial u_1}\right).$$

Using the identities (14.29) and the chain rule, this reduces to

$$\frac{\partial X}{\partial u_1}\,\frac{\partial Y}{\partial u_2} - \frac{\partial X}{\partial u_2}\,\frac{\partial Y}{\partial u_1} = \frac{\partial(x, y)}{\partial(u_1, u_2)},$$

so that

$$\frac{\partial(x, y)}{\partial(s, t)}\,\frac{\partial(s, t)}{\partial(u_1, u_2)} = \frac{\partial(x, y)}{\partial(u_1, u_2)}. \tag{14.30}$$

This result is the analogue of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{\mathrm{d}x}{\mathrm{d}u}$$

for a variable x depending on a variable t which itself depends on a third variable u. Equation (14.30) can be interpreted geometrically thus:

$$\frac{\partial(x, y)}{\partial(s, t)}\,\delta s\,\delta t$$

is the element of area in the xy-plane corresponding to increments δs and δt; if, however, s and t are expressed in terms of u_1, u_2 by (14.28), then

$$\delta s\,\delta t = \frac{\partial(s, t)}{\partial(u_1, u_2)}\,\delta u_1\,\delta u_2,$$

as in (14.24). So the element of area in the xy-plane is

$$\frac{\partial(x, y)}{\partial(s, t)}\,\frac{\partial(s, t)}{\partial(u_1, u_2)}\,\delta u_1\,\delta u_2;$$

it is however equal to

$$\frac{\partial(x, y)}{\partial(u_1, u_2)}\,\delta u_1\,\delta u_2$$

by (14.24), so that (14.30) is verified.

If we choose the variables u_1 and u_2 to be identical with x and y, so that in (14.29)

$$X(u_1, u_2) \equiv u_1, \qquad Y(u_1, u_2) \equiv u_2,$$

then

$$\frac{\partial(x, y)}{\partial(u_1, u_2)} \equiv 1.$$

Then (14.30) becomes

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(x, y)} = 1 \tag{14.31}$$

analogous to the result $(\mathrm{d}x/\mathrm{d}t)(\mathrm{d}t/\mathrm{d}x)=1$ for a change of a single variable. This relation is sometimes useful when s, t are given as functions of x, y, and $\partial(x, y)/\partial(s, t)$ is required: we need not express x, y as functions of s, t, but simply calculate $\partial(s, t)/\partial(x, y)$, the inverse of $\partial(x, y)/\partial(s, t)$.

Example 16

Calculate

$$\iint_S 2xy(x^2 + y^2) \exp[-(x^2 + y^2)^2] \,\mathrm{d}x \,\mathrm{d}y,$$

where S is the shaded region in fig. 14.12.

We choose coordinates $u_1=2xy$, $u_2=x^2-y^2$, whose properties were discussed in Example 11. Then $(x^2+y^2)^2=u_2^2+u_1^2$. Also

$$\frac{\partial(u_1, u_2)}{\partial(x, y)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2 + y^2).$$

Hence by (14.26) and (14.31), the integral is

$$\pm \int_0^{2a^2} \mathrm{d}u_1 \int_{-\infty}^{\infty} \mathrm{d}u_2 \, \frac{2xy(x^2 + y^2)}{4(x^2 + y^2)} \exp[-(x^2 + y^2)^2]$$
$$= \pm\tfrac{1}{4} \int_0^{2a^2} \mathrm{d}u_1 \, u_1 \exp(-u_1^2) \int_{-\infty}^{\infty} \mathrm{d}u_2 \exp(-u_2^2).$$

Using the result of Example 15, we obtain

$$\pm\tfrac{1}{4}[-\tfrac{1}{2}\exp(-u_1^2)]_0^{2a^2} \pi^{\frac{1}{2}} = \pm\tfrac{1}{8}\pi^{\frac{1}{2}}[1 - \exp(-4a^4)].$$

Clearly we choose the positive sign in this result, since the original integrand is positive throughout the region of integration.

EXERCISE 14.2

1. Evaluate

$$\iint x^2 y^2 \,\mathrm{d}x \,\mathrm{d}y$$

over the interior of the circle of radius a with centre at (a, a).

2. If $z = (x/a)^2 + (y/b)^2$, use the coordinates λ, θ defined in Example 14 to evaluate

$$\iint z \, dx \, dy$$

over the area bounded by the ellipse $(x/a)^2 + (y/b)^2 = 1$.

3. Evaluate

$$\iint x^5 y \exp(x^2 y^2) \, dx \, dy$$

over the region bounded by the hyperbola $xy = 1$ and by the straight lines $x = 2$, $x = y$.

4. By using variables (u, v) given by $x = u^2 - v^2$, $y = 2uv$, evaluate

$$\iint (x^2 + y^2)^{-\frac{1}{2}} \, dx \, dy$$

over the region enclosed by the three parabolas

$$y^2 = 4a_r(x + a_r) \qquad (r = 1, 2, 3)$$

where $a_1 > a_2 > 0 > a_3$.

5. Sketch the area of integration for an integral of the form

$$\int_{y=1}^{\sqrt{3}} dy \int_{x=1+y^2}^{4} dx \, f(x, y)$$

and write the integral as a repeated integral over y first and then over x. Evaluate

$$\int_1^{\sqrt{3}} dy \int_{1+y^2}^{4} dx \, \frac{xy \exp(y^2)}{(1 + x)^2} \, .$$

6. Integrate

$$\iint \frac{x^2}{x^2 + y^2} \, dx \, dy$$

over the region lying in the positive quadrant which is bounded by the straight lines $x = 3y$, $x = y\sqrt{3}$ and by the ellipse

$$\frac{x^2}{9a^2} + \frac{y^2}{a^2} = 1.$$

7. Curvilinear coordinates (u, v) are defined by

$$x = c \cosh u \cos v$$
$$y = c \sinh u \sin v.$$

Show that c can be chosen so that the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

is a curve of the family given by taking u constant. Use the coordinates (u, v) to evaluate

$$\iint \left(\frac{x^2}{25} + \frac{y^2}{9}\right) dx\, dy$$

over the area bounded by the ellipse. [Compare the method with that of Question 2.]

8. Evaluate

$$\iint y\,(y-1)\, e^{xy}\, dx\, dy$$

over the triangle bounded by the straight lines $x = 0$, $y = 0$ and $x + y = 2$.

9. Evaluate

$$\int_0^\infty dx \int_0^\infty dy \exp(-x^2 - 2xy\cos\alpha - y^2)\, dx\, dy \qquad (-\pi < \alpha < \pi).$$

§ 3.4. Surface integrals

A surface integral is a natural generalisation of a double integral; surface integrals occur frequently in many branches of physics. In defining the double integral we assume that a function $f(x, y)$ is given over the xy-plane, and we divide the plane into small elements with area denoted by $\delta x\, \delta y$, or into elements defined by curvilinear coordinates u_1, u_2. Likewise for an integral over a more general surface S in 3-space, we assume that a function f is defined at all points on S, and that the surface is divided into small elements of area δS by a 'mesh'. Then the surface integral of f over S is defined by an equation analogous to (14.23).

Let us suppose that u_r $(r=1, 2, 3)$ are curvilinear coordinates in 3-space, defined by equations (14.3):

$$f_k(x, y, z) = u_k \qquad (k = 1, 2, 3).$$

Suppose further that the surface S coincides with a surface of one of these families; for instance, suppose that it has an equation

$$f_3(x, y, z) = u_3 = \alpha \tag{14.32}$$

say, α being a known constant. Then the coordinates of points on S will be given by (14.4) with $u_3=\alpha$:

$$\begin{aligned} x &= X(u_1, u_2, \alpha), \\ y &= Y(u_1, u_2, \alpha), \\ z &= Z(u_1, u_2, \alpha). \end{aligned} \tag{14.33}$$

Different values of u_1, u_2 will give different points on the surface (14.32), so that these variables parametrise the surface. The surfaces

$$f_1(x, y, z) = u_1$$
$$f_2(x, y, z) = u_2,$$

for varying u_1, u_2, will cut the surface (14.32) in a series of curves which will form a mesh, as shown in fig. 14.20. If variations δu_1, δu_2 produce

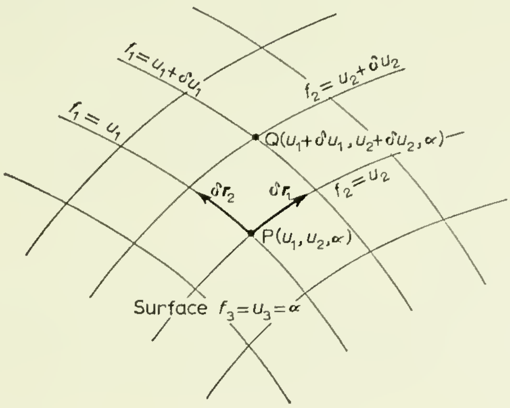

Fig. 14.20

displacements $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$ on the surface S, then as for the mesh in a plane shown in fig. 14.17, the element of area is

$$\delta S = |\delta \boldsymbol{r}_1 \times \delta \boldsymbol{r}_2|; \tag{14.34}$$

the displacements $\delta \boldsymbol{r}_1$, $\delta \boldsymbol{r}_2$ are given by differentiating (14.33):

$$\delta \boldsymbol{r}_k = \left(\frac{\partial X}{\partial u_k}, \frac{\partial Y}{\partial u_k}, \frac{\partial Z}{\partial u_k}\right) \delta u_k \qquad (k = 1, 2). \tag{14.35}$$

For an orthogonal system of coordinates,

$$\delta S = |\delta \boldsymbol{r}_1| \, |\delta \boldsymbol{r}_2| = h_1 h_2 \, \delta u_1 \, \delta u_2, \tag{14.36}$$

where h_r $(r = 1, 2)$ are defined by (14.12).

If $f(x, y, z)$ is defined over the surface S, it can be expressed as a function $F(u_1, u_2)$ of the parameters u_1, u_2 by substituting from (14.33). Then the integral of f over the surface S is defined as

$$\lim_{\delta u_1, \delta u_2 \to 0} \sum_S F(u_1, u_2) \, \delta S \tag{14.37}$$

with δS given in general by (14.34). For an orthogonal system of co-

ordinates, we can use (14.35) to give the double integral as

$$\lim_{\delta u_1, \delta u_2 \to 0} \sum_S F(u_1, u_2) h_1 h_2 \,\delta u_1 \,\delta u_2 = \iint_S F(u_1, u_2) h_1 h_2 \,\mathrm{d}u_1 \,\mathrm{d}u_2. \tag{14.38}$$

The more general integral (14.37), like (14.38), is expressible as a double integral over u_1 and u_2. The ranges of integration of u_1, u_2 must be chosen to cover the surface S.

As for double integrals over a plane, the area of the region of integration on the surface (14.32) is given by taking $F(u_1, u_2) \equiv 1$ in (14.37) or (14.38).

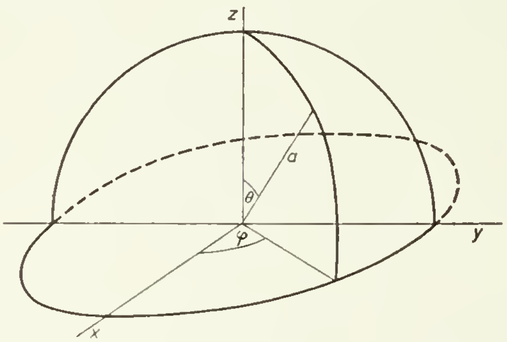

Fig. 14.21

Example 17

Evaluate $\iint_S \sigma(y^2+z^2)\,\mathrm{d}S$ over the curved surface of a hemisphere of radius a with centre at the origin, σ being constant.

We define the points on the surface in terms of spherical polar coordinates, so that the hemisphere has equation $r=a$, as in fig. 14.21:

$$x = a \sin\theta \cos\varphi$$
$$y = a \sin\theta \sin\varphi$$
$$z = a \cos\theta.$$

The whole hemisphere corresponds to the ranges

$$0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \varphi < 2\pi.$$

The element of area is

$$\mathrm{d}S = h_\theta h_\varphi \,\mathrm{d}\theta\,\mathrm{d}\varphi = a^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

while

$$y^2 + z^2 = a^2(\sin^2\theta \sin^2\varphi + \cos^2\theta).$$

Thus the integral becomes

$$\sigma a^4 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta \,(\sin^2\theta \sin^2\varphi + \cos^2\theta)\sin\theta$$

$$= \sigma a^4 \int_0^{2\pi} \sin^2\varphi \,\mathrm{d}\varphi \int_0^{\frac{1}{2}\pi} \sin^3\theta \,\mathrm{d}\theta + \sigma a^4 \cdot 2\pi \int_0^{\frac{1}{2}\pi} \cos^2\theta \sin\theta \,\mathrm{d}\theta,$$

which can be shown to be equal to $\frac{4}{9}\pi\sigma a^4$. The integral is in fact the moment of inertia of the hemispherical shell of mass σ per unit area, about the x-axis.

§ 3.5. Triple integrals

The definition of the triple integral

$$\iiint_\tau f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

over a given region τ of 3-space, in which (x, y, z) form a rectangular set of coordinates, is entirely analogous to the definition (14.15) of a double integral; its equivalence to repeated integrals over the three variables can be proved as for double integrals. If the repeated integration is over z first, then the limits of z are, in general, functions of both x and y. The *volume* of the region of integration τ is given by taking $f(x, y, z) = 1$, and so is just a particular type of triple integral.

Example 18

Write the triple integral $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,f(x, y, z)$, over the tetrahedron bounded by the coordinate planes and the plane P with equation

$$ax + by + cz = 1,$$

as a repeated integral over z, x and y (in that order).

For any values of x and y, the range of z is, as shown in fig. 14.22 between the value $z=0$ and value $z=c^{-1}(1-ax-by)$ lying on the plane P. The possible values of (x, y) are those lying within the triangular face OAB, bounded by the axes $x=0$, $y=0$, and the line $ax+by=1$. The ranges of integration, discussed in Example 10, are $0 \leqslant x \leqslant a^{-1}(1-by)$ and $0 \leqslant y \leqslant b^{-1}$. Hence the integral is

$$\int_0^{b^{-1}} \mathrm{d}y \int_0^{a^{-1}(1-by)} \mathrm{d}x \int_0^{c^{-1}(1-ax-by)} \mathrm{d}z\, f(x, y, z).$$

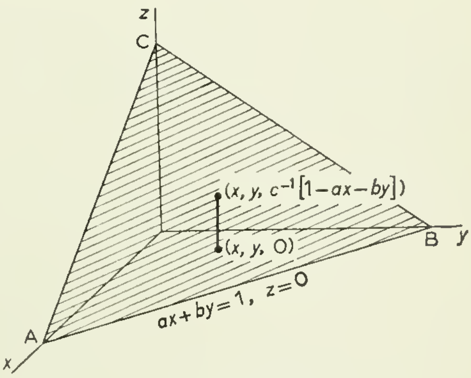

Fig. 14.22

Example 19

Evaluate

$$\iiint x\,\mathrm{e}^{by} \sin(1 - ax - by - cz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

over the tetrahedron defined in Example 18.

Written as a repeated integral, this is

$$\int_0^{b^{-1}} \mathrm{d}y \int_0^{a^{-1}(1-by)} \mathrm{d}x \int_0^{c^{-1}(1-ax-by)} \mathrm{d}z\, x\,\mathrm{e}^{by} \sin(1 - ax - by - cz).$$

In the z-integration, x and y are treated as constant parameters; so the z-integral is

$$\int_0^{c^{-1}(1-ax-by)} dz \sin(1 - ax - by - cz) = c^{-1}[\cos(1 - ax - by) - 1].$$

The x-integral is therefore

$$\int_0^{a^{-1}(1-by)} dx\, c^{-1}x[\cos(1 - ax - by) - 1] = a^{-2}c^{-1}[1 - \tfrac{1}{2}(1 - by)^2 - \cos(1 - by)].$$

The y-integral is then

$$a^{-2}c^{-1} \int_0^{b^{-1}} dy\, e^{by}[1 - \tfrac{1}{2}(1 - by)^2 - \cos(1 - by)],$$

which can be evaluated by ordinary methods.

It is often very useful to use a set of curvilinear coordinates (u_1, u_2, u_3) when evaluating triple integrals. If the function to be integrated is $f(x, y, z)$, we shall suppose that on changing variables to u_k $(k=1, 2, 3)$, $f(x, y, z)$ becomes $F(u_k)$. Further, changes $\delta u_1, \delta u_2, \delta u_3$ in the curvilinear coordinates produce the displacements defined by (14.7) and shown in fig. 14.4. The volume of the small parallelepiped generated by these displacements is given by the formula in Ch. 9 § 6.1:

$$\delta\tau = \pm[\delta\boldsymbol{r}_1, \delta\boldsymbol{r}_2, \delta\boldsymbol{r}_3], \tag{14.39}$$

the sign being chosen to make $\delta\tau$ positive. The definition of the triple integral over τ is then analogous to the definition (14.23) of a double integral:

$$\lim_{\delta u_k \to 0} \sum_\tau F(u_k)\,\delta\tau. \tag{14.40}$$

Using (14.39) and (14.7), the volume element $\delta\tau$ can be expressed as

$$\delta\tau = \pm \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}\,\delta u_1\,\delta u_2\,\delta u_3, \tag{14.41}$$

the Jacobian determinant being defined as

$$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \dfrac{\partial X}{\partial u_1} & \dfrac{\partial Y}{\partial u_1} & \dfrac{\partial Z}{\partial u_1} \\ \dfrac{\partial X}{\partial u_2} & \dfrac{\partial Y}{\partial u_2} & \dfrac{\partial Z}{\partial u_2} \\ \dfrac{\partial X}{\partial u_3} & \dfrac{\partial Y}{\partial u_3} & \dfrac{\partial Z}{\partial u_3} \end{vmatrix}; \tag{14.42}$$

the analogy between (14.42) and (14.25) is obvious. Using (14.41), the integral (14.40) can be written as

$$\pm \iiint_{\tau} F(u_k) \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \, du_1 \, du_2 \, du_3. \tag{14.43}$$

the ranges of the variables u_k being chosen to cover the volume τ.

We note that when (u_k) is an orthogonal set of coordinates, so that $\delta\boldsymbol{r}_1$, $\delta\boldsymbol{r}_2$, $\delta\boldsymbol{r}_3$ are mutually perpendicular, then from (14.11) and (14.39),

$$\delta\tau = |\delta\boldsymbol{r}_1| \cdot |\delta\boldsymbol{r}_2| \cdot |\delta\boldsymbol{r}_3| = h_1 h_2 h_3 \, \delta u_1 \, \delta u_2 \, \delta u_3,$$

the functions h_k being given by (14.12). Then the integral (14.43) reduces to

$$\iiint_{\tau} F(u_k) h_1 h_2 h_3 \, du_1 \, du_2 \, du_3. \tag{14.44}$$

The volume of the region τ is given by putting $F(u_k) \equiv 1$ in (14.40), and hence in (14.43) or (14.44).

Example 20

Evaluate the integral

$$\iiint xz^2 \exp\left(\frac{x^2 + y^2 + z^2}{a^2}\right) dx \, dy \, dz$$

over the octant bounded by the coordinate planes $x=0$, $y=0$, $z=0$ and the sphere $x^2+y^2+z^2=a^2$.

Use spherical polar coordinates (r, θ, φ) defined and discussed in Example 8. The ranges of the variables corresponding to the octant are

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \varphi \leqslant \tfrac{1}{2}\pi.$$

Since the coordinates are orthogonal and $h_r = 1$, $h_\theta = r$, $h_\varphi = r \sin\theta$, the integral, given by (14.44), is

$$\int_0^a dr \int_0^{\frac{1}{2}\pi} d\theta \int_0^{\frac{1}{2}\pi} d\varphi \; r^3 \sin\theta \cos^2\theta \cos\varphi \exp(r^2/a^2) \cdot r^2 \sin\theta.$$

The limits of integration are all constant, so the three integrations can be performed separately:

$$\int_0^{\frac{1}{2}\pi} \cos\varphi \, d\varphi = 1$$

$$\int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta \, d\theta = \frac{\pi}{16}$$

and

$$\int_0^a r^5 \exp(r^2/a^2)\,\mathrm{d}r = [\tfrac{1}{2}a^2r^4 \exp(r^2/a^2)]_0^a - 2a^2 \int_0^a r^3 \exp(r^2/a^2)\,\mathrm{d}r$$

$$= \tfrac{1}{2}a^6\,\mathrm{e} - 2a^2[\tfrac{1}{2}a^2r^2 \exp(r^2/a^2)]_0^a + a^4 \int_0^a 2r \exp(r^2/a^2)\,\mathrm{d}r$$

$$= \tfrac{1}{2}a^6\,\mathrm{e} - a^6\,\mathrm{e} + a^4[a^2 \exp(r^2/a^2)]_0^a = \tfrac{1}{2}a^6\,\mathrm{e}.$$

Hence the whole integral is $\frac{1}{16}\pi \cdot \frac{1}{2}a^6\,\mathrm{e} = \frac{1}{32}\pi a^6\,\mathrm{e}$.

§ 3.6. Integrals over more than three variables

★The definitions and formulae for double and triple integrals can be generalised to integrals over many variables, known as *multiple integrals*. If $f(x_1, x_2, \ldots, x_n) \equiv f(x_k)$ is a piecewise continuous function of n variables $x_1, x_2, \ldots, x_n$, then these variables can be looked upon as rectangular coordinates in n-space. An element of volume in this n-space is

$$\delta\tau = \delta x_1\,\delta x_2 \cdots \delta x_n,$$

and we define the integral of $f(x_k)$ over a region τ to be

$$\lim_{\delta x_r \to 0} \sum_{\tau} f(x_k)\,\delta\tau; \tag{14.45}$$

in each term of the sum in (14.45), $f(x_k)$ is evaluated at a point within the element $\delta\tau$, and the summation is over all elements $\delta\tau$ in the region τ. Just as for double integrals, it can be shown that (14.45) equals each of the repeated integrals, $n!$ in number, over the variables x_k in all possible orders. For instance, the integral is

$$\int \mathrm{d}x_1 \int \mathrm{d}x_2 \ldots \int \mathrm{d}x_n\, f(x_1, x_2, \ldots, x_n), \tag{14.46}$$

each integration being performed treating the remaining variables as parameters. The ranges of integration must be chosen to cover τ; this is often difficult since it is hard to visualise hypervolumes in spaces of more than three dimensions.

When curvilinear coordinates u_l $(l=1, 2, \ldots, n)$ are used, the element of volume can be shown by induction on n to be

$$\delta\tau = \pm \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\,\delta u_1\,\delta u_2 \cdots \delta u_n,$$

the Jacobian determinant

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}$$

of order n being defined by generalising (14.25) and (14.42). The integral (14.45) then becomes

$$\iint \cdots \int \pm F(u_l) \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} \mathrm{d}u_1 \cdots \mathrm{d}u_n, \tag{14.47}$$

where $F(u_l) = f(x_k)$ is found by substituting in f the values of x_k in terms of the curvilinear coordinates u_l. ★

EXERCISE 14.3

1. Evaluate the surface integral

$$\iint_S (x^4 + y^4 + z^4)\, \mathrm{d}S$$

(i) when S is the surface of the sphere of radius a with centre at the origin;
(ii) when S is the surface (including both ends) of the right circular cylinder of radius a and height h defined by the surfaces $x^2+y^2=a^2$, $z=0$ and $z=h$;
(iii) when S is the surface of the oblate spheroid

$$\frac{x^2 + y^2}{c^2 \cosh^2 \alpha} + \frac{z^2}{c^2 \sinh^2 \alpha} = 1.$$

Express (iii) as an integral over a single variable.

2. Evaluate the volume integrals

$$\iiint_\tau (x^4 + y^4 + z^4)\, \mathrm{d}\tau$$

when τ is, in turn, the volume bounded by each of the three surfaces defined in Question 1 above.

3. Find the volume of the cylinder $x^2+y^2=a^2$ contained within the cylinder $y^2+z^2=a^2$. Find also the area on the surface of the first cylinder cut off by the second.

4. If (x, y, z) and (x', y', z') are two sets of rectangular coordinates defining right-handed frames of reference, show that

$$\frac{\partial(x, y, z)}{\partial(x', y', z')} = 1.$$

Find the area of the part of the surface

$$(x \sin \alpha - y \cos \alpha)^2 + (x \cos \alpha + y \sin \alpha)^2 + k^2 z^2 = a^2$$

which has $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, a, k and α being constants.

5. Find the area on the surface of the cone $z=+(x^2+y^2)^{\frac{1}{2}}$ which is cut off by the cylinder $(x-a)^2+y^2=a^2$. Evaluate the integral

$$\iint x^2z^2\,\mathrm{d}S$$

over this area.

6. Evaluate

$$\iiint_{\tau}\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right]^{-\frac{1}{2}} xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where τ is the volume in the positive octant enclosed by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and the coordinate planes.

7. The coordinates of points A, B relative to a rectangular set of axes are $(0, 0, \pm\frac{1}{2}R)$. The distances AP, BP of a general point $\mathrm{P}(x, y, z)$ from A, B are denoted by s, t. Coordinates (ξ, η, φ) are defined by

$$\xi=\frac{s+t}{R}, \qquad \eta=\frac{s-t}{R},$$

with φ equal to the angle between the plane $y=0$ and the plane containing the points A, B and P.

Show that

$$\left|\frac{\partial(\xi, \eta, \varphi)}{\partial(x, y, z)}\right|=\frac{8}{R^3(\xi^2-\eta^2)}$$

and prove that the integral of $(st)^{-1}\exp[-(s+t)/R]$ over all space is equal to $2\pi R/\mathrm{e}$.

8. Evaluate the integral

$$\iiint \log(1-x-y-z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

over the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$.

VECTOR ANALYSIS

§ 1. The gradient of a scalar field

The concept of the 'rate of change' of a function is fundamental in differential calculus. To discuss the change of scalar field $\psi(\boldsymbol{r})$ in a region τ of 3-space, we can define a *vector* field, known as the *gradient* of $\psi(\boldsymbol{r})$ and denoted by grad $\psi(\boldsymbol{r})$, throughout τ. Consider any point P in τ; a unique level surface of $\psi(\boldsymbol{r})$ passes through P, its equation being $\psi(\boldsymbol{r}) = \psi_{\mathrm{P}}$ where ψ_{P} is the value of ψ at P. We draw the normal to this level surface at P, shown in fig. 15.1 as the line with an arrow; the value of ψ on this normal

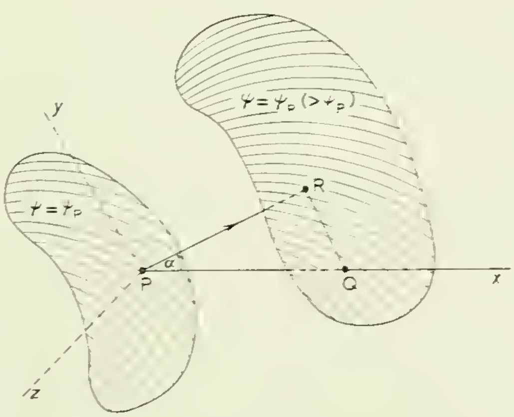

Fig. 15.1

will be less than ψ_{P} on one side of P, and greater than ψ_{P} on the other. Let R be a point near to P on the normal such that $\psi_{\mathrm{R}} > \psi_{\mathrm{P}}$, ψ_{R} being the value of $\psi(\boldsymbol{r})$ at R; the level surface through R has equation $\psi(\boldsymbol{r}) = \psi_{\mathrm{R}}$, and for a well-behaved field the level surfaces of P and R are nearly parallel since R is near to P.

We define grad ψ at P as a *vector*

(i) of magnitude $\displaystyle\lim_{\mathrm{PR}\to 0} \frac{\psi_{\mathrm{R}} - \psi_{\mathrm{P}}}{\mathrm{PR}}$,

(ii) in the direction $\boldsymbol{PR}$.

From (i) we see that in magnitude the gradient closely resembles a first derivative; in fact, if n is a coordinate measuring distance in the direction PR, and if $\delta\psi$ is the small change in ψ corresponding to a small change δn in this coordinate, then by (i) the magnitude of grad ψ is $\lim_{\delta n\to 0}|\delta\psi/\delta n|$, which can be written $|\partial\psi/\partial n|$. The direction of grad ψ is normal to the level surfaces in the direction of ψ increasing, and we can easily see why the word 'gradient' is appropriate if we consider, as in Ch. 14 § 1.1, the analogy of contours on a map. The contour lines $h(x, y)=h_0$ are analogous to the level surfaces of a scalar field in 3-space. The direction corresponding to PR is the normal to the contour lines, and is 'directly uphill'; so, using (ii), the gradient vector grad $h(x, y)$ points in the 'uphill' direction. The magnitude $|\text{grad } h|$, by (i), is the limit of the change in height δh divided by the horizontal distance δn in the uphill direction; thus $|\text{grad } h|$ is simply the slope or gradient of the land in that direction.

§ 1.1. Components of grad ψ

If a set of orthogonal coordinates is given, they define three mutually perpendicular directions at every point in space. Any vector field can be resolved into components in these three directions. Consider first a set of rectangular coordinates (x, y, z), and let us suppose that the vector grad ψ at a point P makes an acute angle α with the x-axis, as shown in fig. 15.1. If the x-axis meets the surface $\psi=\psi_R$ at the point Q, then as PR$\to$0, PQ$=$PR sec α; thus the x-component of grad ψ is

$$|\text{grad }\psi|\cos\alpha = \lim_{\text{PR}\to 0}\frac{\psi_R-\psi_P}{\text{PR}}\cos\alpha = \lim_{\text{PQ}\to 0}\frac{\psi_Q-\psi_P}{\text{PQ}}, \qquad (15.1)$$

since the value ψ_Q of ψ at Q is equal to ψ_R. Now PQ is simply an increment of length in the x-direction, and $\psi_Q-\psi_P$ is the change in ψ due to this incremental change, with y and z being kept constant. So the limit (15.1) is simply the partial derivative

$$\frac{\partial\psi(x, y, z)}{\partial x}. \qquad (15.2)$$

It is easy to check that (15.1), and hence (15.2), also holds when α is an obtuse angle. Similarly the components of grad ψ in the y- and z-directions are $\partial\psi/\partial y$ and $\partial\psi/\partial z$; so when ψ is expressed in terms of rec-

tangular coordinates x, y, z,

$$\operatorname{grad} \psi = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right). \tag{15.3}$$

When a small displacement $\delta \boldsymbol{r} = (\delta x, \delta y, \delta z)$ from P is given we have, to first order, by Taylor's theorem for three variables, which is the obvious extension of that for two variables [eq. (8.87)]:

$$\delta \psi \approx \frac{\partial \psi}{\partial x} \delta x + \frac{\partial \psi}{\partial y} \delta y + \frac{\partial \psi}{\partial z} \delta z. \tag{15.4}$$

Using (15.3), we have

$$\delta \psi \approx \operatorname{grad} \psi \cdot \delta \boldsymbol{r}. \tag{15.5}$$

This expresses the incremental change $\delta \psi$ of the scalar field ψ as a scalar product; so this formula is true for all coordinate systems, although it was derived using a particular set of coordinates.

Now suppose that (u_1, u_2, u_3) are an orthogonal set of coordinates, and the PQ in fig. 15.1 is in the u_1-direction instead of the x-direction. The derivation of (15.1) is unaltered, but the increment of length PQ is $h_1 \, \delta u_1$, where h_1, defined by (14.12), is evaluated at P. Thus the u_1-component of grad ψ is

$$\lim_{\delta u_1 \to 0} \frac{\delta \psi}{h_1 \, \delta u_1} = \frac{1}{h_1} \frac{\partial \psi(u_1, u_2, u_3)}{\partial u_1},$$

since u_2 and u_3 are constant along PQ. Thus if ψ is expressed in terms of orthogonal coordinates u_k, its components at any point in the three orthogonal directions are

$$\operatorname{grad} \psi = \left(\frac{1}{h_1} \frac{\partial \psi}{\partial u_1}, \frac{1}{h_2} \frac{\partial \psi}{\partial u_2}, \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \right). \tag{15.6}$$

Example 1

A scalar field is defined in terms of *rectangular* coordinates by

$$\psi(\boldsymbol{r}) = \frac{xz^2}{x^2 + y^2}.$$

The components of grad ψ in this coordinate system are given by (15.3):

$$\operatorname{grad} \psi = \left(-\frac{z^2(x^2 - y^2)}{(x^2 + y^2)^2}, -\frac{2xyz^2}{(x^2 + y^2)^2}, \frac{2xz}{x^2 + y^2} \right).$$

In *cylindrical polar* coordinates (ρ, φ, z), defined in Ch. 14, Example 7,

$$\psi = \frac{z^2 \cos \varphi}{\rho}.$$

Since $h_\rho = h_z = 1$ and $h_\varphi = \rho$, the components of grad ψ in the ρ-, φ- and z-directions are, by (15.6),

$$\left(\frac{\partial \psi}{\partial \rho}, \frac{1}{\rho}\frac{\partial \psi}{\partial \varphi}, \frac{\partial \psi}{\partial z}\right) = \left(-\frac{z^2}{\rho^2}\cos \varphi, -\frac{z^2}{\rho^2}\sin \varphi, \frac{2z \cos \varphi}{\rho}\right).$$

In *spherical polars* (r, θ, φ), defined in Ch. 14, Example 8,

$$\psi = \frac{r \cos \varphi \cos^2 \theta}{\sin \theta}.$$

Hence the components of grad ψ in (r, θ, φ) directions are

$$\left(\frac{\partial \psi}{\partial r}, \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{1}{r \sin \theta}\frac{\partial \psi}{\partial \varphi}\right) = \left(\frac{\cos \varphi \cos^2 \theta}{\sin \theta}, -\frac{\cos \varphi \cos \theta (1+\sin^2 \theta)}{\sin^2 \theta}, -\frac{\sin \varphi \cos^2 \theta}{\sin^2 \theta}\right).$$

It is easy to see that the z-components in rectangular and cylindrical polar coordinates are the same, as are the φ-components in cylindrical and spherical polars. Note that in this example, ψ has the dimension of a length, so that the components of grad ψ in *all* coordinate systems are dimensionless.

EXERCISE 15.1

1. A scalar field ψ is defined in cylindrical polar coordinates as

$$\psi = \frac{\rho^2 \sec \varphi}{z} \exp\left(1 + \frac{z^2}{\rho^2}\right).$$

Find the components of grad ψ in rectangular, cylindrical polar and spherical polar coordinates. Show that the expressions for grad ψ in rectangular and cylindrical polar coordinates are equivalent.

2. A scalar field ψ in rectangular coordinates is $\psi = xz(x^2+y^2-z^2)$.
Find an expression for grad ψ in the oblate spheroidal coordinates defined in Ch. 14, Example 9, and show that this expression is equivalent to that in rectangular coordinates.

§ 2. The scalar line integral of a vector field

Suppose that $\boldsymbol{w}(\boldsymbol{r})$ is a vector field; suppose also that on a particular continuous path between two points P and Q, as shown in fig. 15.2, $\boldsymbol{t}$ is the unit tangent vector at a point $\boldsymbol{r}$ on the curve, and that s measures the distance along the curve. Then the *scalar line integral* of $\boldsymbol{w}$ along the

curve is defined to be

$$I = \int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{t} \cdot \boldsymbol{w}(\boldsymbol{r})\, \mathrm{d}s \tag{15.7}$$

the integrand $\boldsymbol{t} \cdot \boldsymbol{w}$ being the component of $\boldsymbol{w}$ along the tangent evaluated at the point s on the curve. The integral *is* a scalar, since the integrand is. In rectangular coordinates we have $\boldsymbol{t} = (\mathrm{d}x/\mathrm{d}s,\ \mathrm{d}y/\mathrm{d}s,\ \mathrm{d}z/\mathrm{d}s)$, and if $w = (w_x, w_y, w_z)$ then the integral (15.7) is

$$\int_{\mathrm{P}}^{\mathrm{Q}} (w_x \mathrm{d}x + w_y \mathrm{d}y + w_z \mathrm{d}z),$$

and can be evaluated as in Ch. 11 § 4. Other curvilinear coordinates may also be used, as in Example 3 following.

We define the *differential displacement vector* along the curve to be

$$\mathrm{d}\boldsymbol{s} = \boldsymbol{t}\, \mathrm{d}s, \tag{15.8}$$

so that (15.7) can be written

$$I = \int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{w}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s}. \tag{15.9}$$

Equation (15.8) can be compared with the definition in Ch. 11 § 3, of the unit tangent vector $\boldsymbol{t}$; $\mathrm{d}\boldsymbol{s}$ is the differential corresponding to the differential $\mathrm{d}\boldsymbol{r}$ in this definition.

In general the integral I will depend on the path chosen between P and Q. So if I_1 is the integral along the path 1 (solid line) in fig. 15.2, and I_2 that along path 2 (broken line), then in general $I_1 \neq I_2$.

To demonstrate the importance of the scalar line integral, we consider a force field $\boldsymbol{f}(\boldsymbol{r})$. If the force acts at a point $\boldsymbol{r}$ during a small displacement $\delta\boldsymbol{r}$, it does work $\boldsymbol{f}(\boldsymbol{r}) \cdot \delta\boldsymbol{r}$. So if the force acts during motion from P to Q, the total work done is given by the scalar line integral $\int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{f}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s}$ along the path traversed, $\mathrm{d}\boldsymbol{s}$ being the differential vector corresponding to $\delta\boldsymbol{r}$.

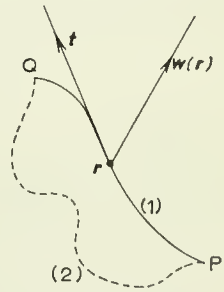

Fig. 15.2

§ 2.1. Conservative vector fields

Many important vector fields that we meet in physics, such as the gravitational and electrostatic fields, can be described in terms of a *scalar potential*. The relation between a field $\boldsymbol{w}$ of this type and its potential ψ is

$$\boldsymbol{w} = -\mathrm{grad}\, \psi \tag{15.10}$$

we say that $\boldsymbol{w}$ is 'derivable from the potential ψ', and that it is a *conservative field*. If we consider the scalar line integral (15.9) of $\boldsymbol{w}(\boldsymbol{r})$ between two points P and Q, then from (15.10)

$$\int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{s} = -\int_{\mathrm{P}}^{\mathrm{Q}} \operatorname{grad} \psi \cdot \mathrm{d}\boldsymbol{s}.$$

Using (15.5) in the limit $\delta \boldsymbol{r} \to 0$, we have

$$\int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{s} = -\int_{\mathrm{P}}^{\mathrm{Q}} \mathrm{d}\psi = \psi_{\mathrm{P}} - \psi_{\mathrm{Q}}. \tag{15.11}$$

Thus the integral is independent of the path traversed between P and Q; in fig. 15.2, the integrals along the paths 1 and 2 will be equal. In other words, the integral along the path 1 from P to Q and back to P along 2 will be zero. Since P, Q may be any points, and the paths 1, 2 any paths joining them we see that the line integral of $\boldsymbol{w}$ round any closed path beginning and ending at the same point is zero; this result is written

$$\oint \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{s} = 0, \tag{15.12}$$

and follows at once from (15.11) if we choose Q to coincide with P.

If $\boldsymbol{w}$ is a conservative force field such as the gravitational or electrostatic field, equation (15.11) says simply that the work done by the force equals the decrease in potential energy. Equation (15.12) tells us that potential energy is conserved in any motion in the field beginning and ending at the same point. This conservation property is the reason why fields satisfying (15.10) are called conservative fields.

Example 2

The gravitational acceleration (or gravitational force per unit mass) locally near the earth's surface is $\boldsymbol{f} = (0, 0, -g)$ using rectangular axes with the z-axis vertically upwards. We have $\boldsymbol{f} = -\operatorname{grad} \psi$, where $\psi = gz$ is the usual gravitational potential. The work done on a mass m by the field in moving from P to Q is $\int_{\mathrm{P}}^{\mathrm{Q}} m\boldsymbol{f}\cdot \mathrm{d}\boldsymbol{s} = m \int_{\mathrm{P}}^{\mathrm{Q}} -g\, \mathrm{d}z = mg(z_{\mathrm{P}} - z_{\mathrm{Q}}) \equiv m(\psi_{\mathrm{P}} - \psi_{\mathrm{Q}})$, since $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$; the work done depends only on the difference of height $z_{\mathrm{P}} - z_{\mathrm{Q}}$, and not on the path.

Example 3

A point charge e produces an electrostatic potential $\psi = e/r$ at a point at a distance r from the charge. Using spherical polar coordinates with origin O at the charge, the electrostatic field is

$$\boldsymbol{E} = -\operatorname{grad} \psi = -\left(\frac{\partial \psi}{\partial r}, \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{1}{r \sin \theta}\frac{\partial \psi}{\partial \varphi}\right) = (e/r^2, 0, 0).$$

In these coordinates $\mathrm{d}\boldsymbol{s} = (\mathrm{d}r,\ r\ \mathrm{d}\theta,\ r \sin\theta\ \mathrm{d}\varphi)$, and so

$$\int_{P}^{Q} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{s} = \int_{P}^{Q} (e/r^2)\,\mathrm{d}r = [-e/r]_{P}^{Q} = e/OP - e/OQ = \psi_P - \psi_Q.$$

The field $\boldsymbol{E}$ is directed away from the charge and obeys the inverse square law.

We have shown that any field $\boldsymbol{w}$ derivable from a potential is conservative; that is, it satisfies (15.12) for all closed paths. The converse is not difficult to prove: given that (15.12) holds for all contours, choose the potential ψ_O arbitrarily at some point O, and define the potential ψ_P at any other point P as

$$\psi_P = \psi_O - \int_{O}^{P} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}; \tag{15.13}$$

by (15.12), the integral in (15.13) is independent of the path from O to P, and the definition of ψ_P is thus unambiguous. Subtracting from (15.13) the corresponding equation for ψ_Q, equation (15.11) follows for all points P and Q. Letting Q approach P, (15.11) gives for the component of grad ψ in direction PQ

$$\lim_{PQ\to 0} \frac{\psi_Q - \psi_P}{PQ} = -\lim_{PQ\to 0} \frac{1}{PQ} \int_{P}^{Q} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}. \tag{15.14}$$

If θ is the angle between $\boldsymbol{w}$ and PQ, and if w is the magnitude of $\boldsymbol{w}$ at P, then in the limit $PQ\to 0$,

$$\frac{1}{PQ}\int_{P}^{Q} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s} = \frac{1}{PQ}\, w\,(PQ)\cos\theta = w\cos\theta,$$

equal to the component of $\boldsymbol{w}$ in direction PQ. This is true for any component of grad ψ, so (15.14) gives

$$\operatorname{grad}\psi = -\boldsymbol{w};$$

thus $\boldsymbol{w}$ is derivable from the potential ψ.

§ 3. The curl of a vector field

For a general vector field $\boldsymbol{w}$, the integral

$$\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s} \tag{15.15}$$

round a closed path measures the 'tendency to circulate' round the path.

We can most easily picture this if we consider the velocity field $\boldsymbol{v}$ in a fluid. If the fluid is swirling round some axis near a whirlpool, then the integral $\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s}$ along a path round this axis will measure the *circulation* of the fluid, as it is called. It is useful to have a function which represents the circulation of a vector field $\boldsymbol{w}$ at a point, and we obtain such a function by letting the closed path of integration in (15.15) become very small. Circulation is a vector of the same type as angular velocity or angular momentum, which are known as 'axial vectors', and we need to define an axis about which we are measuring the circulation. Suppose that we specify an axis at a point P by choosing a unit vector $\boldsymbol{n}$ along it; as shown in fig. 15.3, we let δS be a small surface element, which becomes plane as $\delta S \to 0$, with its normal lying along $\boldsymbol{n}$. The boundary of δS is a simple closed curve whose differential element is $\mathrm{d}\boldsymbol{s}$. Then the circulation vector at P, denoted by curl $\boldsymbol{w}$, is defined to have its component in direction $\boldsymbol{n}$ given by

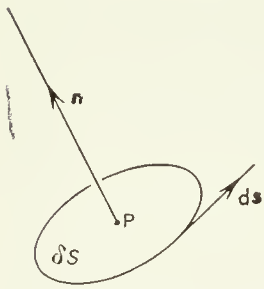

Fig. 15.3

$$\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{w} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{s}, \tag{15.16}$$

the integral being round the bounding curve of δS. We see that (15.16) defines the 'circulation per unit area' about the axis $\boldsymbol{n}$, and for well-behaved vector fields this limit exists, as we shall see in § 3.1. Only the direction of integration round the boundary is undefined in the integral, and it is customary to choose $\boldsymbol{n}$ and $\mathrm{d}\boldsymbol{s}$ to obey the 'right hand rule', as shown in fig. 15.3. Since $\boldsymbol{w} \cdot \mathrm{d}\boldsymbol{s}$ is scalar, the limit (15.16) is a scalar, so that $\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{w}$ is a scalar product.

§ 3.1. Components of curl $\boldsymbol{w}$

The easiest way of showing that the limit (15.16) exists for well-behaved fields $\boldsymbol{w}$ is to evaluate the components of curl $\boldsymbol{w}$ in particular coordinate systems. First let us calculate from (15.16) the components of curl $\boldsymbol{w}$ for rectangular coordinates x, y, z. The z-component $(\operatorname{curl} \boldsymbol{w})_z$ is given by taking $\boldsymbol{n}$ along the z-axis, as in fig. 15.4. We choose the surface δS to be a small rectangle PQRS perpendicular to $\boldsymbol{n}$, the integration being round the boundary in the sense of the arrows. If the components of $\boldsymbol{w}$ are w_x, w_y, w_z, the contribution to $\oint \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{s}$ from PQ and RS is

$$\int_{\mathrm{P}}^{\mathrm{Q}} w_x \, \mathrm{d}x - \int_{\mathrm{R}}^{\mathrm{S}} w_x \, \mathrm{d}x. \tag{15.17}$$

Let the coordinates of P be (x_1, y_1), and those of R be (x_2, y_2), as shown, with $x_2 > x_1$ and $y_2 > y_1$. Then w_x in the integral along PQ in (15.17) means $w_x(x, y_1)$, while w_x in the integral along SR means $w_x(x, y_2)$. Since x varies between limits (x_1, x_2) along both PQ and SR, (15.17) equals

$$-\int_{x_1}^{x_2} [w_x(x, y_2) - w_x(x, y_1)]\mathrm{d}x.$$

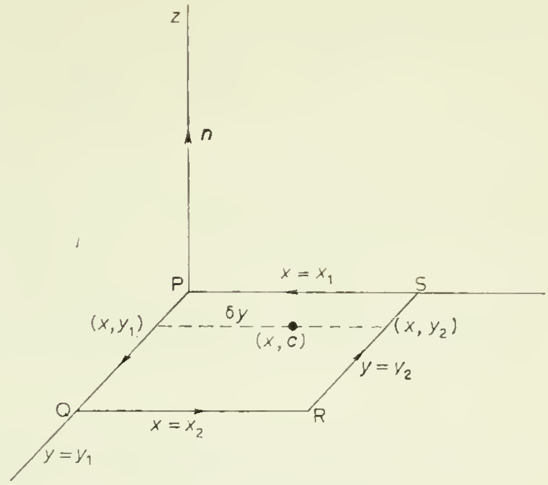

Fig. 15.4

Now assume that, as a function of y, w_x satisfies the conditions of the Mean Value Theorem (Ch. 6 § 1.3), for fixed values of x. Then

$$w_x(x, y_2) - w_x(x, y_1) = (y_2 - y_1)\left[\frac{\partial w_x(x, y)}{\partial y}\right]_{y=c}$$

where $y_1 < c < y_2$. The derivative here is evaluated at the point (x, c) *inside the rectangle* PQRS, as shown in fig. 15.4. Putting $y_2 - y_1 = \delta y$ (15.17) becomes

$$-\delta y \int_{x_1}^{x_2} \frac{\partial w_x}{\partial y}\,\mathrm{d}x.$$

If we assume further that $\partial w_x/\partial y$ is a continuous function of x and y, then as $\delta x = \mathrm{PQ} = x_2 - x_1$ and $\delta y = \mathrm{PS}$ tend to zero, the value of $\partial w_x/\partial y$ at *every* point inside PQRS approaches the values at P. Since the area δS is $\delta x \delta y$, the contribution from PQ and RS to the limit (15.16) are thus

$$\lim_{\delta s \to 0} -\frac{1}{\delta x}\int_{x_1}^{x_2} \frac{\partial w_x}{\partial y}\,\mathrm{d}x = \left[-\frac{\partial w_x}{\partial y}\right]_{\mathrm{P}}$$

the suffix P denoting evaluation at this point. Likewise the contributions from QR and SP give $[+\,\partial w_y/\partial x]$. Hence the definition (15.16) gives for the z-component

$$(\mathrm{curl}\ \boldsymbol{w})_z = \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}; \tag{15.18}$$

the partial derivatives are evaluated at the point P upon which the surface δS shrinks, where they have been assumed to be continuous.

The x- and y-components of curl $\boldsymbol{w}$ may be evaluated in the same way,

and are given by cyclically permuting x, y, z in (15.18); thus

$$\operatorname{curl} \boldsymbol{w} = \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z},\ \frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x},\ \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right). \tag{15.19}$$

The component of curl $\boldsymbol{w}$ in terms of general orthogonal coordinates u_1, u_2, u_3 can be found by a similar method. The line and surface elements are given by (14.11), and it must be remembered that the functions h_k are in general functions of the coordinates. We simply quote the result, leaving its derivation as an exercise to the reader:

$$\begin{aligned} \operatorname{curl} \boldsymbol{w} = \Bigg(& \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 w_3) - \frac{\partial}{\partial u_3}(h_2 w_2)\right], \\ & \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 w_1) - \frac{\partial}{\partial u_1}(h_3 w_3)\right], \\ & \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 w_2) - \frac{\partial}{\partial u_2}(h_1 w_1)\right]\Bigg). \end{aligned} \tag{15.20}$$

Example 4

The components of a vector field $\boldsymbol{w}$ in rectangular coordinates are

$$(w_x, w_y, w_z) = \big(-y(x^2 + y^2), x(x^2 + y^2), \kappa^2 z\big),$$

where κ is constant. Find the components of curl $\boldsymbol{w}$ in rectangular and cylindrical polar coordinates.

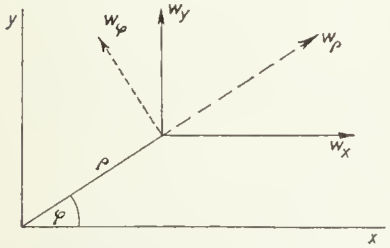

Fig. 15.5

In rectangular coordinates we use (15.19); clearly the x- and y-components of curl $\boldsymbol{w}$ are zero, while

$$\begin{aligned} (\operatorname{curl} \boldsymbol{w})_z &= \frac{\partial}{\partial x}[x(x^2 + y^2)] - \frac{\partial}{\partial y}[-y(x^2 + y^2)] \\ &= (x^2 + y^2) + x \cdot 2x + (x^2 + y^2) + y \cdot 2y \\ &= 4(x^2 + y^2). \end{aligned}$$

Thus curl $\boldsymbol{w} = \big(0, 0, 4(x^2 + y^2)\big)$.

In terms of cylindrical polars (ρ, φ, z), the *rectangular* components of $\boldsymbol{w}$ are

$$(w_x, w_y, w_z) = (-\rho^3 \sin\varphi, \rho^3 \cos\varphi, \kappa^2 z).$$

The components w_ρ, w_φ are in the xy-plane, in the directions shown in fig. 15.5; they can be expressed in terms of w_x, w_y, giving

$$\begin{aligned} w_\rho &= w_x \cos\varphi + w_y \sin\varphi = 0, \\ w_\varphi &= -w_x \sin\varphi + w_y \cos\varphi = \rho^3. \end{aligned}$$

Thus $(w_\rho, w_\varphi, w_z) = (0, \rho^3, \kappa^2 z)$. Using (15.20) with $u_1=\rho$, $u_2=\varphi$, $u_3=z$ and $h_1=h_3=1$, $h_2=\rho$, we find

$$\operatorname{curl} \boldsymbol{w} = \left(0, 0, \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^4)\right) = (0, 0, 4\rho^2).$$

This result clearly agrees with the value of curl $\boldsymbol{w}$ in rectangular coordinates, since $\rho^2 = x^2+y^2$.

EXERCISE 15.2

1. Establish the formula (15.20) for the components of curl $\boldsymbol{w}$ in orthogonal coordinates u_k, starting from the definition (15.16).

2. A vector field $\boldsymbol{w}$ has rectangular components x^2, $xy+xz$, yz. Find the components of curl $\boldsymbol{w}$ in rectangular, cylindrical polar and spherical polar coordinates. Establish the equivalence of the expressions in cylindrical and spherical polar coordinates.

§ 3.2. Stokes' theorem, irrotational fields

The definition (15.16) relates curl $\boldsymbol{w}$ directly to the scalar line integral of $\boldsymbol{w}$ round a closed curve whose size tends to zero. *Stokes' theorem* is essentially the integral of this definition: if S is any *finite* surface element

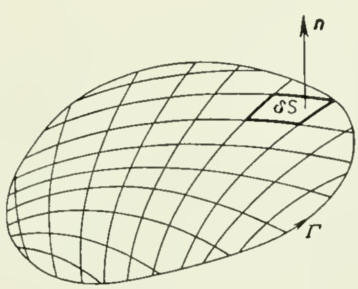

Fig. 15.6

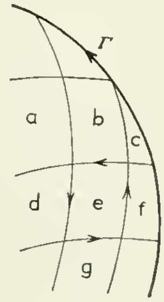

Fig. 15.7

in 3-space bounded by a simple closed curve Γ, then the theorem states that

$$\iint_S (\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{w})\, \mathrm{d}S = \oint_\Gamma \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{s}. \tag{15.21}$$

The integrand in the surface integral is the normal component of curl $\boldsymbol{w}$, $\boldsymbol{n}$ being the normal to S related to the direction of integration round Γ by the right hand rule, as before.

To establish (15.21), we divide the surface S into a large number of small elements by a mesh, as in fig. 15.6: fig. 15.7 shows a portion of S

and Γ enlarged. If δS is the area of an element of S defined by the mesh, then by (15.16) we know that when δS is small,

$$(\boldsymbol{n}\cdot\text{curl }\boldsymbol{w})\,\delta S \approx \oint_{\delta S} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}$$

curl $\boldsymbol{w}$ being evaluated at a point within δS. If we sum this equation over all elements δS of S, we find

$$\sum_{\text{elements}} (\boldsymbol{n}\cdot\text{curl }\boldsymbol{w})\,\delta S \approx \sum_{\text{elements}} \oint_{\delta S} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s} \tag{15.22}$$

the equation becoming an equality in the limit of a fine mesh. Let us examine the right hand sum in (15.22); it consists of a large number of elemental line integrals along segments of the mesh, a great many of which cancel. For example $\boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}$ round the element e in fig. 15.7 can be split into four elemental integrals in the direction of the arrows; each of these exactly cancels another elemental integral in the opposite direction, arising from d, b, f or g. In fact, all integrals along elements of the actual mesh cancel, but integrals along elements of Γ do not cancel; for example, the elemental integrals along two sides of c cancel integrals from b and f, but the integral along the portion Γ is not cancelled. Hence

$$\sum_{\text{elements}} \oint_{\delta S} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s} = \oint_{\Gamma} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}. \tag{15.23}$$

Substituting (15.23) into (15.22) and letting all elemental areas $\delta S \to 0$, we obtain (15.21). Stokes' theorem in essence means that 'circulation' is an additive quantity, the circulation round the circuit being the sum of the circulations round the small circuits of the mesh.

We have shown that a field $\boldsymbol{w}$ is conservative if and only if $\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}=0$ round all closed circuits. We can now express this condition as

$$\text{curl }\boldsymbol{w} \equiv 0 \tag{15.24}$$

throughout space. The proof is immediate; for if $\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}=0$ round all circuits, then curl $\boldsymbol{w}=0$ by the definition (15.16); and if curl $\boldsymbol{w}=0$ everywhere, Stokes' theorem (15.21) tells us that $\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}=0$ round any closed circuit. A field $\boldsymbol{w}$ for which (15.24) holds is called an *irrotational field*, there being no tendency to circulate in any region of space. We have therefore proved the equivalence of three conditions on a field $\boldsymbol{w}$:

(i) $\boldsymbol{w}$ is derivable from a potential, $\boldsymbol{w}=-\text{grad }\psi$.

(ii) $\boldsymbol{w}$ is conservative, $\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}=0$ round all closed circuits.

(iii) $\boldsymbol{w}$ is irrotational, curl $\boldsymbol{w}=0$ everywhere.

The fact that condition (i) implies condition (iii) can be expressed as the mathematical identity

$$\operatorname{curl} \operatorname{grad} \psi \equiv 0 \tag{15.25}$$

for any scalar field ψ. This identity can easily be proved directly for any orthogonal coordinate system, using (15.6) and (15.20).

Example 5

Show that the vector field

$$\boldsymbol{w} = \left(yz^2(x+y)\exp(x/y),\ xz^2(2y-x)\exp(x/y),\ 2xy^2z\exp(x/y)\right)$$

is a conservative field.

We have to show that curl $\boldsymbol{w} \equiv 0$. The rectangular components of curl $\boldsymbol{w}$ are given by (15.19) and are

$$\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} = [4xyz + 2xy^2z(-xy^{-2}) - 2xz(2y-x)]\exp(x/y),$$

$$\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} = [2yz(x+y) - 2y^2z - 2xy^2z\cdot y^{-1}]\exp(x/y),$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = [z^2(2y-2x) + xz^2(2y-x)y^{-1} - z^2(x+2y) - yz^2(x+y)(-xy^{-2})]\exp(x/y);$$

these components are all zero, so $\boldsymbol{w}$ is conservative.

Example 6

The electrostatic field $\boldsymbol{E}$ due to a point charge e was derived in Example 3. Using (15.20) with $u_1=r$, $u_2=\theta$, $u_3=\varphi$ and $h_1=1$, $h_2=r$, $h_3=r\sin\theta$, curl $\boldsymbol{E}$ becomes

$$\left(0,\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(r\cdot\frac{e}{r^2}\right),\ \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\cdot\frac{e}{r^2}\right)\right),$$

and is clearly identically zero.

§ 3.3. Stokes' theorem in two dimensions

Stokes' theorem leads to an important formula connecting line integrals and double integrals of functions of two variables x, y. Suppose that $u(x, y)$ and $v(x, y)$ are functions of x, y, and that S is a region in the xy-plane bounded by a simple closed curve C; then Stokes' theorem becomes

$$\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_C [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y]. \tag{15.26}$$

This result is known as 'Stokes' theorem in two dimensions'. It is obtained from (15.21) by completing the triad of rectangular axes by adding the z-axis and taking $\Gamma = \mathrm{C}$. Then, as in fig. 15.8, the normal $\boldsymbol{n}$ to the surface S in the xy-plane is along the z-axis at every point, or $\boldsymbol{n} = (0, 0, 1)$. The surface element is $\mathrm{d}S = \mathrm{d}x\, \mathrm{d}y$, and the differential displacement vector along the curve C is $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y, 0)$. In (15.21), we take the vector $\boldsymbol{w}$ to be

$$\boldsymbol{w} = \big(v(x, y),\ u(x, y),\ 0\big). \tag{15.27}$$

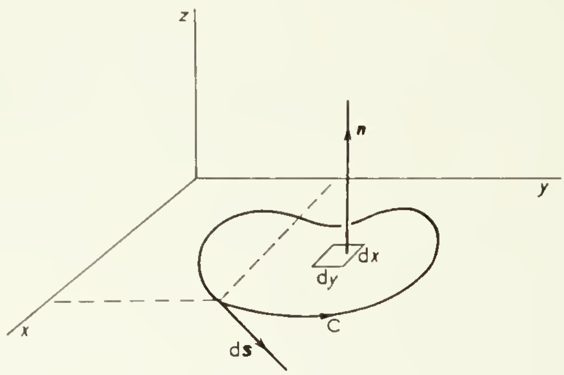

Fig. 15.8

Then since $\partial u/\partial z = \partial v/\partial z = 0$, (15.19) gives

$$\operatorname{curl} w = \left(0, 0, \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right).$$

Substituting these values of $\boldsymbol{n}$, $\mathrm{d}S$, $\mathrm{d}\boldsymbol{s}$, $\boldsymbol{w}$ and curl $\boldsymbol{w}$ in (15.21), we obtain the result (15.26).

We note that we could have assumed the vector field $\boldsymbol{w}$ to have the more general form

$$\boldsymbol{w} = \big(v(x, y),\ u(x, y),\ t(z)\big), \tag{15.28}$$

where $t(z)$ is independent of x, y. In the xy-plane, $t(z)$ is of course just the constant $t(0)$, and it is easy to check that it does not contribute to either integral in (15.19) when S is an area in the xy-plane.

Example 7

Consider the integral $\oint_{\mathrm{C}} \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{s}$ of the vector field

$$\boldsymbol{w} = \big((x^2 + y^2)y,\ -(x^2 + y^2)x,\ a^3 + z^3\big)$$

round the circle $x^2 + y^2 = a^2$ in the xy-plane, denoted in fig. 15.9 by $\mathrm{C} = \Gamma$. The

integral can be evaluated directly, using polar coordinates ρ, φ in the plane: on the circle $\rho=a$, $\mathrm{d}\boldsymbol{s}=(\mathrm{d}x, \mathrm{d}y, 0)=a\,\mathrm{d}\varphi(-\sin\varphi, \cos\varphi, 0)$, while

$$\boldsymbol{w} = (a^2y, -a^2x, a^3) = a^3(\sin\varphi, -\cos\varphi, 1).$$

Since the range of φ required to traverse C is $0<\varphi<2\pi$,

$$\oint \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s} = -a^4 \int_{\theta=0}^{2\pi} (\sin^2\varphi + \cos^2\varphi)\,\mathrm{d}\varphi = -2\pi a^4.$$

Now by (15.26) and (15.28) the integral is equal to the double integral over the interior S of the circle C of

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial x}[-(x^2+y^2)x] - \frac{\partial}{\partial y}[(x^2+y^2)y]$$

$$= -4(x^2+y^2) = -4\rho^2.$$

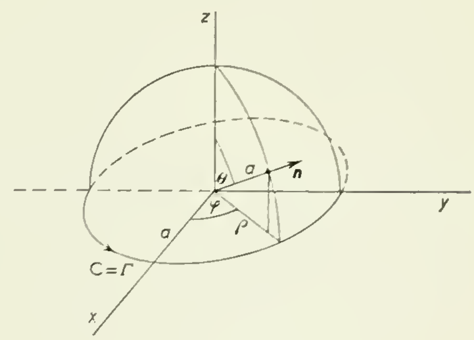

Fig. 15.9

Since $\mathrm{d}S=\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$ in polar coordinates, this double integral is

$$-\int_{\rho=0}^{a}\int_{\varphi=0}^{2\pi} 4\rho^2\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = -2\pi a^4,$$

which is correct.

Using the three-dimensional form of Stokes' theorem (15.21), we know that the integral $\oint_C \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{s}$ is also equal to the integral

$$\iint_S (\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{w})\,\mathrm{d}S$$

over the hemisphere of radius a, bounded by $\mathrm{C}=\Gamma$, and sketched in fig. 15.9. This integral can be evaluated independently using spherical polar coordinates (r, θ, φ). However, to avoid transforming the vector curl $\boldsymbol{w}$ from the rectangular to the spherical polar frame, we evaluate $\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{w}$ using *rectangular components* of the vectors:

$$\boldsymbol{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta).$$

The components of curl $\boldsymbol{w}$ are the same as before, since $\boldsymbol{w}$ is of the form (15.28); thus

$$\operatorname{curl}\boldsymbol{w} = (0, 0, -4\rho^2) = (0, 0, -4a^2\sin^2\theta)$$

on the hemisphere. Hence

$$\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{w} = -4a^2\sin^2\theta\cos\theta.$$

The element of surface of the hemisphere is $\mathrm{d}S=a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. Thus the hemispherical integral is

$$-4a^4\int_0^{\frac{1}{2}\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\,\sin^2\theta\cos\theta\sin\theta = -8\pi a^4\int_0^{\frac{1}{2}\pi}\sin^3\theta\,\mathrm{d}(\sin\theta) = -2\pi a^4,$$

agreeing with the previous two results.

EXERCISE 15.3

1. Show that the following vector fields are conservative:

(i) $(2yz + 4xy,\ 2xz + 2x^2,\ 2xy + z^2)$;

(ii) $\left(\dfrac{2\rho}{z}\sin 2\varphi,\ \dfrac{2\rho}{z}\cos 2\varphi,\ -\dfrac{\rho^2}{z^2}\sin 2\varphi\right)$;

(iii) $(2r\theta \sin\theta \cos^2\varphi,\ r(\sin\theta + \theta\cos\theta)\cos^2\varphi,\ -r\theta\sin 2\varphi)$.

[The standard notation for polar coordinates is used.]

2. A vector field $\boldsymbol{w}$ is defined in terms of rectangular components relative to axes with origin at O by

$$\boldsymbol{w} = (x^2,\ y^2 + z^2,\ x^2 + xy + xz).$$

The sphere $x^2+y^2+z^2=a^2$ intersects the positive x-, y- and z-axes at A, B, C respectively. The simple closed curve Γ consists of the three circular arcs AB, BC and CA. Evaluate the scalar line integral $\oint_\Gamma \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{s}$ directly. Use Stokes' theorem to express this integral as a surface integral over

(i) the surface ABC of the octant of the sphere in the positive quadrant

(ii) the three quadrants of circles in the coordinate planes.

Evaluate these surface integrals by direct integration.

§ 4. The divergence of a vector field

The concept of the *flow* or *flux* of a vector field is fundamental in many branches of physics, most obviously in fluid dynamics and current electricity. If we consider the velocity $\boldsymbol{v}$ of a fluid, then the rate of flow of fluid through a differential surface element $\mathrm{d}S$ is $v\cos\theta\,\mathrm{d}S$, where θ is the angle between $\boldsymbol{v}$ and the normal $\boldsymbol{n}$ to $\mathrm{d}S$, as in fig. 15.10; only the component of $\boldsymbol{v}$ along $\boldsymbol{n}$ contributes to the flow, since any component of $\boldsymbol{v}$ normal to $\boldsymbol{n}$ does not flow *through* $\mathrm{d}S$. The rate of flow can be written $(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S$; $\boldsymbol{n}$ indicates the direction in which we are measuring flow; if the normal component of $\boldsymbol{v}$ is in the opposite direction to $\boldsymbol{n}$, the flow is negative. More generally, we define the flux of any vector field $\boldsymbol{w}$ through an element $\mathrm{d}S$ as

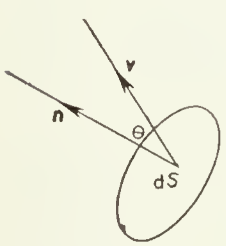

Fig. 15.10

$$(\boldsymbol{w}\cdot\boldsymbol{n})\,\mathrm{d}S; \tag{15.29}$$

it is clearly a scalar quantity, but is dependent both on the magnitude of $\mathrm{d}S$ and the direction of its normal $\boldsymbol{n}$.

If we consider now a finite region τ of 3-space bounded by a simple closed surface S, then the flux of $\boldsymbol{w}$ out of the region τ is given by summing

(15.29) over all small elements δS of the surface S. In the limit with every $\delta S \to 0$, the outward flux becomes the surface integral

$$\iint_S (\boldsymbol{w}\cdot\boldsymbol{n})\,\mathrm{d}S, \tag{15.30}$$

the normal $\boldsymbol{n}$ pointing *out* of τ at every point of S. Just as it was convenient to define curl $\boldsymbol{w}$ to measure the 'circulation of $\boldsymbol{w}$ at a point', it is also convenient to define a scalar field to measure the 'outflow at a point'. This scalar field is the *divergence* of $\boldsymbol{w}$, and is denoted by div $\boldsymbol{w}$; it is defined as

$$\operatorname{div}\boldsymbol{w} = \lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\iint_{\delta S}(\boldsymbol{w}\cdot\boldsymbol{n})\,\mathrm{d}S, \tag{15.31}$$

$\delta\tau$ being the small volume enclosed by the small closed surface δS. Comparing with (15.30), we see that div $\boldsymbol{w}$ measures the 'outflow per unit volume'.

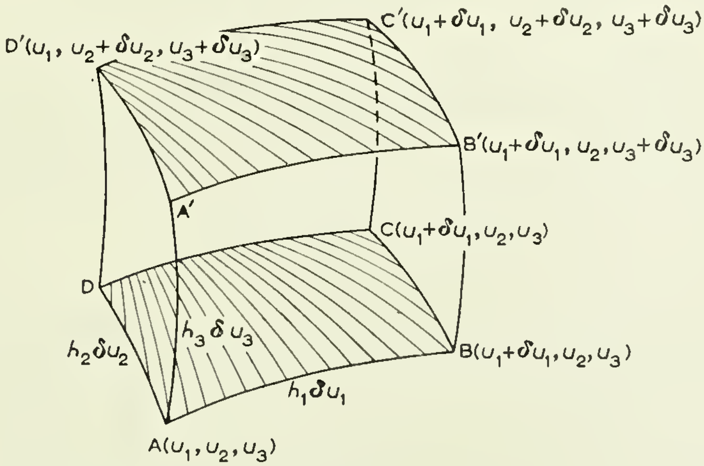

Fig. 15.11

§ 4.1. Divergence in orthogonal coordinate systems

An expression for div $\boldsymbol{w}$ can be found in terms of arbitrary orthogonal coordinates u_k ($k=1, 2, 3$). We choose the volume $\delta\tau$ in (15.31) to be a nearly rectangular solid with opposite corners at points A(u_1, u_2, u_3) and C'($u_1+\delta u_1$, $u_2+\delta u_2$, $u_3+\delta u_3$), as shown in fig. 15.11. The edges AB, AD, AA' are along the orthogonal coordinate axes, and so are of length $h_1\,\delta u_1$, $h_2\,\delta u_2$, $h_3\,\delta u_3$ approximately. Let us calculate the contribution to $\iint(\boldsymbol{w}\cdot\boldsymbol{n})\,\mathrm{d}S$ from the shaded faces ABCD and A'B'C'D'. If w_1, w_2, w_3

are the components of $\boldsymbol{w}$ along the orthogonal axes, then $\boldsymbol{w}\cdot\boldsymbol{n}=-w_3$ on ABCD, and $\boldsymbol{w}\cdot\boldsymbol{n}=+w_3$ on A′B′C′D′. Hence the contribution is

$$\iint\limits_{\mathrm{A'B'C'D'}} w_3h_1h_2\,\mathrm{d}u_1\,\mathrm{d}u_2 - \iint\limits_{\mathrm{ABCD}} w_3h_1h_2\,\mathrm{d}u_1\,\mathrm{d}u_2$$

$$\approx \iint\limits_{\mathrm{ABCD}} \delta u_3\,\frac{\partial}{\partial u_3}(w_3h_1h_2)\,\mathrm{d}u_1\,\mathrm{d}u_2, \qquad (15.32)$$

since corresponding points on ABCD and A′B′C′D′ differ only by a change δu_3 in the coordinate u_3. To lowest order in δu_1, δu_2, δu_3, the contribution (15.32) becomes

$$\delta u_1\,\delta u_2\,\delta u_3\,\frac{\partial}{\partial u_3}(w_3h_1h_2),$$

the derivative being evaluated at the point A. Since the volume $\delta\tau\approx\prod_{k=1}^{3} h_k\,\delta u_k$, this can be written to lowest order as

$$\frac{\delta\tau}{h_1h_2h_3}\,\frac{\partial}{\partial u_3}(w_3h_1h_2).$$

Thus the contribution of the two faces to the limit (15.31) is

$$\frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial u_3}(w_3h_1h_2).$$

Similar contributions arise from the other two pairs of faces, so that we have finally

$$\operatorname{div}\boldsymbol{w}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(w_1h_2h_3)+\frac{\partial}{\partial u_2}(w_2h_3h_1)+\frac{\partial}{\partial u_3}(w_3h_1h_2)\right]. \qquad (15.33)$$

For rectangular coordinates x, y, z we put $h_1=h_2=h_3=1$ in (15.33); this gives

$$\operatorname{div}\boldsymbol{w}=\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}. \qquad (15.34)$$

Example 8

Evaluate div $\boldsymbol{w}$ in cylindrical polar coordinates.

We put $u_1=\rho$, $u_2=\varphi$, $u_3=z$, and $h_1=h_3=1$, $h_2=\rho$ in (15.33); this gives

$$\operatorname{div}\boldsymbol{w}=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho w_\rho)+\frac{\partial w_\varphi}{\partial\varphi}+\rho\,\frac{\partial w_z}{\partial z}\right].$$

Example 9

Find the divergence of the vector field whose rectangular components are

$$(w_x,\ w_y,\ w_z) = \left(-y(x^2+y^2),\ x(x^2+y^2),\ \kappa^2 z\right).$$

We can either use (15.34) to give

$$\operatorname{div} \boldsymbol{w} = -y\cdot 2x + x\cdot 2y + \kappa^2 = \kappa^2$$

or we can write $\boldsymbol{w}$ in cylindrical polar coordinates, as in Example 4:

$$(w_\rho,\ w_\varphi,\ w_z) = (0,\ \rho^3,\ \kappa^2 z).$$

Then the result of Example 8 gives

$$\operatorname{div} \boldsymbol{w} = \frac{\partial w_z}{\partial z} = \kappa^2.$$

§ 4.2. Green's theorem

In working with integrals of the type (15.30) it is convenient to introduce a vector differential $\mathrm{d}\boldsymbol{S}$ defined by

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S, \tag{15.35}$$

known often as the *vector element of surface*; the vector $\mathrm{d}\boldsymbol{S}$ indicates both the magnitude and the orientation of the element $\mathrm{d}S$. The definition (15.31) is then written as

$$\operatorname{div} \boldsymbol{w} = \lim_{\delta\tau\to 0} \frac{1}{\delta\tau} \iint\limits_{\delta S} \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{S}. \tag{15.36}$$

In § 3.2, we established Stokes' theorem by integrating the definition (15.16) of curl $\boldsymbol{w}$ over a finite surface S. We can similarly integrate the definition (15.36) over a finite volume τ to establish *Green's theorem*, which states that

$$\iiint\limits_{\tau} \operatorname{div} \boldsymbol{w}\,\mathrm{d}\tau = \iint\limits_{S} \boldsymbol{w}\cdot\mathrm{d}\boldsymbol{S}, \tag{15.37}$$

where S is the surface enclosing the volume τ. The physical meaning of Green's theorem is clear: $\operatorname{div} \boldsymbol{w}\,\mathrm{d}\tau$ is the flux out of a differential element $\mathrm{d}\tau$, so that the volume integral on the left of (15.37) measures the total flux out of τ. This is equal to the surface integral of $\boldsymbol{w}\cdot\mathrm{d}\boldsymbol{S}$, giving the flux through the bounding surface S. The proof of the theorem, which is closely analogous to the proof of Stokes' theorem given in § 3.2, is left as an exercise to the reader.

EXERCISE 15.4

1. Starting from the definition (15.36), establish Green's theorem, equation (15.37).

2. The components of a vector field $\boldsymbol{w}$ along ρ-, φ-, and z-axes in cylindrical polar coordinates are

$$(\rho z \cos \varphi,\ \rho^2 \sin \varphi,\ \rho^2 \sin 2\varphi).$$

Calculate div $\boldsymbol{w}$ using (i) rectangular, (ii) cylindrical polar, (iii) spherical polar coordinates.

3. Apply Green's theorem to the vector field

$$\boldsymbol{w} = (v(x, y),\ -u(x, y),\ 0),$$

the volume of integration τ being a right cylinder of height h whose base is an area bounded by a simple closed curve C in the xy-plane. Deduce Stokes' theorem in two dimensions.

§ 5. Rectangular coordinates; vector operators

§ 5.1. The operator ∇

The most frequently used system of coordinates is the rectangular system (x, y, z). The field quantities grad ψ, curl $\boldsymbol{w}$ and div $\boldsymbol{w}$ are especially simple in this system since the elements of distance along the axes are simply δx, δy and δz; in the notation of curvilinear coordinates,

$$h_1 \equiv h_2 \equiv h_3 = 1 \tag{15.38}$$

at all points of space and (15.3), (15.19) and (15.34) give for rectangular coordinates

$$\text{grad}\ \psi = \left(\frac{\partial \psi}{\partial x},\ \frac{\partial \psi}{\partial y},\ \frac{\partial \psi}{\partial z}\right), \tag{15.39}$$

$$\text{curl}\ \boldsymbol{w} = \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z},\ \frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x},\ \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right), \tag{15.40}$$

$$\text{div}\ \boldsymbol{w} = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}. \tag{15.41}$$

These three formulae can be put into a very neat form if we introduce the vector differential operator *del*, defined as

$$\nabla = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right). \tag{15.42}$$

The three components of $\boldsymbol{\nabla}$ are treated in vector formulae as though they were components of a vector, and they 'operate on' (that is, they differentiate) any function written to their right, as single differential operators do. We then see that (15.39), (15.40) and (15.41) can be written

$$\operatorname{grad} \psi = \boldsymbol{\nabla}\psi, \tag{15.43}$$

$$\operatorname{curl} \boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{w}, \tag{15.44}$$

$$\operatorname{div} \boldsymbol{w} = \boldsymbol{\nabla} \cdot \boldsymbol{w}. \tag{15.45}$$

Since in (15.43) for example, grad ψ transforms like a vector and ψ as a scalar under rotations of axes, the operator $\boldsymbol{\nabla}$ must transform like a vector; this justifies the description 'vector differential operator'.

The introduction of $\boldsymbol{\nabla}$ facilitates work when we are using rectangular coordinates. But it must be remembered that the formulae (15.43)–(15.45) apply *only for rectangular coordinates*, and mistakes have occurred even in serious research work through using the $\boldsymbol{\nabla}$ operator in other coordinate systems. For general orthogonal coordinates, we see from (15.6) and (15.33) that the gradient and divergence operators satisfying (15.43) and (15.45) are

$$\operatorname{div} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\, h_2h_3,\ \frac{\partial}{\partial u_2}\, h_3h_1,\ \frac{\partial}{\partial u_3}\, h_1h_2\right)$$

and

$$\operatorname{grad} = \left(\frac{1}{h_1}\,\frac{\partial}{\partial u_1},\ \frac{1}{h_2}\,\frac{\partial}{\partial u_2},\ \frac{1}{h_3}\,\frac{\partial}{\partial u_3}\right).$$

These two operators are different in general since in the first component for example of the divergence operator, (h_2h_3) is differentiated with respect to u_1, which does not happen in the gradient operator. Furtheir more, if we compare the expression (15.20) with (15.44), we see that it is not possible in general to write curl $\boldsymbol{w}$ in the form $\boldsymbol{\nabla} \times \boldsymbol{w}$.

§ 5.2. Formulae involving vector operators

Using the operator $\boldsymbol{\nabla}$ and the properties of scalar and vector products, it is easy to prove several very useful formulae involving the operators grad, curl and div. Five of these formulae are

(i) $\operatorname{div}(\psi\boldsymbol{w}) = \psi \operatorname{div} \boldsymbol{w} + \boldsymbol{w} \cdot \operatorname{grad} \psi$,

(ii) $\operatorname{curl}(\psi\boldsymbol{w}) = \psi \operatorname{curl} \boldsymbol{w} + \operatorname{grad} \psi \times \boldsymbol{w}$,

(iii) $\operatorname{div}(\boldsymbol{w}_1 \times \boldsymbol{w}_2) = \boldsymbol{w}_2 \cdot \operatorname{curl} \boldsymbol{w}_1 - \boldsymbol{w}_1 \cdot \operatorname{curl} \boldsymbol{w}_2$,
(iv) $\operatorname{curl} \operatorname{grad} \psi = 0$,
(v) $\operatorname{div} \operatorname{curl} \boldsymbol{w} = 0$.

Formula (ii), for example, follows from the rule for differentiating a product:

$$\boldsymbol{\nabla} \times \psi \boldsymbol{w} = \psi(\boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{\nabla}\psi) \times \boldsymbol{w}.$$

Formula (iv) merely states that

$$(\boldsymbol{\nabla} \times \boldsymbol{\nabla})\psi = 0,$$

which is obvious from the definition of a vector product. The serious student is advised to check the validity of the other three formulae.

Although the formulae (i)–(v) have been established using a particular system of coordinates (rectangular ones), we can deduce that they are true for all coordinate systems. This follows from the fact that we defined the operators grad, curl and div *in a way which was independent of any coordinate system*. So when an equation involving only scalars, vectors and these operators has been established in one coordinate system, we may change to any other coordinate systems, knowing that the quantities on each side of the equation will transform in the same way – for example, scalar quantities will be unchanged by any change of coordinate system. Therefore the equation will hold in any coordinate system. This argument applies to each of the formulae (i)–(v), which are therefore true in any coordinate system.

We can in addition establish a formula for curl curl $\boldsymbol{w}$ in rectangular coordinates. Using the $\boldsymbol{\nabla}$ operator and formula (9.81) for a triple vector product, we find

$$\begin{aligned} \operatorname{curl} \operatorname{curl} \boldsymbol{w} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{w}) \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{w}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{w}. \end{aligned} \tag{15.46}$$

In the first terms, $\boldsymbol{\nabla} \cdot \boldsymbol{w}$ is the scalar div $\boldsymbol{w}$, so the term is grad div $\boldsymbol{w}$. The second term is, however, not identifiable using only (15.43), (15.44) and (15.45). The operator scalar product $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is often abbreviated as $\boldsymbol{\nabla}^2$, and has the form

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \tag{15.47}$$

in (15.46), $\boldsymbol{\nabla}^2$ operates on the three components of $\boldsymbol{w}$. We can therefore write (15.46) as

$$\operatorname{curl} \operatorname{curl} \boldsymbol{w} = \operatorname{grad} \operatorname{div} \boldsymbol{w} - \boldsymbol{\nabla}^2 \boldsymbol{w}. \tag{15.48}$$

We note that this formula applies *only for rectangular coordinates*, since the term $\boldsymbol{\nabla}^2 \boldsymbol{w}$ is defined in terms of these coordinates and is not expressible in terms of the operators grad, curl and div; the argument which ensures the general validity of formulae (i)–(v) therefore fails for (15.48).

If the operator $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ occurring in (15.46) is applied to a *scalar* field ψ, then from (15.43) and (15.34),

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi = \operatorname{div}\operatorname{grad}\psi. \tag{15.49}$$

The operator $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ can thus be written in a form which is independent of coordinate systems *when it operates on a scalar*; it is then usually abbreviated as ∇^2, and is known as the *Laplace operator* or the *Laplacian.* In rectangular coordinates, ∇^2 coincides with $\boldsymbol{\nabla}^2$, given by (15.47); in other orthogonal systems, $\nabla^2\psi \equiv \operatorname{div}\operatorname{grad}\psi$ is given by (15.6) and (15.33). The Laplacian ∇^2 is very important in many branches of mathematical physics. In the next two sections we shall see how it arises in fluid mechanics and in electrostatics.

EXERCISE 15.5

1. By applying Green's theorem to the vector $\psi\boldsymbol{a}$, where ψ is a scalar field and $\boldsymbol{a}$ is any constant vector, show that

$$\iiint_\tau \operatorname{grad}\psi\,\mathrm{d}\tau = \iint_S \psi\,\mathrm{d}\boldsymbol{S}$$

where S is the simple closed boundary of the volume τ.

2. Apply Green's theorem to the vector field $\boldsymbol{w}\times\boldsymbol{a}$, where $\boldsymbol{a}$ is constant, to prove that

$$\iiint_\tau \operatorname{curl}\boldsymbol{w}\,\mathrm{d}\tau = \iint_S \mathrm{d}\boldsymbol{S}\times\boldsymbol{w}.$$

§ 6. The kinematics of fluids

We introduced the field operators div and curl by associating them with the physical concepts of 'outflow' and 'circulation', which are most easily pictured as properties of fluids. In this section we shall describe the kinematic properties of fluids in terms of the field operators.

§ 6.1. The equation of continuity

Let us consider a fluid of density ρ; in general ρ is a function of the position vector $\boldsymbol{r}$ in the fluid, and will also vary with the time t; we

assume that $\rho(\boldsymbol{r}, t)$ is a 'well-behaved' function of position, being continuous and having continuous partial derivatives. The velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ of the fluid in general varies with position and time, and so is a vector field. At a given time, the volume of fluid crossing a surface element $\mathrm{d}S$ per unit time is $\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$, so that the mass of fluid crossing per unit time is $\rho\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$. Thus if τ is any volume within the fluid bounded by a closed surface S, the mass of fluid flowing out of the region is

$$\iint_S \rho\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$$

which can be written, using (15.37), as

$$\iiint_\tau \operatorname{div}(\rho\boldsymbol{v})\,\mathrm{d}\tau. \tag{15.50}$$

Now the mass within the volume τ at time t is

$$M_\tau = \iiint_\tau \rho(\boldsymbol{r}, t)\,\mathrm{d}\tau;$$

if the volume τ does not vary with time, the rate of change of mass within τ is governed by the rate of change of the density of the fluid inside τ, and is

$$\frac{\mathrm{d}M_\tau}{\mathrm{d}t} = \iiint_\tau \frac{\partial\rho(\boldsymbol{r}, t)}{\partial t}\,\mathrm{d}\tau. \tag{15.51}$$

Now any increase in mass within τ must arise either from given sources of fluid within τ or by fluid flowing in over the boundary S. The amount flowing *in* over S is the negative of (15.50); so if we assume that the rate of production of fluid within τ is $m(\boldsymbol{r}, t)$ per unit volume, then the rate of increase of mass is

$$\iiint_\tau m(\boldsymbol{r}, t)\,\mathrm{d}\tau - \iiint_\tau \operatorname{div}(\rho\boldsymbol{v})\,\mathrm{d}\tau. \tag{15.52}$$

The integrals (15.51) and (15.52) are equal at all times; further, since they are equal for all fixed volumes τ within the fluid, the integrands must be equal everywhere. So

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = m \tag{15.53}$$

at all points within the fluid, and at all times t. Equation (15.53) is known as the *equation of continuity*, and expresses the law of conservation of mass.

A fluid is *incompressible* when $\rho(r, t)$ is a constant ρ_0, independent of

both $\boldsymbol{r}$ and t. Then and by the identity (i) in § 5.2,

$$\operatorname{div} \rho\boldsymbol{v} = \rho \operatorname{div} \boldsymbol{v} + \boldsymbol{v}\cdot\operatorname{grad}\rho = \rho_0 \operatorname{div} \boldsymbol{v}.$$

Thus equation (15.53) reduces to

$$\rho_0 \operatorname{div} \boldsymbol{v} = m \tag{15.54}$$

for incompressible fluids. In a region of the fluid where there are no sources, so that $m(r, t) = 0$, then (15.54) reduces simply to

$$\operatorname{div} \boldsymbol{v} = 0, \tag{15.55}$$

directly expressing the fact that the net outflow from any volume element in the region is zero.

§ 6.2. Circulation and irrotational flow

If C is a simple closed curve within the fluid, then the scalar line integral round C

$$\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s}$$

is the *circulation* of the fluid round this circuit. The circulation per unit area around an axis $\boldsymbol{n}$ at a point in the fluid is measured by the component

$$\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{v},$$

from the definition (15.16) of curl $\boldsymbol{v}$, and we say that curl $\boldsymbol{v}$ measures the *vorticity* at any point in the fluid. In general the vorticity is non-zero, but the study of *irrotational flow*, which has zero vorticity everywhere, is of fundamental importance in fluid dynamics. If we assume that curl $\boldsymbol{v} = \boldsymbol{0}$ throughout the fluid, then we know that $\boldsymbol{v}$ is a conservative vector field and is derivable from a potential. That is to say, there is a scalar field $\psi(\boldsymbol{r}, t)$ such that

$$\boldsymbol{v} = -\operatorname{grad}\psi. \tag{15.56}$$

The function $\psi(\boldsymbol{r}, t)$ is called the *velocity potential* for irrotational flow.

§ 6.3. The Laplace and Poisson equations

If the fluid is incompressible and the flow irrotational, then from (15.54) and (15.56) we have

$$\operatorname{div}\operatorname{grad}\psi = -\frac{m}{\rho_0}; \tag{15.57}$$

the operator div grad is the Laplace operator introduced at the end of § 5.2, and denoted by ∇^2. Thus (15.57) can be written

$$\nabla^2\psi = -\, m/\rho_0. \tag{15.58}$$

Equation (15.58) is *Poisson's equation.* In a region of the fluid where there are no sources of fluid, the equation reduces to *Laplace's equation*

$$\nabla^2\psi = 0 \tag{15.59}$$

which is one of the most important equations in the whole field of applied mathematics.

We often need to know solutions of Laplace's equation in various coordinate systems, so that it is useful to find an expression for ∇^2 in different coordinate systems. From (15.3) and (15.34) we see that

$$\nabla^2\psi \equiv \operatorname{div}\operatorname{grad}\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{15.60}$$

in rectangular coordinates. Also from (15.5) and (15.33) we see that for a system of orthogonal coordinates u_k,

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]. \tag{15.61}$$

It follows that for cylindrical polar coordinates ρ, φ, z, with $h_1 = h_3 = 1$ and $h_2 = \rho$,

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}. \tag{15.62}$$

For spherical polar coordinates r, θ, φ, with $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$,

$$\nabla^2\psi = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]. \tag{15.63}$$

Compare eqs. (15.62) and (15.63) with the result of transformations (i) and (ii) in Exercise 8.4, No. 19.

§ 7. The electrostatic field

It is an experimental fact that the force between two electric charges e and e_1 acts along the line joining the charges, obeys the inverse square law and is proportional to the two charges. If the charges are at positions $\boldsymbol{r}$ and $\boldsymbol{a}_1$, and $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{a}_1$ as the displacement from e_1 to e, as in fig. 15.12,

then the force exerted by e_1 on e, according to this law, is

$$f_1(r) = \frac{ee_1}{r_1^3}\, \boldsymbol{r}_1 \qquad (r_1 = |\boldsymbol{r}_1|).$$

We say that the force on e is due to the electric field

$$\boldsymbol{E}_1(\boldsymbol{r}) = \frac{e_1}{r_1^3}\, \boldsymbol{r}_1 \tag{15.64}$$

produced by the charge e_1, the force $\boldsymbol{f}_1$ being the product of the electric field $\boldsymbol{E}_1$ and the charge e. Generally, if charges e_k ($k=1, 2, \ldots n$) are at positions $\boldsymbol{a}_k$, then the force on a charge e at $\boldsymbol{r}$ is

$$\boldsymbol{f}(\boldsymbol{r}) = e \sum_{k=1}^{n} \frac{e_k}{r_k^3}\, \boldsymbol{r}_k,$$

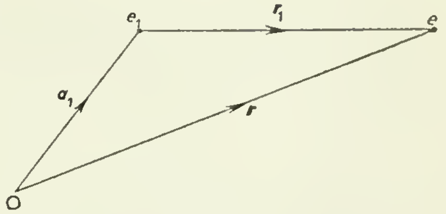

Fig. 15.12

where $\boldsymbol{r}_k = \boldsymbol{r} - \boldsymbol{a}_k$ is the displacement from e_k to e. That is to say, the charge e is in an electric field

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{k=1}^{n} \frac{e_k}{r_k^3}\, \boldsymbol{r}_k. \tag{15.65}$$

produced by the other charges.

§ 7.1. The electrostatic potential

The electric field $\boldsymbol{E}_1$, given by (15.64), is a conservative field since, as in Example 3,

$$\boldsymbol{E}_1(\boldsymbol{r}) = -\text{grad}\, \psi_1(\boldsymbol{r})$$

where $\psi_1 = e_1/r_1$; the gradient is taken, of course, at the field point $\boldsymbol{r}$. So as we checked in Example 6, $\boldsymbol{E}_1$ is irrotational. Generally, the electric field $\boldsymbol{E}$ of a number of point charges can be written

$$\boldsymbol{E}(\boldsymbol{r}) = -\text{grad}\, \psi(\boldsymbol{r}) \tag{15.66}$$

where

$$\psi(\boldsymbol{r}) = \sum_{k=1}^{n} \frac{e_k}{r_k}. \tag{15.67}$$

Thus the electrostatic field due to a number of point charges is conservative and irrotational, the potential being given by (15.67).

Small numbers of charged particles are often treated as point charges. But a large number of charged particles is more conveniently treated by defining a volume density of charge $\sigma(\boldsymbol{r})$; then $\sigma(\boldsymbol{a})\,\delta\tau$ is the total charge in a small element of volume $\delta\tau$ containing the point $\boldsymbol{a}$. The electrostatic potential $\psi(\boldsymbol{r})$ for a volume distribution is found by replacing the sum over charges in (15.67) by an integral over the charge density,

$$\psi(\boldsymbol{r}) = \iiint\limits_{(\boldsymbol{a})} \frac{\sigma(\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|}\,\mathrm{d}\tau. \tag{15.68}$$

The symbol $(\boldsymbol{a})$ attached to the integral sign indicates that the integration is over the coordinates of the charge density. The electric field at $\boldsymbol{r}$ is given again by (15.66).

§ 7.2. Gauss's theorem and Poisson's equation

The flux of the electric field $\boldsymbol{E}_1(\boldsymbol{r})$ through a differential element of surface $\mathrm{d}S$ is, by (15.29) and (15.64),

$$\boldsymbol{E}_1\cdot\mathrm{d}\boldsymbol{S} = \frac{e_1}{r_1^3}\,(\boldsymbol{r}_1\cdot\boldsymbol{n})\,\mathrm{d}S.$$

If θ is the angle between $\boldsymbol{r}_1$ and the unit normal $\boldsymbol{n}$, as in fig. 15.13, this flux is equal to

$$e_1\,\frac{\mathrm{d}S\cos\theta}{r_1^2} = e_1\,\mathrm{d}\omega_1$$

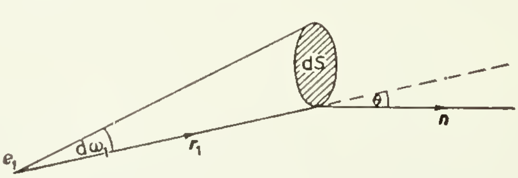

Fig. 15.13

where $\mathrm{d}\omega_1$ is the element of solid angle subtended at e_1 by $\mathrm{d}S$. So if e_1 lies *inside* a closed surface S, the total outward flux of $\boldsymbol{E}_1$ through S is

$$\iint e_1\,\mathrm{d}\omega_1 = 4\pi e_1. \tag{15.69}$$

If e_1 lies outside S, the total flux is zero. Equation (15.69) applies for all charges in the volume enclosed by S. So we can say quite generally:

the total outward flux of the electric field $\boldsymbol{E}$ *through a closed surface* S *equals* 4π *times the electric charge contained within* S. This is known as *Gauss's theorem.*

Let us apply Gauss's theorem to a volume distribution $\sigma(\boldsymbol{r})$, and a small closed surface δS enclosing a small volume $\delta\tau$. Assuming $\sigma(\boldsymbol{r})$ to be continuous, then as $\delta\tau \to 0$, the charge enclosed by δS is $\sigma(\boldsymbol{r})\,\delta\tau$, where $\boldsymbol{r}$ is inside $\delta\tau$. So the flux of $\boldsymbol{E}$ through δS is $4\pi\sigma(\boldsymbol{r})\,\delta\tau$, by Gauss's theorem. But by the definition (15.31), this flux is div $\boldsymbol{E}(\boldsymbol{r})\,\delta\tau$. Therefore

$$\operatorname{div} \boldsymbol{E}(\boldsymbol{r}) = 4\pi\sigma(\boldsymbol{r}). \tag{15.70}$$

This equation holds at all points within the charge distribution. Where there is no charge,

$$\operatorname{div} \boldsymbol{E} = 0, \tag{15.71}$$

implying that the net outward flux of $\boldsymbol{E}$ is zero for any region containing no charge; this is a law of conservation of flux, directly analogous to the law of conservation (15.55) of an incompressible fluid. Comparing (15.70) with (15.54), we see that apart from the factor 4π, the charge density $\sigma(\boldsymbol{r})$ is the 'source' of the electric field in the same way that $m(\boldsymbol{r})/\rho_0$ is the source of fluid. Equation (15.71) can be checked directly by using (15.65), or (15.66) and (15.68).

The field $\boldsymbol{E}$, unlike the velocity field $\boldsymbol{v}$ in an incompressible fluid, is always irrotational. From (15.70) and (15.66) the electrostatic potential satisfies Poisson's equation in the form

$$\nabla^2\psi = -4\pi\sigma; \qquad (15.71)$$

where the charge density σ is zero, this reduces to Laplace's equation (15.59).

ORDINARY DIFFERENTIAL EQUATIONS. THE LAPLACE TRANSFORM

§ 1. Introduction

An ordinary differential equation for a variable y as a function of a real variable x is an equality involving y and its derivatives with respect to x. The simplest type of differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \tag{16.1}$$

where $g(x)$ is a given function of x. From Ch. 2 § 2 we know that

$$y = \int_a^x g(t)\,\mathrm{d}t \tag{16.2}$$

is a solution of (16.1); this solution contains one arbitrary constant, which appears in (16.2) as the lower limit of integration a. Equation (16.1) contains only the first derivative of y, and is called a *first order differential equation*. An *n-th order differential equation* contains terms involving the nth derivative $\mathrm{d}^n y/\mathrm{d}x^n$, but no higher derivatives. Equation (16.1) is also a *linear equation* for y, meaning that each term in the equation involves y through at most a single factor y or $\mathrm{d}y/\mathrm{d}x$ or $\mathrm{d}^2y/\mathrm{d}x^2$ or .... Equations containing terms such as y^3, $y\,\mathrm{d}y/\mathrm{d}x$ or $(1+y)^{-1}$ are *non-linear*.

In this chapter we shall be mainly concerned with linear differential equations, though in § 2 we shall discuss a special class of non-linear equations. It can be shown that the most general solution of an nth order linear differential equation normally contains exactly n arbitrary constants, and later in the chapter we shall prove this theorem for a class of linear equations known as constant coefficient equations. When we know that a physical quantity y obeys a certain differential equation, these n constants are usually fixed by knowledge of certain particular values of y and its derivates. To take a concrete example, suppose that

y in equation (16.1) is the speed of a vehicle and x is the time. Then we are told that the acceleration $\mathrm{d}y/\mathrm{d}x$ is equal to the function $g(x)$; this is not sufficient information to determine the velocity y: we need also to be told the velocity at one particular time. If we know that y takes the value y_0 when $x=x_0$, then the arbitrary constant in (16.2) is determined, and the solution required is

$$y = \int_{x_0}^{x} g(t)\,\mathrm{d}t + y_0. \tag{16.3}$$

Conditions such as '$y=y_0$ when $x=x_0$' are called 'boundary conditions' or 'initial conditions'. In a particular physical problem, we often know that a differential equation is satisfied over a certain range of values of x, and the point x_0 where the boundary conditions are given is very often one of the end points of the range. The differential equation and appropriate boundary conditions determine the solution throughout the given range of x.

Example 1

A quantity y obeys the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{-x},$$

and $y=2$ when $x=0$. Find the solution.

The solution is given by (16.3) with $y_0=2$ and $x_0=0$:

$$y = \int_0^x t\mathrm{e}^{-t}\,\mathrm{d}t + 2 = [-(1+t)\mathrm{e}^{-t}]_0^x + 2 = -(1+x)\mathrm{e}^{-x} + 3.$$

§ 2. Separable equations of first degree

We have discussed the solution of equations of type (16.1). Almost as simple is the wider class of equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \beta(x)\,\gamma(y) \tag{16.4}$$

where $\beta(x)$ and $\gamma(y)$ are given functions of x and y. In general, this equation is non-linear since $\gamma(y)$ is not necessarily a linear function of y; the equation is however linear in $\mathrm{d}y/\mathrm{d}x$, and is therefore said to be a *first degree equation*. In general, an nth order differential equation is *of degree* r if it contains $\mathrm{d}^ny/\mathrm{d}x^n$ raised to the rth power, but not to any higher power.

The first degree equation (16.4) is of a particularly simple type known as *separable equations*, since it can be written as

$$\frac{\mathrm{d}y}{\gamma(y)} = \beta(x)\,\mathrm{d}x. \tag{16.5}$$

This equation connects the differentials $\mathrm{d}x$ and $\mathrm{d}y$ over some range of values of x. We can therefore integrate both sides of (16.5) over any part of this range, giving

$$\int_b^y \frac{\mathrm{d}u}{\gamma(u)} = \int_a^x \beta(t)\,\mathrm{d}t. \tag{16.6}$$

This solution contains an arbitrary constant of integration arising from the choice of lower limits a and b. If we are given the boundary condition $y=y_0$ when $x=x_0$, this constant is fixed, and the solution is

$$\int_{y_0}^y \frac{\mathrm{d}u}{\gamma(u)} = \int_{x_0}^x \beta(t)\,\mathrm{d}t. \tag{16.7}$$

Example 2

Find the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y^4,$$

and the solution for which $y=1$ when $x=2$.

We have $y^{-4}\,\mathrm{d}y = x^2\,\mathrm{d}x$. Integrating, $-y^{-3}=x^3-C$, where C is an arbitrary constant. So $y^{-3}=C-x^3$ is the general solution. If $y=1$ when $x=2$, then $1=C-8$, or $C=9$. Hence the solution is $y^{-3}=9-x^3$. This can also be found by using (16.7).

Example 3

Find the function $y(x)$ which obeys

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y^2)x\cos x,$$

and which has value -1 when $x=\frac{3}{2}\pi$.

Putting $\beta(t)=t\cos t$, $\gamma(u)=1+u^2$, $y_0=-1$, $x_0=\frac{3}{2}\pi$ in (16.7),

$$\int_{-1}^y \frac{\mathrm{d}u}{1+u^2} = \int_{\frac{3}{2}\pi}^x t\cos t\,\mathrm{d}t,$$

or $\tan^{-1}y-\tan^{-1}(-1)=(x\sin x+\cos x)+\frac{3}{2}\pi$. Now $\tan^{-1}(-1)=-\frac{1}{4}\pi+n\pi$, where n

is an arbitrary integer. So

$$\begin{aligned} y &= \tan[x \sin x + \cos x + \tfrac{7}{4}\pi + n\pi] \\ &= \tan[x \sin x + \cos x + \tfrac{7}{4}\pi]. \end{aligned}$$

Note that the arbitrary integer n does not affect the solution.

EXERCISE 16.1

1. Find the general solution of each of the following equations:

(i) $\dfrac{dy}{dx} = 4x^3 \cos 2x;$

(ii) $\dfrac{dy}{dx} = x \sin^{-1} x;$

(iii) $\dfrac{dy}{dx} = \dfrac{xy}{1 + x^2};$

(iv) $\dfrac{dy}{dx} = \dfrac{\sec y}{(1 + x^2)^{\frac{1}{2}}};$

(v) $\dfrac{dy}{dx} = (1 + y^2)(1 + x^2)^{\frac{1}{2}}.$

2. Find the solution of each of the following equations with the stated initial conditions:

(i) $\dfrac{dy}{dx} = \cos^{-1} x, \quad y = \dfrac{\pi}{2\sqrt{2}}$ when $x = -\frac{1}{4}\pi;$

(ii) $\dfrac{dy}{dx} = \dfrac{xy}{(1 + x^2)^{\frac{1}{2}}}, \quad y = 1$ when $x = 2\sqrt{2};$

(iii) $\dfrac{dy}{dx} = \dfrac{4 + y^2}{(1 + x^2)^{\frac{1}{2}}}, \quad y = \frac{5}{2}\pi$ when $x = -\frac{5}{2}\pi.$

§ 3. First order linear equations

We shall next discuss the most general first order linear equation, which is

$$\frac{dy}{dx} + \alpha(x)y = g(x), \tag{16.8}$$

$\alpha(x)$ and $g(x)$ being given functions of x. There are several methods of solving this type of equation. We shall first give a method which leads to a complete solution of (16.8), and which exemplifies a technique used to solve far more complicated equations. If we omit the term $g(x)$ from

(16.8) we obtain the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}x} + \alpha(x)Y = 0 \tag{16.9}$$

for a function $Y(x)$ of x. We say that (16.9) is the *homogeneous equation* associated with (16.8), since it contains *only* first degree terms in Y and $\mathrm{d}Y/\mathrm{d}x$; $Y(x)$ is called the *complementary function* of y. Now, equation (16.9) is separable, and so is soluble; we have in fact

$$Y^{-1}\,\mathrm{d}Y = -\alpha(x)\,\mathrm{d}x. \tag{16.10}$$

We shall find it convenient to specify the arbitrary constant in $Y(x)$ by choosing $Y(x_0)=1$ for some value x_0 of x. Then since $\log Y(x_0)=0$, the solution of (16.10) is

$$\log Y = -\int_{x_0}^{x} \alpha(t)\,\mathrm{d}t \quad \text{or} \quad Y(x) = \exp[-\int_{x_0}^{x} \alpha(t)\,\mathrm{d}t]. \tag{16.11}$$

Provided we can calculate the integral $\int_{x_0}^{x} \alpha(t)\,\mathrm{d}t$, $Y(x)$ is known. The method used to solve (16.8) is to treat the complementary function $Y(x)$ as a 'first approximation' to y, putting

$$y = Y(x)z, \tag{16.12}$$

where z is to be found. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha(x)y = \frac{\mathrm{d}Y}{\mathrm{d}x}z + Y\frac{\mathrm{d}z}{\mathrm{d}x} + \alpha(x)Yz = Y(x)\frac{\mathrm{d}z}{\mathrm{d}x}, \tag{16.13}$$

since Y satisfies equation (16.9). Using (16.13), equation (16.8) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{g(x)}{Y(x)},$$

with solution

$$z = \int_{a}^{x} \frac{g(t)}{Y(t)}\,\mathrm{d}t, \tag{16.14}$$

the lower limit of integration a being arbitrary. If $Y(x)$ has been calculated by eq. (16.11), the integrand in (16.14) is known, and we can calculate z. Then y is found immediately from (16.12). Suppose that we are given the boundary condition $y=y_0$ when $x=x_0$; since we have chosen $Y(x_0)=1$, then when $x=x_0$ (16.12) gives $z=y=y_0$. As in (16.3), this

condition is satisfied if we choose the solution (16.14) as

$$z = \int_{x_0}^{x} \frac{g(t)}{Y(t)}\,\mathrm{d}t + y_0. \tag{16.15}$$

Equations (16.12) and (16.15) give

$$y = Y(x)\left[\int_{x_0}^{x} \frac{g(t)}{Y(t)}\,\mathrm{d}t + y_0\right] \tag{16.16}$$

as the solution of (16.8) which has $y = y_0$ when $x = x_0$, the complementary function $Y(x)$ being defined by (16.11).

This method of solution is often called the *integrating factor method,* since multiplication of (16.8) by $1/Y(x)$ yields

$$\frac{1}{Y(x)}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha(x)\,y\right] = \frac{g(x)}{Y(x)},$$

which can be expressed as

$$\frac{\mathrm{d}(y/Y)}{\mathrm{d}x} = \frac{g(x)}{Y(x)}$$

by virtue of (16.9). This equation is a directly integrable equation of type (16.1) for $z = y/Y$; the solution for z, given by (16.15), leads at once to (16.16).

Example 4

Find the solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 3x\mathrm{e}^{-x}$$

for which $y = y_0$ when $x = x_0$.

First calculate the complementary function $Y(x)$ from (16.11) with $\alpha(t) = 4$:

$$Y(x) = \exp[-4(x - x_0)].$$

Now calculate z from (16.15). Since $g(x) = 3x\mathrm{e}^{-x}$

$$\begin{aligned} z &= \int_{x_0}^{x} \frac{g(t)}{Y(t)}\,\mathrm{d}t + y_0 \\ &= \exp(-4x_0) \int_{x_0}^{x} 3t \exp 3t\,\mathrm{d}t + y_0 \\ &= \exp(-4x_0)[(x - \tfrac{1}{3}) \exp 3x - (x_0 - \tfrac{1}{3}) \exp 3x_0 + y_0 \exp 4x_0]. \end{aligned}$$

Hence the solution (16.16) is

$$y = \exp(-4x)[(x - \tfrac{1}{3})\exp 3x - (x_0 - \tfrac{1}{3})\exp 3x_0 + y_0 \exp 4x_0].$$

We may of course insert any numerical values we wish for x_0 and y_0, giving y as a definite function of x.

Example 5

Find the solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = x$$

such that $y = -1$ when $x = 1$.

The complementary function $Y(x)$, given by (16.11) with $\alpha(t) = -2/t$ and $x_0 = 1$, is

$$Y(x) = \exp[2 \int_1^x t^{-1}\,\mathrm{d}t] = x^2.$$

Since $g(t) = t$,

$$\int_1^x \frac{g(t)}{Y(t)}\,\mathrm{d}t = \int_1^x t^{-1}\,\mathrm{d}t = \log x.$$

Hence the solution (16.16) is, putting $y_0 = -1$,

$$y = x^2(\log x - 1).$$

§ 3.1. Alternative method of solution

Equations (16.16) and (16.11) give the complete solution of equation (16.8). Regarding x_0 as fixed, the arbitrary constant appears as y_0 in the second term of (16.16). If we choose a particular value of y_0, then (16.16) gives us a *particular solution* of (16.8), and any other solution is found by adding to this a multiple of $Y(x)$, the complementary function, since it satisfies (16.9). So if a particular solution $Y_1(x)$ of (16.8) is known, the general solution is

$$y = Y_1(x) + CY(x) \tag{16.17}$$

where C is an arbitrary constant. A common method of solving equations of type (16.8) is to guess a particular solution $Y_1(x)$ and use (16.17). We shall solve Example 4 by this method.

Example 6

Find the solution $y(x)$ of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 3x\mathrm{e}^{-x}$$

which has $y(x) = y_0$ when $x = x_0$.

We look for a particular solution Y_1 of the equation which we guess takes the form

$$Y_1(x) = (Ax + B)e^{-x}.$$

We have

$$\frac{dY_1}{dx} + 4Y_1 = (3Ax + A + 3B)e^{-x},$$

so that taking $A=1$, $B=-\frac{1}{3}A=-\frac{1}{3}$ gives a solution. The complementary function, as before, is a multiple of e^{-4x}, so that by (16.17) the general solution is

$$y = (x - \tfrac{1}{3})e^{-x} + Ce^{-4x}.$$

The condition $y=y_0$ when $x=x_0$ means that C must satisfy the equation

$$y_0 = (x_0 - \tfrac{1}{3})\exp(-x_0) + C\exp(-4x_0).$$

To eliminate C we multiply this equation by $\exp[4(x_0-x)]$ and subtract from the previous equation, obtaining the solution

$$y = \exp(-4x)[(x - \tfrac{1}{3})\exp 3x - (x_0 - \tfrac{1}{3})\exp 3x_0 + y_0 \exp 4x_0].$$

This method of solution is inferior to evaluating y by (16.11) and (16.16) for two reasons: first, we have to be able to guess the form of a particular solution, and this may not be at all easy; second, the boundary condition has to be satisfied after the general solution (16.17) has been found. By contrast, (16.16) gives a general solution incorporating the boundary condition, and even if the integral in (16.16) cannot be evaluated explicitly, numerical methods can be used to approximate to the solution.

EXERCISE 16.2

1. Find the solution of each of the following equations, given that $y=y_0$ when $x=x_0$:

(i) $\dfrac{dy}{dx} - 2y = x^3 + 4x;$

(ii) $\dfrac{dy}{dx} + y = x\cos x + 2e^{-x};$

(iii) $\dfrac{dy}{dx} + \dfrac{xy}{1+x^2} = \dfrac{4}{(1+x^2)^2};$

(iv) $\dfrac{dy}{dx} + \dfrac{y}{x^3} = \dfrac{1}{x^3}\cosh\left(\dfrac{1}{2x^2}\right);$

(v) $\dfrac{dy}{dx} + y\cot x = 4(x^2 + e^{-2x}).$

§ 4. The Laplace transform and constant coefficient equations

In this section we are concerned with the solution of linear differential equations of arbitrary order in which the coefficients of y and its derivatives are constants. These equations are called linear *constant coefficient equations*; if we write the symbol D for the differential operator $\mathrm{d}/\mathrm{d}x$, so the D^n stands for $\mathrm{d}^n/\mathrm{d}x^n$, then the general equation of this type is

$$D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y = g(x), \qquad (16.18)$$

where $a_1, a_2, \dots, a_{n-1}, a_n$ are constants.

There are several methods of tackling the problem of solving (16.18) when $n > 1$. One method is an extension of that discussed in § 3.1 for first order equations; it involves finding the complementary function $Y(x)$ by solving (16.18) with $g(x)=0$, and then adding an arbitrary multiple of $Y(x)$ to any particular solution. This method has two major disadvantages.

(i) No systematic method of finding particular solutions of (16.18) is prescribed; and even if we can guess the general form of the particular solution, finding it is generally a complex business.

(ii) The complementary function $Y(x)$ will contain n arbitrary constants, which then have to be chosen to satisfy certain boundary conditions; this procedure is in general clumsy and complicated, involving the solution of n linear equations for the n constants.

There are three other commonly used methods of solving equations of type (16.18), involving respectively differential operator techniques, Laplace transforms and Fourier transforms. It is important to realise that these three techniques are very closely linked mathematically, although notational differences may conceal their similarity. We shall use the Laplace transform method, since we feel that it is a little simpler conceptually and a little more systematic in practice than operator methods; but we re-iterate that the two methods are closely linked, and one cannot reasonably argue that one is better than the other. Fourier transforms will be defined in Ch. 18, and they bear a very close relation to Laplace transforms; both mathematically and physically Fourier transforms are of great importance, but the solution of equations of type (16.18) is rather neater if we use Laplace transforms.

If $f(x)$ is a given function of x defined for all positive values of x, its Laplace transform is defined as the function

$$\mathscr{F}(p) = \int_0^\infty \mathrm{e}^{-px} f(x)\,\mathrm{d}x \qquad (16.19)$$

of a variable p. The function $\mathscr{F}(p)$ is only defined for values of p for which the integral (16.19) converges. If we assume that for all values of x in the range $(0, \infty)$,

$$|f(x)| \leqslant K\,\mathrm{e}^{\alpha x}, \tag{16.20}$$

where K and α are some constants, then

$$|f(x)\,\mathrm{e}^{-px}| \leqslant K\,\mathrm{e}^{(\alpha-p)x}.$$

Hence the integral (16.19) is dominated by

$$K\int_0^\infty \mathrm{e}^{-(p-\alpha)x}\,\mathrm{d}x$$

which converges provided $p-\alpha>0$. So we assume that $f(x)$ satisfies a condition of type (16.20), ensuring the existence of $\mathscr{F}(p)$ when $p>\alpha$.

We shall now evaluate some simple Laplace transforms and establish a series of properties of Laplace transforms which will be useful in applications. In the remainder of the chapter, whenever functions denoted by $\mathscr{F}(p)$ and $f(x)$, occur, we shall understand that they are related by equation (16.19); this relation is often written

$$\mathscr{F}(p) = \mathscr{L}[f(x)].$$

One elementary property of Laplace transforms is the property of linearity: if A and B are constants and $f_1(x)$ and $f_2(x)$ are functions possessing Laplace transforms, then it follows at once from (16.19) that

$$\mathscr{L}[Af_1(x) + Bf_2(x)] = A\mathscr{L}[f_1(x)] + B\mathscr{L}[f_2(x)]. \tag{16.21}$$

Because (16.21) expresses an 'obvious' property which is very frequently used in all work involving Laplace transforms, we shall use it in future without referring back to this equation.

Table 16.1 lists a few elementary functions $f(x)$ in the first column, their Laplace transforms $\mathscr{F}(p)$ in the second column, and the range of convergence of the integral (16.19) in the third. In the table, a denotes a real constant and n denotes a non-negative integer.

It is not difficult to prove the results of Table 16.1. The evaluation of $\mathscr{L}(\mathrm{e}^{ax})$, using (16.19), is trivial. Since $\cosh ax = \frac{1}{2}(\mathrm{e}^{ax}+\mathrm{e}^{-ax})$, $\mathscr{L}(\cosh ax)$ follows at once, as does $\mathscr{L}(\sinh ax)$. $\mathscr{L}(\cos ax)$ and $\mathscr{L}(\sin ax)$ are most easily evaluated by observing that $\mathscr{L}(\mathrm{e}^{iax}) = (p-ia)^{-1}$, provided $p>0$, and then taking real and imaginary parts of this result. The reader is advised to check these five results for himself.

TABLE 16.1

$f(x)$	$\mathscr{F}(p)$	Range of p
e^{ax}	$\dfrac{1}{p-a}$	$p > a$
$\cosh ax$	$\dfrac{p}{p^2-a^2}$	$p > \lvert a\rvert$
$\sinh ax$	$\dfrac{a}{p^2-a^2}$	$p > \lvert a\rvert$
$\cos ax$	$\dfrac{p}{p^2+a^2}$	$p > 0$
$\sin ax$	$\dfrac{a}{p^2+a^2}$	$p > 0$
x^n	$\dfrac{n!}{p^{n+1}}$	$p > 0$
$\dfrac{x}{2a} \sin ax$	$\dfrac{p}{(p^2+a^2)^2}$	$p > \lvert a\rvert$
$\dfrac{1}{2a^3}(\sin ax - ax\cos ax)$	$\dfrac{1}{(p^2+a^2)^2}$	$p > \lvert a\rvert$

The formula for $\mathscr{L}(x^n)$ is proved inductively. It is certainly true when $n=0$. We therefore assume that

$$\int_0^\infty e^{-px}\, x^{n-1}\, dx = \frac{(n-1)!}{p^n}$$

for $n \geqslant 1$; integrating by parts we find that

$$\frac{1}{n}\left[e^{-px}x^n\right]_0^\infty + \frac{p}{n}\int_0^\infty e^{-px}x^n\, dx = \frac{(n-1)!}{p^n};$$

since $n \geqslant 1$, the first term on the left is zero, so that

$$\int_0^\infty e^{-px}\, x^n\, dx = \frac{n!}{p^{n+1}},$$

completing the proof by induction.

Since we have shown that

$$\int_0^\infty e^{-px} \cos ax \, dx = \frac{p}{p^2 + a^2}$$

for all values of a, we may differentiate both sides of the equation partially with respect to a, giving the Laplace transform of $(x \sin ax)/2a$. The reader can check that the last result in Table 16.1 is likewise given by differentiating the equation giving the transform of $\sin ax$.

§ 4.1. Some properties of Laplace transforms

There are a number of useful properties of Laplace transforms which we need to establish. In this section we shall state and prove the first three of these properties.

Property 1. If $\mathscr{F}(p)$ exists for $p > \alpha$, then there is only one function $f(x)$ which satisfies (16.19) for all $p > \alpha$.

This is the most important property of Laplace transforms, implying that in (16.19), $f(x)$ is uniquely determined by $\mathscr{F}(p)$. We say that $f(x)$ is the *inverse (Laplace) transform of* $\mathscr{F}(p)$, written

$$f(x) = \mathscr{L}^{-1}[\mathscr{F}(p)].$$

Because of this property, if we wish to determine a function $f(x)$, we can determine its transform $\mathscr{F}(p)$ instead, knowing that $f(x)$ is thereby uniquely fixed.

The proof of property 1 is moderately long, and can be omitted if the reader wishes.

★We wish to prove that if

$$\int_0^\infty f(x) e^{-px} \, dx = \int_0^\infty g(x) e^{-px} \, dx \tag{16.22}$$

for all values of p greater than some constant α, then $f(x) \equiv g(x)$ for $x \geqslant 0$. To ensure the convergence of the integrals in (16.22), we assume that both $f(x)$ and $g(x)$ satisfy a condition of the form (16.20). So if we define a function

$$h(x) \equiv e^{-\alpha x} [f(x) - g(x)], \tag{16.23}$$

it will satisfy a condition of type

$$|h(x)| < K_1, \tag{16.24}$$

for some constant K_1. Putting $q=p-\alpha$, and using (16.23), we can express condition (16.22) in the form

$$\int_0^\infty h(x)\,\mathrm{e}^{-qx}\,\mathrm{d}x = 0 \tag{16.25}$$

for all $q>0$. Now, from (16.23), the theorem will be established if we can show that $h(x)\equiv 0$, given that (16.24) and (16.25) hold.

First, we remark that for every x, $h(x)$ is either positive, negative, or zero. So we can divide the range $(0, \infty)$ of x into intervals in which $h(x)\geqslant 0$ and $h(x)\leqslant 0$ alternately. If for instance $h(x)\geqslant 0$ for small values of x, we assume that $h(x)\geqslant 0$ in the intervals $(0, x_1)$, $(x_2, x_3), \ldots, (x_{2n-2}, x_{2n-1}), \ldots$, with $0<x_1<x_2\ldots<x_{2n-2}<x_{2n-1}<\ldots$, while $h(x)\leqslant 0$ in the intervals (x_1, x_2) $(x_3, x_4), \ldots, (x_{2n-1}, x_{2n})\ldots$. We shall prove by induction on n that $h(x)\equiv 0$ in every interval. We therefore assume that $h(x)=0$ in the range $(0, x_{2n-1})$, and shall prove that it is zero in the next interval (x_{2n-1}, x_{2n}). Using this induction hypothesis, the range of integration $(0, \infty)$ in (16.25) can be replaced by the range (x_{2n-1}, ∞); splitting this range into the ranges (x_{2n-1}, x_{2n}) and (x_{2n}, ∞), (16.25) gives

$$\left|\int_{x_{2n-1}}^{x_{2n}} h(x)\,\mathrm{e}^{-qx}\,\mathrm{d}x\right| = \left|\int_{x_{2n}}^{\infty} h(x)\,\mathrm{e}^{-qx}\,\mathrm{d}x\right| \tag{16.26}$$

for $q>0$. Using (16.24), we have

$$\begin{aligned}\left|\int_{x_{2n}}^{\infty} h(x)\,\mathrm{e}^{-qx}\,\mathrm{d}x\right| &\leqslant K_1 \int_{x_{2n}}^{\infty} \mathrm{e}^{-qx}\,\mathrm{d}x \\ &= \frac{K_1\,\mathrm{e}^{-qx_{2n}}}{q}.\end{aligned} \tag{16.27}$$

Also, since $h(x)\leqslant 0$ and $\mathrm{e}^{-qx}>\mathrm{e}^{-qx_{2n}}>0$ in (x_{2n-1}, x_{2n}),

$$\begin{aligned}\left|\int_{x_{2n-1}}^{x_{2n}} h(x)\,\mathrm{e}^{-qx}\,\mathrm{d}x\right| &\equiv \int_{x_{2n-1}}^{x_{2n}} [-h(x)]\,\mathrm{e}^{-qx}\,\mathrm{d}x \\ &> \mathrm{e}^{-qx_{2n}} \int_{x_{2n-1}}^{x_{2n}} [-h(x)]\,\mathrm{d}x.\end{aligned} \tag{16.28}$$

Substituting the inequalities (16.27) and (16.28) in (16.26) gives

$$\int_{x_{2n-1}}^{x_{2n}} [-h(x)]\,\mathrm{d}x < \frac{K_1}{q}.$$

This is true for all $q > 0$, however large, so

$$\int_{x_{2n-1}}^{x_{2n}} [-h(x)]\,\mathrm{d}x = 0 \tag{16.29}$$

since $[-h(x)] \geqslant 0$ in the range; this condition implies further that (16.29) cannot hold unless $h(x) \equiv 0$ in (x_{2n-1}, x_{2n}). A similar proof, with $h(x)$ replacing $[-h(x)]$, shows that $h(x)$ is zero throughout (x_{2n}, x_{2n+1}) if it is zero in $(0, x_{2n})$.

To complete the inductive proof, we must show that $h(x) = 0$ in $(0, x_1)$. We split the range of integration $(0, x)$ in (16.25) into the ranges $(0, x_1)$ and (x_1, ∞); then replacing x_{2n-1} by 0, x_{2n} by x_1 and $[-h(x)]$ by $h(x)$ in the proof above, the result follows at once, and the inductive proof is complete. The assumption that $h(x) > 0$ in $(0, x_1)$ clearly does not affect the proof, the only important assumption being that $h(x)$ is of uniform sign throughout the interval.

The proof given here fails if the sequence of points $x_1, x_2, x_3, \ldots$ has a finite limit point. This cannot happen if we assume that $f(x)$ only oscillates finitely in any finite interval. ★

PROPERTY 2. The Laplace transform of the mth derivative $D^m f(x)$ is given by

$$\mathscr{L}[D^m f(x)] = p^m \mathscr{F}(p) - \sum_{r=0}^{m-1} p^{m-r-1} f^{(r)}(0), \tag{16.30}$$

where $f^{(r)}(0)$, $(r = 0, 1, \ldots, m-1)$ denote the values of the rth derivatives $D^r f(x)$ when $x = 0$:

$$f^{(r)}(0) = [D^r f(x)]_{x=0}; \tag{16.31}$$

these m quantities are constants.

When $m = 1$, this property states that

$$\mathscr{L}[Df(x)] = p\mathscr{F}(p) - f(0), \tag{16.32}$$

since there is only one term $(r = 0)$ in the series $\sum_{r=0}^{m-1} p^{m-r-1} f^{(r)}(0)$. Equation (16.32) is easy to establish by integrating by parts. Using (16.19)

$$\begin{aligned}\mathscr{L}[Df(x)] &= \int_0^\infty \mathrm{e}^{-px} \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x \\ &= [\mathrm{e}^{-px} f(x)]_0^\infty + p \int_0^\infty \mathrm{e}^{-px} f(x)\,\mathrm{d}x = -f(0) + p\mathscr{F}(p).\end{aligned}$$

Thus (16.30) is true when $m=1$. We shall establish (16.30) generally by induction on m. Assume that (16.30) is true for a particular value of $m(\geqslant 1)$:

$$\int_0^\infty e^{-px}\{D^m f(x)\}\,dx = p^m \mathscr{F}(p) - \sum_{r=0}^{m-1} p^{m-r-1} f^{(r)}(0).$$

Integrating the left-hand member by parts, we have

$$\left[-\frac{1}{p}e^{-px}\{D^m f(x)\}\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-px}\{D^{m+1}f(x)\}\,dx$$
$$= p^{-1}\{f^{(m)}(0) + \mathscr{L}[D^{m+1}f(x)]\},$$

using the definition (16.31). Thus

$$\mathscr{L}[D^{m+1}f(x)] + f^{(m)}(0) = p\{p^m\mathscr{F}(p) - \sum_{r=0}^{m-1} p^{m-r-1}f^{(r)}(0)\},$$

establishing (16.30) with m replaced by $m+1$. This completes the inductive proof.

PROPERTY 3.

$$\mathscr{L}[\int_0^x f(t)\,dt] = p^{-1}\mathscr{F}(p). \tag{16.33}$$

Expressed loosely, this property states that 'integration of $f(x)$ from $x=0$ is equivalent to dividing $\mathscr{F}(p)$ by p'.

To establish (16.33), we integrate by parts the definition

$$\int_0^\infty e^{-px}f(x)dx = \mathscr{F}(p),$$

giving

$$[e^{-px}\{\int_0^x f(t)dt\}]_0^\infty + p\int_0^\infty e^{-px}\{\int_0^x f(t)dt\}\ dx = \mathscr{F}(p).$$

Since $\int_0^x f(t)dt$ vanishes when $x=0$, and e^{-px} vanishes when $x=\infty$, the property is established.

Property 3 is often used to find a function whose Laplace transform is given and contains a factor p^{-1}. The property is then expressed

$$\mathscr{L}^{-1}[p^{-1}\mathscr{F}(p)] = \int_0^x f(t)dt.$$

Example 7

Find

$$\mathscr{L}^{-1}\left[\frac{1}{p(p^2+9)}\right] \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{1}{p^2(p^2+9)}\right].$$

From Table 16.1, we know that

$$\mathscr{L}^{-1}\left(\frac{1}{p^2+9}\right) = \tfrac{1}{3}\sin 3x.$$

So using property 3,

$$\mathscr{L}^{-1}\left[\frac{1}{p(p^2+9)}\right] = \int_0^x \tfrac{1}{3}\sin 3t \, dt$$

$$= \tfrac{1}{9}(1 - \cos 3x).$$

Using this result and property 3 again,

$$\mathscr{L}^{-1}\left[\frac{1}{p^2(p^2+9)}\right] = \tfrac{1}{9}\int_0^x (1 - \cos 3t)\, dt$$

$$= \tfrac{1}{27}(3x - \sin 3x).$$

These results could also be obtained by writing

$$\frac{1}{p(p^2+9)} = \frac{1}{9}\left(\frac{1}{p} - \frac{p}{p^2+9}\right),$$

$$\frac{1}{p^2(p^2+9)} = \frac{1}{9}\left(\frac{1}{p^2} - \frac{1}{p^2+9}\right),$$

and then using Table 16.1.

EXERCISE 16.3

1. Write down the Laplace transforms of

$$e^{3x}, \qquad \cosh 2x, \qquad x^3 + 4x + 2, \qquad D^3 f, \qquad D^2 f + a_1 Df + a_2 f.$$

2. Find the inverse Laplace transforms of

$$\frac{1}{p^2-3}, \qquad \frac{1}{(p-a)(p-b)}, \qquad \frac{1}{p(p^2-4)}, \qquad p^{-n} \quad (n \geqslant 1),$$

$$\frac{1}{p^4-16}, \qquad \frac{1}{(p^2+a^2)(p^2+b^2)}, \qquad \frac{1}{p(p^2+a^2)(p^2+b^2)}.$$

§ 4.2. Solution of constant coefficient equations

We shall now show how Laplace transforms can be used to solve constant coefficient equations of the form (16.18).

In § 1 and § 3 we have seen that first order equations of this type do not determine y uniquely as a function of x; we also need to be given the value y_0 of y corresponding to a value x_0 of x. We are now in a position to show that a solution $y=f(x)$ of the nth order equation (16.18) is uniquely determined if the values of y and its first $n-1$ derivatives are given for some value x_0 of x. We shall make the assumption that $x_0=0$; for if it is not zero, we can always re-define the zero of x to be at x_0. We shall also assume that we wish to solve (16.18) for positive values of x; identical methods can be used to find a solution for $x<0$.

Suppose that $y, Dy, D^2y, \ldots, D^{n-1}y$ take values $y_0, y_1, y_2, \ldots, y_{n-1}$ when $x=0$. Then the solution $y=f(x)$ of (16.18) must satisfy

$$D^nf(x)+a_1D^{n-1}f(x)+\ldots+a_{n-1}Df(x)+a_nf(x)=g(x) \tag{16.34}$$

for $x\geqslant 0$, and

$$f(0)=y_0, \qquad f'(0)=y_1, \quad \ldots, \quad f^{(n-1)}(0)=y_{n-1}. \tag{16.35}$$

Let us take the Laplace transform of both sides of (16.34) by multiplying throughout by e^{-px} and integrating over x from 0 to ∞; we get

$$\mathscr{L}[D^nf]+a_1\mathscr{L}[D^{n-1}f]+\ldots+a_{n-1}\mathscr{L}[Df]+a_n\mathscr{L}[f]=\mathscr{L}[g].$$

If $\mathscr{F}(p)$ and $\mathscr{G}(p)$ are the transforms of $f(x)$ and $g(x)$, then using (16.30) with $m=n$, $m=n-1$, ..., $m=1$, this equation gives

$$[p^n\mathscr{F}(p)-\sum_{r=0}^{n-1}p^{n-r-1}f^{(r)}(0)]+a_1[p^{n-1}\mathscr{F}(p)-\sum_{r=0}^{n-2}p^{n-r-2}f^{(r)}(0)]$$
$$+\ldots+a_{n-1}[p\mathscr{F}(p)-f(0)]+a_n\mathscr{F}(p)=\mathscr{G}(p);$$

putting in the initial values $f(0), f'(0), \ldots$, given by (16.35), we have

$$(p^n+a_1p^{n-1}+\ldots+a_{n-1}p+a_n)\mathscr{F}(p)$$
$$=\mathscr{G}(p)+\sum_{r=0}^{n-1}p^{n-r-1}y_r+a_1\sum_{r=0}^{n-2}p^{n-r-2}y_r+\ldots+a_{n-1}y_0. \tag{16.36}$$

Since the function $\mathscr{G}(p)$ and the constants y_r ($r=0, 1, 2,\ldots, n-1$) are given, (16.36) is an algebraic equation which determines $\mathscr{F}(p)$ uniquely. But $\mathscr{F}(p)$ is the Laplace transform of $f(x)$, and by property 1 determines

it completely. Thus the solution $y=f(x)$ is determined uniquely by (16.34) and the initial conditions (16.35).

Equation (16.36) is known as the *subsidiary equation* of (16.34). The function $\mathscr{F}(p)$ which it defines may be quite complicated, and some ingenuity is required before the inverse transform $f(x)$ can be found. However, a large class of fairly simple equations can be solved by using Table 16.1 and properties 2 and 3.

We note that the left-hand side of (16.36) is obtained from (16.34) by replacing $f(x)$ by $\mathscr{F}(p)$ and D by p.

Example 8

Solve $Dy-3y=2e^x$, given that $y=y_0$ when $x=0$.

If $y=f(x)$ is the solution, then $f(0)=y_0$. Take Laplace transforms; then using Table 16.1 and (16.32)

$$(p-3)\mathscr{F}(p) - y_0 = \frac{2}{p-1},$$

which is just equation (16.36). So

$$\mathscr{F}(p) = \frac{2}{(p-1)(p-3)} + \frac{y_0}{p-3}$$
$$= \frac{1+y_0}{p-3} - \frac{1}{p-1},$$

using partial fractions. Property 1 tells us that $f(x)$ is uniquely determined by $\mathscr{F}(p)$, so we can simply use Table 16.1 to give us the inverse transform of $\mathscr{F}(p)$; thus

$$f(x) = (1+y_0)e^{3x} - e^x.$$

It is of interest to note that the complementary function $Y(x)$ of y is given by omitting the term $2e^x$ from the original equation. Then only the term containing y_0 survives in $\mathscr{F}(p)$, so that $Y(x)=y_0e^{3x}$. Thus in the solution $f(x)$, the term containing e^{3x} could be called the complementary function, the term $-e^x$ being the simplest particular solution.

Example 9

Solve $D^2y-2Dy=x^2$, given that the initial values of y and dy/dx are $y_0=3$ and $y_1=2$.

Using Table 16.1 and (16.30) with $m=1$, 2, we obtain (16.36):

$$(p^2-2p)\mathscr{F}(p) = 2p^{-3} + (py_0+y_1) - 2y_0$$

or

$$\mathscr{F}(p) = \frac{2}{p^4(p-2)} + \frac{y_0}{p-2} + \frac{y_1-2y_0}{p(p-2)}.$$

We know that

$$\mathscr{L}^{-1}\left[\frac{1}{p-2}\right] = e^{2x}.$$

Property 3 tells us that

$$\mathcal{L}^{-1}\left[\frac{1}{p(p-2)}\right] = \int_0^x e^{2t}\,dt = \tfrac{1}{2}(e^{2x}-1),$$

dealing with the third term in $\mathcal{F}(p)$. Applying property 3 again we get

$$\mathcal{L}^{-1}\left[\frac{1}{p^2(p-2)}\right] = \int_0^x \tfrac{1}{2}(e^{2x}-1) = \tfrac{1}{4}(e^{2x}-1) - \tfrac{1}{2}x.$$

A further two integrations give us

$$\mathcal{L}^{-1}\left[\frac{1}{p^4(p-2)}\right] = \tfrac{1}{16}(e^{2x}-1) - \tfrac{1}{12}x^3 - \tfrac{1}{8}x^2 - \tfrac{1}{8}x.$$

Hence the solution is

$$\begin{aligned} y = f(x) &= \mathcal{L}^{-1}[\mathcal{F}(p)] \\ &= \tfrac{1}{8}e^{2x}(1+4y_1) - \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 - \tfrac{1}{4}x - \tfrac{1}{8} + \tfrac{1}{2}(2y_0 - y_1). \end{aligned}$$

Putting $y_0=3$ and $y_1=2$, we find

$$y = \tfrac{9}{8}e^{2x} - \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 - \tfrac{1}{4}x + \tfrac{15}{8}.$$

We note that the complementary function is found by omitting the term $2/p^4(p-2)$ from $\mathcal{F}(p)$, leaving only the terms containing y_0 and y_1. So the complementary function is of the form

$$Ae^{2x} + B,$$

where A and B are constants depending on y_0 and y_1. We wish to emphasize that the Laplace transform method expresses the complementary function directly in terms of the initial conditions y_0 and y_1; we do not need to fit the constants A and B to satisfy the initial conditions.

EXERCISE 16.4

1. Find solutions for $x>0$ of the following equations which satisfy the given initial conditions.

(i) $Dy + 2y = 4$; $\quad y_0 = 0$.
(ii) $Dy + 4y = x^2 + 2$; $\quad y_0 = -3$.
(iii) $D^2y + 4y = 3\sin x$; $\quad y_0 = 1, \quad y_1 = -2$.
(iv) $D^2y - 3Dy + 2y = 6x^3$; $\quad y_0 = 0, \quad y_1 = 4$.
(v) $D^4y + 8D^2y + 16y = 0$; $\quad y_0 = y_1 = 0, \quad y_2 = 2, \quad y_3 = 4$.

2. Find solutions for $x>0$ of the following equations as functions of the initial conditions y_r $(r=1, 2,\ldots)$: write down the complementary function for each equation.

(i) $Dy - 3y = 6x^3$;
(ii) $Dy - y = e^x$;

(iii) $D^2y - 4y = 4\cosh x$;
(iv) $D^3y + D^2y = 3e^{-4x}$;
(v) $D^4y - 2D^2y + y = 0$.

§ 4.3. Further properties of Laplace transforms

Examples 8 and 9 indicate the general method of solving a constant coefficient equation by forming the subsidiary equation (16.36) for the Laplace transform of the solution. These particular equations would, however, be easy to solve by the methods of § 3 and § 3.1, guessing the form of the particular solution in Example 9. Before we attempt to solve more difficult equations, we shall establish three further properties of Laplace transforms which will enable us to deal with quite complex equations.

Property 4. If b is a constant then

$$\mathscr{L}[e^{bx}f(x)] = \mathscr{F}(p - b).$$

Using the definition (16.19), the proof of this property is trivial; nevertheless the result is of great importance, as the following examples show.

Example 10

Find $\mathscr{L}^{-1}[(p-b)^{-n}]$, where b is constant and n is a positive integer.

Property 4 tells us that the function is

$$e^{bx}\mathscr{L}^{-1}[p^{-n}] = \frac{1}{(n-1)!}e^{bx}x^{n-1}.$$

Example 11

Using property 4 and Laplace transforms from table 16.1, we have

$$\mathscr{L}[e^{bx}\cos ax] = \frac{p-b}{(p-b)^2 + a^2}, \tag{16.37}$$

$$\mathscr{L}[e^{bx}\sin ax] = \frac{a}{(p-b)^2 + a^2}. \tag{16.38}$$

Results (16.37) and (16.38) enable us to deal with Laplace transforms with quadratic denominators which have no real factors. For instance,

$$\mathscr{L}^{-1}\left[\frac{p}{p^2 + 4p + 5}\right] = \mathscr{L}^{-1}\left[\frac{(p+2) - 2}{(p+2)^2 + 1}\right];$$

then using (16.37) and (16.38) with $a = 1$, $b = -2$ gives the result

$$e^{-2x}(\cos x - 2\sin x).$$

Example 12

Using property 4,

$$\mathscr{L}^{-1}\left[\frac{a}{(p-c)\{(p-b)^2+a^2\}}\right] = \mathrm{e}^{cx}\mathscr{L}^{-1}\left[\frac{a}{p\{(p-b+c)^2+a^2\}}\right].$$

Now using property 3 and property 4 again, we obtain

$$\begin{aligned}
&\mathrm{e}^{cx}\int_0^x \mathrm{e}^{(b-c)t}\sin at\,\mathrm{d}t\\
&\qquad = \mathrm{e}^{cx}\,\mathrm{Im}\left[\int_0^x \mathrm{e}^{t(b-c+ia)}\,\mathrm{d}t\right]\\
&\qquad = \mathrm{e}^{cx}\,\mathrm{Im}\left[\frac{\mathrm{e}^{x(b-c+ia)}-1}{b-c+ia}\right]\\
&\qquad = \mathrm{Im}\left[\frac{\{\mathrm{e}^{x(b+ia)}-\mathrm{e}^{cx}\}(b-c-ia)}{(b-c)^2+a^2}\right].
\end{aligned}$$

Thus the inverse transform is

$$\frac{(b-c)\mathrm{e}^{bx}\sin ax + a(e^{cx}-e^{bx}\cos ax)}{(b-c)^2+a^2}.$$

Similar results to those in Example 11 hold for the hyperbolic functions:

$$\mathscr{L}[\mathrm{e}^{bx}\cosh ax] = \frac{p-b}{(p-b)^2-a^2}, \tag{16.39}$$

$$\mathscr{L}[\mathrm{e}^{bx}\sinh ax] = \frac{a}{(p-b)^2-a^2}. \tag{16.40}$$

These formulae are less valuable than (16.37) and (16.38), since the denominators have the real factors $p-b\pm a$, and the partial fraction method can be used.

PROPERTY 5. If λ is a positive constant, then

$$\mathscr{L}^{-1}[\mathrm{e}^{-\lambda p}\mathscr{F}(p)] = h(x)$$

where

$$\begin{aligned}
h(x) &= f(x-\lambda) &&\text{for}\quad x \geqslant \lambda,\\
h(x) &= 0 &&\text{for}\quad x < \lambda.
\end{aligned}$$

Remembering that in the definition (16.19) of $\mathscr{F}(p)$, $f(x)$ has to be defined only for $x>0$, we can without contradiction take $f(x)\equiv 0$ for $x<0$. Then we can look upon $h(x)$ as the function formed by shifting the graph of $f(x)$ a distance λ along the x-axis, as shown in fig. 16.1. Property 5 is

easily established: by the definition of $h(x)$

$$\mathscr{L}[h(x)] = \int_{x=\lambda}^{\infty} \mathrm{e}^{-px} f(x-\lambda)\,\mathrm{d}x.$$

Changing variable in the integral to $x'=x-\lambda$ gives $\mathrm{e}^{-\lambda p}\mathscr{F}(p)$, establishing the result.

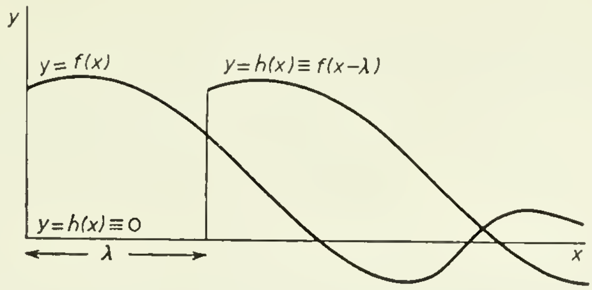

Fig. 16.1

Example 13

If $\lambda>0$, then

$$\mathscr{L}^{-1}\left[\frac{\mathrm{e}^{-\lambda p}}{p^2+a^2}\right] = h(x)$$

where

$$\begin{aligned} h(x) &= 0 && \text{for} \quad x<\lambda, \\ h(x) &= a^{-1}\sin a(x-\lambda) && \text{for} \quad x \geqslant \lambda. \end{aligned}$$

PROPERTY 6. If $\mathscr{L}[f_1(x)]=\mathscr{F}_1(p)$ and $\mathscr{L}[f_2(x)]=\mathscr{F}_2(p)$, then

$$\mathscr{L}[\int_0^x f_1(t)\,f_2(x-t)\,\mathrm{d}t] = \mathscr{F}_1(p)\,\mathscr{F}_2(p).$$

This property is useful when we have to find the function whose transform is the product of two known transforms $\mathscr{F}_1(p)$ and $\mathscr{F}_2(p)$. The Laplace transform above is

$$\int_0^\infty \mathrm{e}^{-px}\,\mathrm{d}x \int_0^x f_1(t)\,f_2(x-t)\,\mathrm{d}t$$

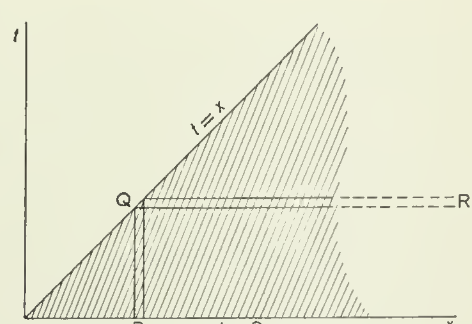

Fig. 16.2

and is a double integral over x and t. As shown in fig. 16.2, the region of integration in the xt-plane is over the shaded octant between $t=0$ and $t=x$, the integral over t being along strips PQ. Changing the order of

integration so that we integrate over x first along strips QR, the integral becomes

$$\int_0^\infty f_1(t)\,\mathrm{d}t \int_t^\infty \mathrm{e}^{-px} f_2(x-t)\,\mathrm{d}x.$$

In the x-integral, t appears as a constant parameter; so we can change the variable of integration to $x'=x-t$, giving

$$\begin{aligned}\int_0^\infty f_1(t)\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-p(x'+t)} f_2(x')\,\mathrm{d}x' \\ &= \int_0^\infty f_1(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-px'} f_2(x')\,\mathrm{d}x' \\ &= \mathscr{F}_1(p)\,\mathscr{F}_2(p).\end{aligned}$$

This property is often useful for finding the inverse transform of a product of known transforms. It is then expressed

$$\mathscr{L}^{-1}[\mathscr{F}_1(p)\,\mathscr{F}_2(p)] = \int_0^x f_1(t)\,f_2(x-t)\,\mathrm{d}t. \tag{16.41}$$

Example 14

The inverse transform evaluated in Example 12 can be more simply performed by using (16.41). Since by (16.38),

$$\mathscr{L}^{-1}\left[\frac{a}{(p-b)^2+a^2}\right] = \mathrm{e}^{bx}\sin ax \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{1}{p-c}\right] = \mathrm{e}^{cx},$$

then

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{a}{(p-c)\{(p-b)^2+a^2\}}\right] &= \int_0^x \mathrm{e}^{bt}\sin at \cdot \mathrm{e}^{c(x-t)}\,\mathrm{d}t \\ &= \mathrm{e}^{cx}\int_0^x \mathrm{e}^{(b-c)t}\sin at\,\mathrm{d}t.\end{aligned}$$

This integral can be evaluated as in Example 12, giving

$$\frac{(b-c)\mathrm{e}^{bx}\sin ax + a(\mathrm{e}^{cx} - \mathrm{e}^{bx}\cos ax)}{(b-c)^2+a^2}.$$

§ 4.4. Solution of more complex equations

We are now in a position to solve almost any constant coefficient equation (16.18) that is likely to arise in practice, by finding $\mathscr{F}(p)$ from the sub-

sidiary equation (16.36) and then calculating its inverse transform. The method extends in an obvious way to simultaneous constant coefficient equations for two variables y_1 and y_2 as functions of x. In Examples 15–17 we shall solve some harder equations for a single function and a pair of simultaneous equations for two functions.

Example 15

Find the solution of $D^2y-2Dy+5y=e^{-2x}$ with initial conditions $y_0=4$, $y_1=0$.

The subsidiary equation (16.36) becomes

$$(p^2-2p+5)\mathscr{F}(p) = \frac{1}{p+2} + (p-2)y_0,$$

so that

$$\mathscr{F}(p) = \frac{1}{(p+2)\{(p-1)^2+2^2\}} + \frac{4(p-1)-4}{(p-1)^2+2^2}.$$

The inverse transform of the first term here is found by putting $a=2$, $b=1$, $c=-2$ in the result of Example 14, giving

$$\frac{3e^x \sin 2x + 2(e^{-2x} - e^x \cos 2x)}{2(3^2+2^2)}.$$

The inverse transforms of the other two terms are given by (16.37) and (16.38). Thus

$$\begin{aligned} f(x) &= \mathscr{L}^{-1}[\mathscr{F}(p)] \\ &= \tfrac{1}{26}(3e^x \sin 2x + 2e^{-2x} - e^x \cos 2x) + 2e^x(2\cos 2x - \sin 2x) \\ &= \tfrac{1}{13}(e^{-2x} - e^x \cos 2x) + 4e^x \cos 2x - \tfrac{49}{26}e^x \sin 2x. \end{aligned}$$

Example 16

Find the general solution of the equation $D^4y+8D^2y+16y=e^{-x}(\cos x - \sin x)$.

The subsidiary equation (16.36) is, using property 4,

$$(p^2+4)^2\mathscr{F}(p) = \frac{p}{(p+1)^2+1} + (y_3+py_2+p^2y_1+p^3y_0) + 8(y_1+py_0);$$

thus

$$\mathscr{F}(p) = \frac{p}{(p^2+4)^2\{(p+1)^2+1\}} + \frac{(y_3+4y_1)+p(y_2+4y_0)}{(p^2+4)^2} + \frac{y_1+py_0}{p^2+4}.$$

Now $\mathscr{L}^{-1}\left[\dfrac{1}{(p+1)^2+1}\right] = e^{-x}\sin x$ and from Table 16.1,

$$\mathscr{L}^{-1}\left[\frac{p}{(p^2+4)^2}\right] = \tfrac{1}{4}x \sin 2x.$$

So using (16.41),

$$\mathscr{L}^{-1}\left[\frac{p}{(p^2+4)^2\{(p+1)^2+1\}}\right]$$

$$= e^{-x}\int_0^x \tfrac{1}{4}t\sin 2t\cdot e^t \sin(x-t)\,dt$$

$$= \tfrac{1}{8}e^{-x}\int_0^x te^t\{\cos(3t-x)-\cos(t+x)\}\,dt.$$

Now

$$\int_0^t te^t\cos(at+b) = \operatorname{Re}\left[e^{ib}\int_0^x te^{t(1+ia)}\,dt\right]$$

$$= \operatorname{Re}\left[e^{ib}\left\{\frac{xe^{x(1+ia)}}{1+ia}-\frac{e^{x(1+ia)}-1}{(1+ia)^2}\right\}\right]$$

$$= \frac{xe^x}{1+a^2}\{\cos(ax+b)+a\sin(ax+b)\}+$$

$$-\frac{1}{(1+a^2)^2}\{(1-a^2)[e^x\cos(ax+b)-\cos b]+2a[e^x\sin(ax+b)-\sin b]\}.$$

Putting in turn $a=3$, $b=-x$ and $a=1$, $b=x$, we find

$$\mathscr{L}^{-1}\left[\frac{p}{(p^2+4)^2\{(p+1)^2+1\}}\right]$$

$$= \frac{x}{80}(\cos 2x+3\sin 2x)+\frac{1}{400}(4\cos 2x-3\sin 2x-3e^{-x}\sin x-4e^{-x}\cos x)$$

$$-\frac{x}{16}(\cos 2x+\sin 2x)+\frac{1}{16}(\sin 2x-e^{-x}\sin x)$$

$$= \frac{1}{200}(11\sin 2x+2\cos 2x-14e^{-x}\sin x-2e^{-x}\cos x)-\frac{x}{40}(2\cos 2x+\sin 2x)$$

The inverse transforms of the other terms in $\mathscr{F}(p)$ are known from Table 16.1:

$$\mathscr{L}^{-1}\left[\frac{(y_3+4y_1)+p(y_2+4y_0)}{(p^2+4)^2}\right] = \tfrac{1}{16}(y_3+4y_1)(\sin 2x-2x\cos 2x)+$$
$$+\tfrac{1}{4}(y_2+4y_0)\sin 2x,$$

$$\mathscr{L}^{-1}\left[\frac{y_1+py_0}{p^2+4}\right] = \tfrac{1}{2}y_1\sin 2x+y_0\cos 2x.$$

The solution $f(x)=\mathscr{L}^{-1}[\mathscr{F}(p)]$ is found by summing these three inverse transforms.

Example 17

Solve the simultaneous equations

$$Dy + 2y + z = 1$$
$$2Dy + Dz + 10y + 3z = 0$$

for y, z as functions of x, given the initial conditions $y_0 = 1$, $z_0 = -3$.

If the solution is $y = f(x)$, $z = k(x)$, and the Laplace transforms of f and k are $\mathscr{F}(p)$ and $\mathscr{K}(p)$ then taking Laplace transforms, we find the two subsidiary equations analogous to (16.36):

$$(p + 2)\mathscr{F}(p) + \mathscr{K}(p) = p^{-1} + y_0$$
$$(2p + 10)\mathscr{F}(p) + (p + 3)\mathscr{K}(p) = 2y_0 + z_0.$$

Thus

$$[(p + 2)(p + 3) - 2(p + 5)]\mathscr{F}(p) = (p + 3)(p^{-1} + y_0) - (2y_0 + z_0)$$

or

$$\mathscr{F}(p) = \frac{3}{p(p - 1)(p + 4)} + \frac{(1 + y_0 - z_0) + py_0}{(p - 1)(p + 4)}.$$

Using partial fractions,

$$\mathscr{L}^{-1}\left[\frac{p}{(p - 1)(p + 4)}\right] = \tfrac{1}{5}(e^x + 4e^{-4x})$$

and

$$\mathscr{L}^{-1}\left[\frac{1}{(p - 1)(p + 4)}\right] = \tfrac{1}{5}(e^x - e^{-4x}).$$

By property 3,

$$\mathscr{L}^{-1}\left[\frac{1}{p(p - 1)(p + 4)}\right] = \frac{1}{5}\int_0^x (e^t - e^{-4t})\,dt$$
$$= \tfrac{1}{5}(e^x + \tfrac{1}{4}e^{-4x} - \tfrac{5}{4}).$$

Thus

$$\begin{aligned} f(x) &= \mathscr{L}^{-1}[\mathscr{F}(p)] \\ &= \tfrac{1}{5}[3(e^x + \tfrac{1}{4}e^{-4x} - \tfrac{5}{4}) + (1 + y_0 - z_0)(e^x - e^{-4x}) + y_0(e^x + 4e^{-4x})] \\ &= \tfrac{9}{5}e^x - \tfrac{1}{20}e^{-4x} - \tfrac{3}{4}, \end{aligned}$$

putting $y_0 = 1$ and $z_0 = -3$. We could likewise solve for $\mathscr{K}(p)$, but in this particular example it is easier to find $k(x)$ by using the first differential equation:

$$\begin{aligned} k(x) &= 1 - Df(x) - 2f(x) \\ &= \tfrac{5}{2} - \tfrac{27}{5}e^x - \tfrac{1}{10}e^{-4x}. \end{aligned}$$

At first sight it seems that the Laplace transform method fails when the term $g(x)$ on the right of (16.34) contains terms such as $e^{\alpha x^2}$, where $\alpha > 0$. For then the integral

$$\mathscr{G}(p) = \int_0^\infty e^{-px} g(x)\,dx$$

diverges for all values of p. The reason for this divergence is that in real physical situations one never has a term such as $e^{\alpha x^2}$ acting as $x \to \infty$, becoming indefinitely large. When such a term arises, we can assume that for x greater than some large value X, the term is zero. Then the contribution to $\mathscr{G}(p)$ from $e^{\alpha x^2}$, for instance, would be

$$\int_0^X e^{-px}\, e^{\alpha x^2}\, dx,$$

which is convergent since it is an integral over a finite range. If we are solving (16.34) in a given finite range $(0, x_0)$ we can always choose X to be outside this range $(X > x_0)$; the solution in the range $(0, x_0)$ is then independent of X.

EXERCISE 16.5

1. Find the solutions of the following equations which satisfy the given initial conditions:

(i) $D^2y + n^2y = A\cos(nx + \lambda)$; $\quad y_0 = 2A/n^2, \quad y_1 = A/n$.
(ii) $D^2y - n^2y = e^{mx} + 2e^{nx}$; $\quad y_0 = y_1 = 0$.
(iii) $D^3y - y = 2x\cos x$; $\quad y_0 = y_1 = 0, \quad y_2 = 4$.
(iv) $D^3y - y = e^x \sin 2x$; $\quad y_0 = y_2 = 0, \quad y_1 = -2$.
(v) $D^2y + 2aDy + (a^2 + b^2)y = t^2e^{at}$; $\quad y_0 = 0, \quad y_1 = 1$.
(vi) $D^2y + (m + n)Dy + mny = Ae^{-mx}$; $\quad y_0 = A/m^2, \quad y_1 = 0$.
(vii) $D^4y + 2D^2y + y = A\sin x$; $\quad y_0 = y_1 = y_2 = 0, \quad y_3 = 2A$.
(viii) $D^4y + 5D^2y + 4y = g(x)$, where $g(x) = 0$ for $x < 2$, $g(x) = \sin x$ for $x \geqslant 2$; $y_0 = 4$, $y_1 = y_2 = y_3 = 0$.

2. Solve the following equations, expressing the solutions in terms of the initial values of y and its derivatives:

(i) $D^2y + y = xe^x \cos x$.
(ii) $D^4y + 4D^2y + 4y = \sin x \sinh 2x$.
(iii) $D^4y - y = 4e^x \sin x$.
(iv) $D^4y + 7D^3y + 17D^2y + 17Dy + 6y = g(x)$, where $g(x) = 0$ for $x < 3$, $g(x) = 2xe^{-2(x-3)}$ for $x \geqslant 3$.

3. Solve the following sets of simultaneous equations, taking the initial values of y and its derivatives as $y_0, y_1, \ldots$, unless otherwise stated:

(i) $Dz + z + 4y = 0$
$Dy + 2z + 3y = 0$.

$D^2y - 6Dz - 8y = \cos x$
$D^2z + Dy + 2z = e^{2x}$;
$y_0 = y_1 = z_1 = 0, \qquad z_0 = 8$.

(iii) $D^2y - aDz - by = 0$
$D^2z + aDy - bz = 0;$

$y_0 = z_0 = 0$. Consider all possible values of a, b.

(iv) $Df_r = \lambda f_{r-1} - f_r \qquad (r = 1, 2, \ldots, n)$
$Df_0 = -f_0,$

where $f_r(x)$ $(r = 0, 1, \ldots, n)$ are functions of x whose values at $x = 0$ are μ^r, λ and μ being constants.

FUNCTIONS OF A COMPLEX VARIABLE

§ 1. Regular functions

If x and y are real variables, then as we explained in Ch. 7 § 5, $z=x+iy$ is called a *complex variable*; x and y are the real and imaginary parts of z. In the Argand diagram, z can be looked upon as the variable point whose rectangular coordinates are (x, y); in complex function theory the Argand diagram is usually called the *z-plane*.

A *regular function* $w(z)$ of the complex variable z has two fundamental properties:

(i) the variables x and y appear in $w(z)$ only in the particular combination $x+iy$;

(ii) a unique and finite first derivative of $w(z)$ exists and is defined, as for real variables, by

$$\frac{\mathrm{d}w(z)}{\mathrm{d}z} = \lim_{\delta z \to 0} \frac{w(z+\delta z) - w(z)}{\delta z}. \tag{17.1}$$

Simple examples of regular functions are

$$z^2 \equiv (x+iy)^2 \equiv (x^2 - y^2) + 2ixy$$

and

$$z^3 \equiv (x+iy)^3 \equiv (x^3 - 3xy^2) + i(3x^2y - y^3),$$

with derivatives equal to $2z$ and $3z^2$. The functions $\bar{z}=x-iy$, x and $z\bar{z}\equiv x^2+y^2$ are not regular, since they violate condition (i).

It is possible for a function to satisfy condition (i), but not condition (ii); consider for example the function

$$w(z) = \begin{cases} z^2 & \text{when} \quad y \geqslant 0, \\ z & \text{when} \quad y < 0. \end{cases} \tag{17.2}$$

Let us use (17.1) to find $\mathrm{d}w/\mathrm{d}z$ at a point z on the x-axis ($y=0$). We find

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\delta z \to 0} \frac{(z+\delta z)^2 - z^2}{\delta z} = 2z,$$

provided $z+\delta z$, and hence δz, has a *positive* imaginary part. If δz and $z+\delta z$ have a negative imaginary part, (17.1) gives

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\delta z \to 0} \frac{(z+\delta z) - z^2}{\delta z};$$

this limit is infinite and non-unique when $z \neq 0$, since it depends specifically on δz, whose real and imaginary parts can tend to zero independently. Hence $w(z)$ defined by (17.2) is not regular on the x-axis; it is, however, regular at all other points, since it behaves just like z^2 or z. Condition (ii) may be satisfied when (i) is not; for example $|z|^2 = z\bar{z}$ has a unique derivative (zero) at $z=0$ found by putting $w(z)=|z|^2$ in (17.1).

Many important functions are regular except at a few isolated points. Consider for example

$$w(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}.$$

The derivative $\mathrm{d}w/\mathrm{d}z$ is $-z^{-2}$, except at $z=0$ (the origin in the z-plane). If we write $x = r\mathrm{e}^{i\theta}$, so that (r, θ) are polar coordinates in the z-plane, then $r=0$ at the origin; so $w(z)$ is clearly infinite there. Points where a function $w(z)$ is not regular are called *singularities* of $w(z)$. A function is not necessarily infinite at a singularity; for instance, the function (17.2) is singular at all points on the x-axis, but has the value $z^2 = x^2$ at these points.

§ 1.1. The Cauchy–Riemann equations

A regular function $w(z)$ can be looked upon as a function of x and y. Since x occurs only in the combination $z = x+iy$, the partial derivative with respect to x is given, as in Example 42 of Ch. 1, by

$$\frac{\partial w}{\partial x} = \frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\partial z}{\partial x} = \frac{\mathrm{d}w}{\mathrm{d}z}.$$

Since y also occurs only in this combination,

$$\frac{\partial w}{\partial y} = \frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\partial z}{\partial y} = i\,\frac{\mathrm{d}w}{\mathrm{d}z}.$$

Thus the mathematical expression of the conditions for regularity is

$$\frac{\partial w}{\partial x} = -i\,\frac{\partial w}{\partial y}. \tag{17.3}$$

Suppose that the real and imaginary parts of $w(z)$ are $u(x, y)$ and $v(x, y)$:

$$w(z) = u(x, y) + iv(x, y), \tag{17.4}$$

where u and v are real functions. For example, when $w(z)=z^3$ the real and imaginary parts, evaluated in § 1, are

$$u(x, y) = x^3 - 3xy^2 \quad \text{and} \quad v(x, y) = 3x^2y - y^3.$$

Substituting (17.4) into (17.3),

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{17.5}$$

The partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ are all real functions of real variables. So if we equate real and imaginary parts in equation (17.5), we find

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{17.6}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{17.7}$$

These equations are the *Cauchy–Riemann equations*; they are satisfied whenever a unique finite derivative dw/dz exists, and can be regarded as the criterion of regularity of $w(z)$. Functions u and v satisfying the Cauchy–Riemann equations are called *conjugate functions*.

Example 1

When $w(z)=z^{-1}$, then

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = -\frac{y}{x^2 + y^2}$$

So unless $x=y=0$,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} - \frac{x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial v}{\partial y} = -\frac{1}{x^2 + y^2} + \frac{y \cdot 2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

So, as we expect, equation (17.6) is satisfied except when $z=0$. It is clear that (17.7) is also satisfied.

If $g(z)$ and $h(z)$ are two regular functions satisfying the Cauchy–Riemann equations, it is easy to show that the sum $g(z)+h(z)$ and the product $g(z)h(z)$ are also regular; further, the quotient $g(z)/h(z)$ is regular except where $h(z)=0$. Thus we may add, multiply and divide regular functions, obtaining regular functions except where a denominator vanishes.

Generally, we shall be dealing with complex functions which are regular except at isolated points or along certain lines in the z-plane. Points where such functions are not regular are known as *singular points* or *singularities*. In the next section we shall discuss several standard types of singularity.

§ 2. Transformations and singularities

Just as we represent $z=x+iy$ as a point in the z-plane, equation (17.4) allows us to represent the complex function w as the point (u, v) in the w-plane, the coordinates $u(x, y)$ and $v(x, y)$ being determined by the values of x and y. In other words, each point (x, y) in the z-plane determines a point (u, v) in the w-plane through equation (17.4). We say that (17.4) defines *a correspondence* between points in the z-plane and the w-plane, or that it *transforms* the z-plane into the w-plane. When $w(z)$ is regular, the transformation is called *conformal.*

Example 2

If $w(z)=z^2$, then any point (x, y) in the z-plane corresponds to the point $(u, v)=(x^2-y^2, 2xy)$ in the w-plane, as in fig. 17.1. The point corresponding to

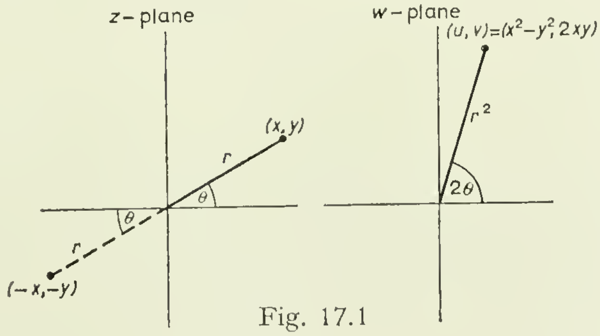

Fig. 17.1

(x, y) can be specified more readily if we use polar coordinates in the z-plane, putting $z=re^{i\theta}$; the coordinate r is called the *modulus* and θ the *argument* of z. Then $w=r^2e^{2i\theta}$ so that the polar coordinates in the w-plane (that is, the modulus and argument of w) are r^2 and 2θ, as shown. Note that points (x, y) and $(-x, -y)$ in the z-plane correspond to the *same* point (u, v) in the w-plane; we say that there is a 2:1 correspondence between the planes.

If we restrict the value of θ to the range $0 \leqslant \theta < \pi$, so that we consider points z in the 'upper half plane', then there is *exactly one point* z corresponding to each point w. So there is 1 : 1 correspondence between the upper half of the z-plane and the whole w-plane.

Example 3

If $w = z^{\alpha}$, where α is real, then putting $z = re^{i\theta}$ gives $w = r^{\alpha}e^{i\alpha\theta}$, where r^{α} is the real number whose logarithm is α times the logarithm of r; so w has modulus r^{α} and argument $\alpha\theta$. Corresponding points z and w are shown in fig. 17.2, with $\alpha \approx \frac{3}{8}$.

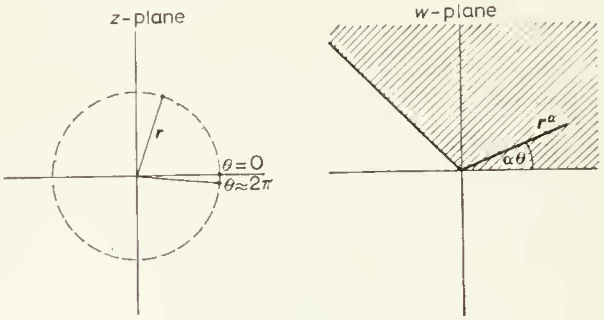

Fig. 17.2

The whole z-plane corresponds to the ranges $r \geqslant 0$, $0 \leqslant \theta < 2\pi$; so the argument of w lies in the range $(0, 2\pi\alpha)$, and the shaded region in the w-plane is in 1 : 1 correspondence with the whole z-plane. If $\alpha > 1$, the whole w-plane corresponds to part of the z-plane, as in Example 2 where $\alpha = 2$.

If $\alpha < 0$, $w(z) = z^{\alpha}$ is infinite at the origin $z = 0$. At first sight it seems that $w(z)$ is regular at all points in the z-plane except possibly $z = 0$, having derivative $dw/dz = \alpha z^{\alpha - 1}$. Consider, however, two points in the z-plane with the same argument r, and amplitudes $\theta = 0$ and $\theta \approx 2\pi$, as shown in fig. 17.2. These points are close together in the z-plane, but $w = r^{\alpha}$ and $w \approx r^{\alpha}e^{2\pi i\alpha}$ there; so $w(z)$ is not continuous at points with $\theta = 0$ in the z-plane unless $e^{2\pi i\alpha} = 1$, that is, unless α is an integer.

If α is non-integral, then $w(z)$ is discontinuous and therefore non-regular along the positive x-axis $\theta = 0$. We say that there is a *cut* or a *branch line* in the plane along this line; the cut ends at the origin in a *branch point*. If we had taken the range of θ to be $\lambda \leqslant \theta < 2\pi + \lambda$, the discontinuity in $w(z)$ would have been along $\theta = \lambda$ instead of $\theta = 0$. We cannot avoid, however, having a cut in the plane ending at the branch point.

Example 4

If $w = z^n$, n integral, then if we write $z = re^{i\theta}$, $w = r^n e^{in\theta}$. As in Example 2, where $n = 2$, the ranges $0 < \theta < 2\pi/n$, $r > 0$, correspond to the whole w-plane. If $0 \leqslant \theta < 2\pi/n$, then the n distinct points z with modulus r and arguments $\theta + 2k\pi/n$ ($k = 0, 1, 2, \ldots, n-1$) all correspond to the same point w. These n values of z are the nth roots of the complex number w, discussed in Ch. 7 § 4.4. In particular, the square roots of

$w=r^2e^{2i\theta}$ are $re^{i\theta}$ and $re^{i(\theta+\pi)}=-re^{i\theta}$, and the cube roots of $w=r^3e^{3i\theta}$ are $re^{i\theta}$, $re^{i(\theta+2\pi/3)}$ and $re^{i(\theta+4\pi/3)}$.

If we increase the argument θ of z by 2π, then $w=z^n$ becomes $r^ne^{in(\theta+2\pi)}=r^ne^{in\theta}$, returning to its original value, as we noted in Example 3. So there is no line of discontinuities of $w=z^n$ ending at the origin, and the origin is not a branch point. If $n>0$, w is finite, unique and continuous everywhere, and possesses the unique finite derivative nz^{n-1}. Thus z^n $(n>0)$ is regular for all values of z.

If $n<0$, $w=z^n$ is regular except at the origin, where it is infinite. An isolated singularity of this kind is called a *pole of order* $|n|$. More generally, the function $w=(z-\zeta)^n$, where $n<0$ and ζ is complex, is regular except at $z=\zeta$, where it has a pole of order $|n|$.

Example 5

If $w(z)=e^z=e^x\,e^{iy}=e^x(\cos y+i\sin y)$, then $u=e^x\cos y$, $v=e^x\sin y$. Thus w has modulus e^x and argument y, as shown in fig. 17.3. We note that if n is an integer, $e^{z+2in\pi}=e^x\,e^{iy+2in\pi}=e^z$, so that the points $(x, y+2n\pi)$ in the z-plane all correspond to the same point in the w-plane; several of these points are shown. If $0\leqslant y<2\pi$, there is *exactly one point* in the z-plane corresponding to (u, v) in the w-plane; this point has $e^{2x}=u^2+v^2$ or $x=\frac{1}{2}\log(u^2+v^2)$, y being the polar angle in the w-plane. So there is a 1 : 1 correspondence between points in the region $0\leqslant y<2\pi$ in the z-plane, shaded in fig. 17.3, and the points in the w-plane.

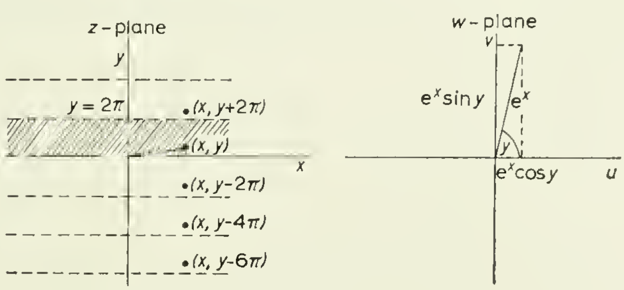

Fig. 17.3

Example 6

If $w(z)=\log z$, then $e^w=z$, and we simply interchange the roles of z and w in Example 5. Thus $x=e^u\cos v$ and $y=e^u\sin v$, so that $u=\frac{1}{2}\log(x^2+y^2)$, while $v=\tan^{-1}(y/x)$ is the polar angle in the xy-plane. Since v can be altered by $2n\pi$ (n integral) without changing x and y, there are an infinite number of values of w for a given z; in other words, there are an infinite number of logarithms of a complex variable, differing by integral multiples of $2\pi i$. If we restrict v to a range of length 2π, for example if $0\leqslant v<2\pi$ or $-\pi\leqslant v<\pi$, then v and hence w is uniquely specified if z is given.

If we write $z=re^{i\theta}$, then $w=\log r+i\theta$. So if we fix r and let θ increase from 0 to 2π, w changes by an amount $2\pi i$. Thus $w(z)$ is not continuous at points on $\theta=0$ in the z-plane, and $z=0$ is a branch point of the function $\log z$. We often take the range of θ to be $(-\pi, \pi)$; then the cut in the z-plane is along the negative real axis $\theta=\pi$.

EXERCISE 17.1

1. Examine the correspondence between the z-plane and the w-plane defined by the following relations:

(i) $w(z) = (z - \zeta)^{\frac{1}{3}}$;
(ii) $w(z) = \cosh nz$;
(iii) $w(z) = z^{\frac{1}{2}}/(z - 1)$;
(iv) $w(z) = (z^2 - \zeta^2)^{-1}$;
(v) $w(z) = \log(z^2 - \zeta^2)$;
(vi) $w(z) = \sec \alpha z$;
(vii) $w(z) = (\cos z - \zeta)^{-1}$.

[n is a real integer, α is real and non-integral, ζ is complex.]

2. Evaluate the real and imaginary parts of the functions defined in parts (ii), (iii), (v) and (vi) of Question 1. Show directly that the Cauchy–Riemann equations are satisfied where the functions are regular.

§ 3. The two-dimensional Laplace equation

In § 4.3, we shall prove that in a region where a complex function $w(z)$ is regular, all of its derivatives $\mathrm{d}^n w/\mathrm{d}z^n$ ($n=1, 2, \ldots$) exist and are regular. For any particular regular function, such as z^α or $e^{\lambda z}$, this is usually fairly obvious. The real and imaginary parts $u(x, y)$ and $v(x, y)$ therefore have derivatives of all orders also. Using the Cauchy–Riemann equations (17.6) and (17.7), we find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 u}{\partial y^2},$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{17.8}$$

Similarly we can show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{17.9}$$

If we regard x and y as two of a triad of rectangular coordinates, then equations (17.8) and (17.9) tell us that $u(x, y)$ and $v(x, y)$ are each solutions of Laplace's equation which are independent of the third coordinate. (Unfortunately this third coordinate is always called 'z'; it must not be confused with the complex variable $z=x+iy$.) Solutions which are independent of one coordinate are called 'two-dimensional', and correspond to physical field quantities which are the same for all values of this coordinate.

§ 3.1. Application to incompressible irrotational fluid motion

In Ch. 15 § 6, we have seen that the velocity potential ψ of an incompressible irrotational fluid motion satisfies Laplace's equation. Since any regular function $w(z)=u(x, y)+iv(x, y)$ provides us with two solutions u and v of Laplace's equations, we can consider a situation in which, e.g.,

$$\psi = u(x, y) \tag{17.10}$$

is the velocity potential of a fluid. By (15.56), the velocity of the fluid is

$$-\text{grad}\, u = -\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right). \tag{17.11}$$

If we consider plane cross-sections of the fluid parallel to the xy-plane, we see that the velocity vector (17.11) is (i) parallel to the plane, having zero component along the third axis; (ii) the same for every plane cross-section, since $u(x, y)$ is independent of the third coordinate. The restrictions (i) and (ii) imply that velocity potentials derived from functions of a complex variable only apply to a restricted class of fluid motions, known as 'two-dimensional'. In practical applications, it happens fairly frequently (often by design) that fluid motions are almost two-dimensional in certain regions, so that the velocity potential is of the form (17.10).

The equipotential surfaces for the potential (17.10) are

$$u(x, y) = u_0, \tag{17.12}$$

where u_0 is a constant parameter equal to the potential. These surfaces are uniform cylinders whose axes are perpendicular to the xy-plane; in fig. 17.4 the cross-sections of several cylinders are shown as continuous lines.

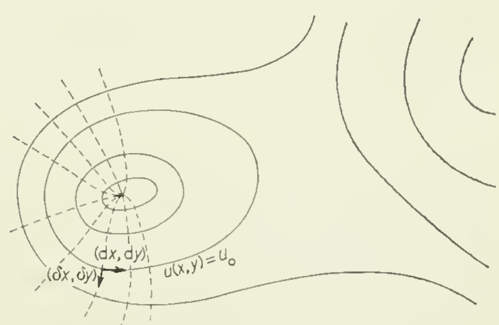

Fig. 17.4

§ 3.2. Stream lines

Let us consider a differential displacement $(\mathrm{d}x, \mathrm{d}y)$ along the tangent to one of these cross-sections; since (17.12) holds with fixed u_0 at every point on the curve, then by differentiating (17.12) we find

$$\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = 0. \tag{17.13}$$

Now suppose that $v(x, y)$ is the imaginary part of $w(z)$, conjugate to $u(x, y)$. We can draw curves, shown as dotted lines in fig. 17.4, in the xy-plane of the form

$$v(x, y) = v_0, \tag{17.14}$$

where v_0 is a constant parameter; a differential displacement $(\delta x, \delta y)$ along a tangent to a curve (17.14) satisfies

$$\frac{\partial v}{\partial x}\,\delta x + \frac{\partial v}{\partial y}\,\delta y = 0.$$

Using the Cauchy–Riemann equations (17.6) and (17.7), this equation becomes

$$\frac{\partial u}{\partial y}\,\delta x - \frac{\partial u}{\partial x}\,\delta y = 0. \tag{17.15}$$

From (17.13) and (17.15) we have

$$\mathrm{d}x\,\delta x + \mathrm{d}y\,\delta y = 0, \tag{17.16}$$

telling us that the vectors $(\mathrm{d}x, \mathrm{d}y)$ and $(\delta x, \delta y)$ are perpendicular. This is true at every point (x, y), so the two families of curves (17.12) and (17.14) are mutually orthogonal. From the definition of the gradient vector, we know that the velocity

$$-\mathrm{grad}\,\psi = -\mathrm{grad}\,u(x, y)$$

is everywhere perpendicular to the equipotentials (17.12); so the velocity is everywhere parallel to the curves (17.14), which are known as the *stream lines* of the fluid; in a fairly slow steadily moving liquid, the stream lines can be shown up by dropping a potassium permanganate crystal into the liquid. When $u(x, y)$ is the velocity potential, $v(x, y)$ is known as the *stream function.*

We note that we could equally well have chosen $v(x, y)$ to be the velocity potential; then $u(x, y)$ would be the stream function, and the lines of flow would be given by (17.12). The function $w(z) = u + iv$ which gives the potential and the stream function for a particular two-dimensional flow is called the *complex potential.*

Example 7

If $w=z^3$, then $u=x^3-3xy^2$, $v=3x^2y-y^3$. Thus

$$\mathrm{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right) = 3(x^2 - y^2, -2xy)$$

and $\operatorname{grad} v = 3(2xy, x^2 - y^2)$, and it is clear that $\operatorname{grad} u \cdot \operatorname{grad} v = 0$ at all points (x, y). Thus we have checked that the families with u constant and with v constant are everywhere orthogonal.

Example 8

If $w(z) = az^2$ is the complex potential, where a is real, then $u(x, y) = a(x^2 - y^2)$ can be taken as the velocity potential; $u(x, y)$ clearly satisfies the two-dimensional Laplace equation. The equipotential surfaces $a(x^2 - y^2) = u_0$ are rectangular hyperbolae (dashed lines in fig. 17.5). The stream lines $v(x, y) = 2axy = v_0$ are the conjugate set of rectangular hyperbolae, drawn as solid lines in the first quadrant. One particular stream line is $xy = 0$, comprising the two coordinate axes; this means that the velocity of the fluid is tangential to these two axes at all points on them, so that the fluid motion defined by this particular potential is that of irrotational flow between two flat plates meeting at right angles, with equations $x = 0$, and $y = 0$. The velocity at any point is

$$-\operatorname{grad} u = 2a(-x, y);$$

the parameter a can be chosen to give the correct magnitude of the velocity. We note that the flow pattern occupying the first quadrant of the xy-plane corresponds to the region $v \geqslant 0$ of the uv-plane.

If $v(x, y)$ were taken as the velocity potential, the dashed lines $u(x, y) = u_0$ would be the stream lines. The flow pattern is unchanged, but merely rotated through $\frac{1}{4}\pi$ in the xy-plane.

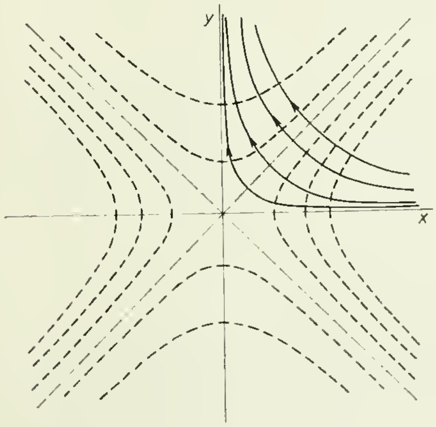

Fig. 17.5

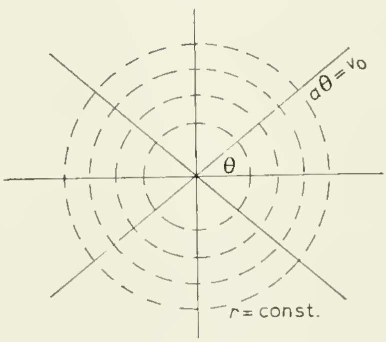

Fig. 17.6

Example 9

If $w(z) = a \log z$, where a is real, then $u = a \log r$ can be taken as the velocity potential; then the equipotentials are the circular cylinders with r constant, shown as dashed lines in fig. 17.6. The stream lines are $v = a\theta = v_0$ or $\theta =$ constant, radiating outwards from the origin (solid lines). The only non-zero component of the velocity vector $-\operatorname{grad} u$ in cylindrical polars is the radial component $-u/r = -a/r$. Thus the outward rate of flow of fluid over a length l of a circular cylinder of radius r is

$-(a/r)\cdot 2\pi lr = -2\pi la$; this is the same for all radii r, and accords with the fact that the amount of fluid in any region is conserved. If a is negative, the outflow $(-2\pi la)$ is positive and must arise from a 'line source' along the common axis of the cylinders; this axis is at $z=0$, the only singularity of $w(z)$. If a is positive, fluid flows inward and has to be absorbed at a 'line sink' at $z=0$.

We can also take $v=a\theta$ as the velocity potential. The stream lines are then the circles with r constant, and represent a fluid circulating about the origin $z=0$ with velocity $|\text{grad } v|=v/r\theta=a/r$. We notice at once that the circulation of the fluid is non-zero; in fact, the scalar line integral $\oint \boldsymbol{v}\cdot d\boldsymbol{s}$ around any of these circles is $2\pi r\cdot a/r=2\pi a$, so that the velocity field $\boldsymbol{v}$ is not conservative. We cannot therefore expect there to be a unique velocity potential; the potential $v=a\theta$ increases steadily from zero to $2\pi a$ as we encircle the origin, returning to the initial point. But, provided we accept a discontinuity in the potential at $\theta=0$, $v=a\theta$ has the properties of a potential everywhere else. This discontinuity is of course along the branch line of the complex potential, discussed in Example 6.

EXERCISE 17.2

1. Show that irrotational two-dimensional flow near the junction of two large flat plates can be described by a complex potential of the form $w=az^{\alpha}$.

2. Discuss the equipotentials and stream lines in a fluid with complex potential

$$w = a \log \frac{z+b}{z-b} \qquad (a,\ b \text{ real})$$

both when u and when v is taken as the potential.

3. Show that the complex potential $w=\cosh^{-1}(z/a)$ can be used to describe the irrotational two-dimensional flow of fluid through the gap created by cutting a strip of uniform width $2a$ out of a large flat plate.

§ 4. Complex integrals and Cauchy's theorem

A curve C drawn in the z-plane is known as a *contour*. If the curve is continuous everywhere and possesses a tangent except at a finite number of points, we can define an incremental vector $d\boldsymbol{s}=(dx,\ dy)$ along the curve in the direction of the tangent. If $w(z)=u(x,y)+iv(x,y)$ is a function regular on a contour C, the integral of $w(z)$ along the contour is defined as

$$\int_C w(z)\,dz = \int_C [u(x,y)+iv(x,y)](dx+i\,dy); \tag{17.17}$$

the integrals in (17.17), for example $\int_C u(x,y)\,dx$ are defined in the way we discussed in Ch. 11 § 4.1, the values of x and y at any point on the contour C being the coordinates at that point.

If the contour C consists of a closed loop which does not cross itself, it is called a *simple closed contour*, abbreviated to 's.c.c.'. The integral round a s.c.c. is denoted by

$$\oint_{\mathrm{C}} w(z)\,\mathrm{d}z\,;$$

the integral is taken in an anti-clockwise sense round C, to avoid ambiguity. Let us suppose now that the function $w(z)$ is regular *both inside and on* the contour C, so that $\mathrm{d}w/\mathrm{d}z$ exists and the Cauchy–Riemann equations (17.6) and (17.7) hold throughout the area S enclosed by C. Consider for example the imaginary part of the integral (17.17)

$$\oint_{\mathrm{C}} [v(x, y)\,\mathrm{d}x + u(x, y)\,\mathrm{d}y]\,;$$

using Stokes' theorem in two dimensions, equation (15.26), this can be transformed into the double integral

$$\iint_{\mathrm{S}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y. \tag{17.18}$$

Since (17.6) holds throughout S, the integral (17.18) is zero. Likewise the real part of (17.17) is

$$-\iint_{\mathrm{S}} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y,$$

which is zero by virtue of (17.7). This establishes *Cauchy's theorem*: "If a function $w(z)$ is regular in and on a s.c.c. C, then

$$\oint_{\mathrm{C}} w(z)\,\mathrm{d}z = 0.\text{''} \tag{17.19}$$

§ 4.1. The deformation property

Suppose that $w(z)$ is a given function which is regular apart from certain poles P_1, P_2, ... and certain branch points B_1, B_2, ..., and let us suppose that cuts have been made in the plane ending at the branch points. Then $w(z)$ is regular except at the poles and along the branch lines; in fig. 17.7, we have shown the singularities of a function with five poles and two branch points.

Let us consider the integral

$$\oint_{\mathrm{C}_1} w(z)\,\mathrm{d}z$$

round a s.c.c. C_1 along which $w(z)$ is regular. As in fig. 17.7, C_1 may enclose some of the poles (P_2, P_3 and P_4), but it cannot enclose any branch points. Now let C_2 be a second s.c.c. which contains *exactly* the same poles as C_1, and for the present suppose that C_1 and C_2 do not intersect. In fig. 17.7, small breaks have been made in the contours C_1 and C_2, the ends being joined by two adjacent lines l and l'. If we consider that contour C formed by C_1, l, C_2 and l', directed in the sense of

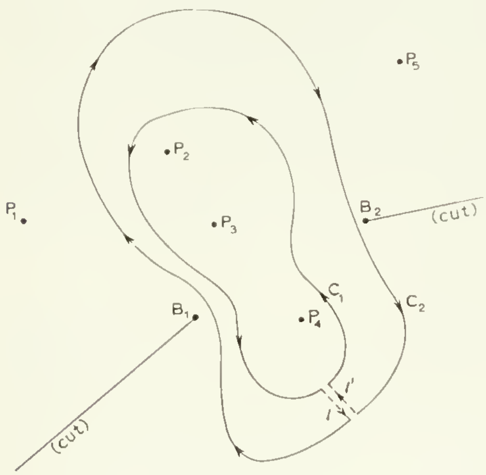

Fig. 17.7

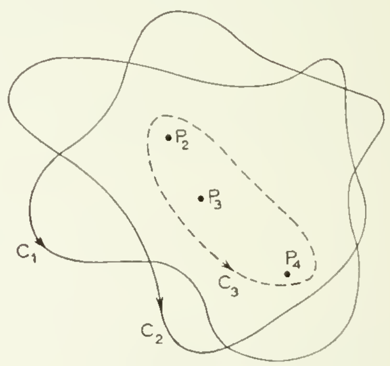

Fig. 17.8

the arrows, we see that C is a s.c.c. containing no singularities, so that (17.19) is satisfied. The integral along C can be divided into integrals along C_1, l, C_2 and l'; we notice that the integral around C_2 is in a clockwise direction, and so is counted negatively according to the usual convention. Thus (17.19) becomes

$$\int_{C_1} w(z)\,dz + \int_{l} w(z)\,dz - \int_{C_2} w(z)\,dz + \int_{l'} w(z)\,dz = 0. \tag{17.20}$$

If we let the lines l and l' come closer together and eventually coincide, then the integrals in (17.20) along l and l' will exactly cancel, since they are in opposite directions. At the same time, the breaks in C_1 and C_2 will close up, so that (17.20) gives

$$\oint_{C_1} w(z)\,dz = \oint_{C_2} w(z)\,dz, \tag{17.21}$$

the integrals being anticlockwise round the s.c.c.'s.

If C_1 and C_2 contain the same poles but intersect, as in fig. 17.8, equation (17.21) still holds. We can always draw a third s.c.c. C_3 which contains the same poles, but which is entirely inside both C_1 and C_2.

Then the result (17.21) for non-intersecting contours tells us that the integrals round C_1 and C_2 are each equal to the integral round C_3, and hence are equal.

Equation (17.21) embodies the *deformation property*:

The integral $\oint w(z)\,\mathrm{d}z$ of a function $w(z)$ round a s.c.c. is unchanged if we deform the s.c.c. in any way, provided that in doing so the contour does not cross any singularities of $w(z)$.

§ 4.2. Cauchy's integral

This integral is one of the fundamental results in the complex integral calculus. Suppose that a function $w(z)$ is regular in and on a s.c.c. C, and ζ is a point inside C. Then

$$w(\zeta) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z-\zeta}\,\mathrm{d}z. \qquad (17.22)$$

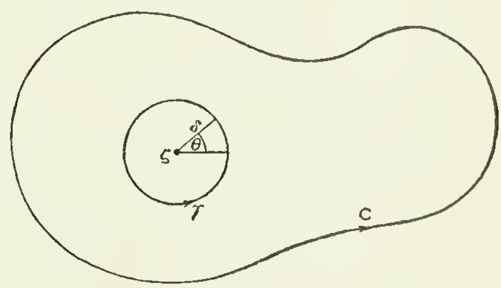

Fig. 17.9

The only singularity of the integrand $w(z)/(z-\zeta)$ is the first order pole at $z=\zeta$. So we can use the deformation property to replace C by a small circle γ of radius δ, with centre at ζ, as shown in fig. 17.9. If θ is the polar angle of a point z on γ relative to ζ, then $z=\zeta+\delta e^{i\theta}$, so that $\mathrm{d}z=i\delta e^{i\theta}\,\mathrm{d}\theta$ on γ. As z transverses γ, θ varies from 0 to 2π, so that the right-hand member of (17.22) becomes

$$\frac{1}{2\pi i}\int_{\theta=0}^{2\pi} \frac{w(\zeta+\delta e^{i\theta})}{\delta e^{i\theta}}\, i\delta e^{i\theta}\,\mathrm{d}\theta. \qquad (17.23)$$

Now let $\delta\to 0$, so that $w(\zeta+\delta e^{i\theta})\to w(\zeta)$ for all values of θ. Thus (17.23) becomes simply

$$\frac{w(\zeta)}{2\pi}\int_{\theta=0}^{2\pi} \mathrm{d}\theta = w(\zeta),$$

establishing (17.22).

Example 10

The function $z^2 \sin \pi z$ is regular in and on the square contour formed by the lines $x=\pm 1$ and $y=\pm 1$. Since the point $z=\frac{1}{2}$ lies inside this square, the integral

$$\oint \frac{z^2 \sin \pi z}{z-\frac{1}{2}} \mathrm{d}z$$

taken round the contour is given by (17.22) with $w(z)=z^2 \sin \pi z$ and $\zeta=\frac{1}{2}$; thus it is

$$2\pi i \left[z^2 \sin \pi z\right]_{z=\frac{1}{2}} = \tfrac{1}{2}\pi i \; .$$

§ 4.3. Derivatives of a regular function. Residues

In equation (17.22) the complex quantity ζ appears as a parameter. Suppose that $w(z)$ is regular within a region or *domain* D of the z-plane, and that ζ is a point inside this domain. Then we can always find a contour C which lies in D and encloses ζ as in fig. 17.10, so that (17.22) holds. Now suppose $\delta\zeta$ is a small complex increment such that $\zeta+\delta\zeta$ also lies in C. Then (17.22) also gives

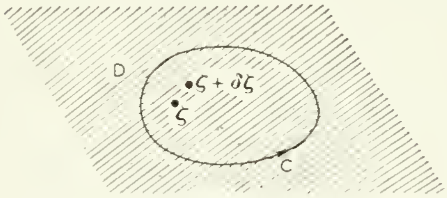

Fig. 17.10

$$w(\zeta+\delta\zeta) = \frac{1}{2\pi i} \oint_{C} \frac{w(z)}{z-(\zeta+\delta\zeta)} \mathrm{d}z.$$

Subtracting (17.22) from this equation we find

$$\frac{w(\zeta+\delta\zeta)-w(\zeta)}{\mathrm{d}\zeta} = \frac{1}{2\pi i} \oint_{C} \frac{w(z)}{(z-\zeta)(z-\zeta-\delta\zeta)} \mathrm{d}z.$$

Now let $\delta\zeta \to 0$; the integrand is finite everywhere on the contour C, so we obtain

$$\frac{\mathrm{d}w(\zeta)}{\mathrm{d}\zeta} = \frac{1}{2\pi i} \oint_{C} \frac{w(z)}{(z-\zeta)^2} \mathrm{d}z. \qquad (17.24)$$

What we have done is to show that we can differentiate equation (17.22) directly with respect to the parameter ζ to obtain $\mathrm{d}w(\zeta)/\mathrm{d}\zeta$. We can repeat this process indefinitely to establish the formula

$$\frac{\mathrm{d}^n w(\zeta)}{\mathrm{d}\zeta^n} = \frac{n!}{2\pi i} \oint_{C} \frac{w(z)}{(z-\zeta)^{n+1}} \mathrm{d}z \qquad (17.25)$$

for the nth derivative of $w(z)$ at an interior point ζ of the domain D in which $w(z)$ is regular. Previously we have only known that a regula- function possesses a unique *first* derivative; we have now shown that *unique derivatives of all orders exist at interior points of a domain of regur larity, and are given by* (17.25).

Equation (17.25) enables us to evaluate integrals round contours which encircle poles of arbitrary order. If $w(z)$ is regular in and on a s.c.c. C, then

$$\oint_C \frac{w(z)}{(z-\zeta)^{n+1}}\,\mathrm{d}z = 2\pi i\mathscr{R}(\zeta) \tag{17.26}$$

where the quantity

$$\mathscr{R}(\zeta) = \frac{1}{n!}\,\frac{\mathrm{d}^n w(\zeta)}{\mathrm{d}\zeta^n} \tag{17.27}$$

is known as the *residue* of $w(z)/(z-\zeta)^{n+1}$ at $z=\zeta$. For a first order pole ($n=0$) the residue of $w(z)/(z-\zeta)$ is just

$$\mathscr{R}(\zeta) = w(\zeta), \tag{17.28}$$

and the integral (17.26) reduces to (17.22).

Example 11

Evaluate the integral

$$\oint_C \frac{z^p}{(z-\zeta)^{n+1}}\,\mathrm{d}z$$

where the point ζ lies inside the s.c.c. C, and p, n are integers.

The residue at $z=\zeta$, given by (17.27), is

$$\mathscr{R}(\zeta) = \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\,(\zeta^p) = \binom{p}{n}\zeta^{p-n} \quad \text{if} \quad n \leqslant p,$$

while $\mathscr{R}(\zeta)=0$ if $n>p$. The value of the integral is given by (17.26).

EXERCISE 17.3

Find the residues of the following functions at their poles:

1. $\dfrac{\cos 2z}{z-\frac{1}{2}\pi}$.
2. $\dfrac{z}{(z-a)(z-b)(z-c)}$.
3. $\dfrac{1}{z^2(z-1)}$.
4. $\dfrac{\cosh z}{(z^2-a^2)^2}$.
5. $\dfrac{1}{(z^2+a^2)^n}$.
6. $\dfrac{\exp z}{(z^2+2z+2)^3}$.

§ 5. Complex power series

Consider now a function $w(z)$ defined as an infinite series in z with complex coefficients:

$$w(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{17.29}$$

We shall first of all investigate the convergence of the series (17.29). Putting $z = re^{i\theta}$ and $c_n = |c_n| e^{i\alpha_n}$ $(n=0, 1, 2, \ldots)$ in (17.29), and equating real and imaginary parts, we find

$$\left.\begin{aligned} u(x, y) &= \sum_{n=0}^{\infty} |c_n| r^n \cos(n\theta + \alpha_n), \\ v(x, y) &= \sum_{n=0}^{\infty} |c_n| r^n \sin(n\theta + \alpha_n). \end{aligned}\right\} \tag{17.30}$$

Whatever the value of the amplitudes θ and α_n,

$$|\cos(n\theta + \alpha_n)| \leqslant 1 \quad \text{and} \quad |\sin(n\theta + \alpha_n)| \leqslant 1.$$

So the series (17.30) are absolutely convergent provided that

$$\sum_{n=0}^{\infty} |c_n| r^n \equiv \sum_{n=0}^{\infty} |c_n| \, |z|^n \tag{17.31}$$

is convergent, by the comparison test given in Ch. 3 § 6.1. Further, by Cauchy's root test, Ch. 3 § 8, we know that the series (17.31) is convergent whenever there is a number $k < 1$ such that

$$|c_n|^{\frac{1}{n}} \, |z| < k.$$

Thus if the limit

$$R^{-1} = \lim_{n\to\infty} |c_n|^{\frac{1}{n}} \tag{17.32}$$

exists, the series (17.30) are absolutely convergent whenever $|z| < R$; we then say that (17.29) is *absolutely convergent* within the *circle of convergence* $|z| = R$.

★ For more general series, for which the limit (17.32) does not exist, we can always find a number $R \geqslant 0$ such that

$$|c_n|^{\frac{1}{n}} < R^{-1} + \varepsilon.$$

where $\varepsilon \to 0$ as $n \to \infty$. If R is the largest such number, then by (3.30), the series (17.29) is convergent if $|z| < R$. This defines a *circle of convergence* for an arbitrary power (17.29); it must be remembered, however, that the radius of the circle might be zero, (17.29) being absolutely convergent only when $z = 0$. ★

Example 12

Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\lambda z)^n.$$

This series is equal to $\exp(\lambda z)$ when λz is real, and in § 5.2 we shall show that this is true for complex λz. The series is absolutely convergent when the comparison series (17.31), equal to

$$\sum_{n=0}^{\infty} \frac{1}{n!} |\lambda|^n r^n$$

is convergent; but this is simply the real series expansion for $\exp(|\lambda|r)$, which converges for all r. So the exponential series is absolutely convergent in the whole z-plane; we say that it has an infinite radius of convergence.

It follows at once that the series for $\cos \lambda z$, $\sin \lambda z$, $\cosh \lambda z$ and $\sinh \lambda z$ all have infinite radii of convergence.

Example 13

Find the radius of convergence of the series

$$-\sum_{n=1}^{\infty} \frac{1}{n} (-\lambda z)^n.$$

When λz is real, this is the series expansion of $\log(1+\lambda z)$ provided $|\lambda z|<1$; in § 5.2 we shall justify this identification for complex values of λz when $|\lambda z|<1$. The comparison series (17.31) is

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda|^n r^n;$$

this is the real series expansion for $\log(1-|\lambda|r)$, which we know to be convergent for $r<|\lambda|^{-1}$. Therefore the radius of convergence of the complex variable series is $|\lambda|^{-1}$.

§ 5.1. Regularity of convergent power series

If we differentiate the series (17.29) term by term, we obtain the formal result

$$\frac{\mathrm{d}w(z)}{\mathrm{d}z} = \sum_{n=0}^{\infty} n c_n z^n. \tag{17.33}$$

This series will be absolutely convergent if

$$\sum_{n=0}^{\infty} n|c_n|\,|z|^n$$

is convergent, and by Cauchy's root test, Ch. 3 § 8, this happens if

$$|z| < \lim_{n\to\infty} (n|c_n|)^{\frac{1}{n}} = \lim |c_n|^{\frac{1}{n}}$$

since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ (see Ch. 6, Exercise 6.1, No. 15). So using (17.32), we know that (17.33) is absolutely convergent when $|z| < R$, and therefore has the same radius of convergence as the original series (17.29). Thus $\mathrm{d}w/\mathrm{d}z$ is defined by (17.33) within the radius of convergence of $w(z)$. Since $w(z)$ contains x and y only in the combination $z = x+iy$, equation (17.3) holds; therefore the real and imaginary parts u and v, defined by (17.4), will satisfy the Cauchy–Riemann equations (17.6) and (17.7). The convergence of (17.33) ensures the convergence of the series $\partial u/\partial x$, $\partial v/\partial y$, $\partial u/\partial y$ and $\partial v/\partial x$.

We have therefore proved that *a series of the form* (17.29) *is regular within its circle of convergence.* Further, since (17.33) is convergent, it too is regular; hence the series $\mathrm{d}^2w/\mathrm{d}z^2$ is convergent and regular, and so on; thus *all derivatives of the series* (17.29) *are regular within its circle of convergence.*

Example 14

If we differentiate term by term the series given in Example 13, we obtain

$$\lambda \sum_{n=1}^{\infty} (-\lambda z)^{n-1}$$

which is absolutely convergent when $|\lambda z| < 1$, or $|z| < |\lambda|^{-1}$. The series is in fact the expansion of $\lambda(1+\lambda z)^{-1}$, which, as we expect, is the derivative of $\log(1+\lambda z)$ defined by (17.1).

§ 5.2. Complex Taylor series

Suppose now that a function $w(z)$ is given in closed form, and that we wish to expand it in a power series. An expansion of the form (17.29) is known as an "expansion about the origin". More generally,

$$w(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n, \tag{17.34}$$

where ζ is a complex number, is *an expansion about* $z = \zeta$. The series in (17.34) is convergent for values of z lying inside the circle

$$|z-\zeta| = R$$

with centre at ζ, the radius R again being given by (17.32).

Suppose now that the given function $w(z)$ is regular in and on a circle C with centre at ζ and radius ρ; this is the shaded domain in fig. 17.11. We shall show that the series expansion (17.34) is valid at all points z inside C, and that the coefficients c_n are given, as for real Taylor series in (6.35), by

$$c_n = \frac{1}{n!}\,\frac{\mathrm{d}^n w(\zeta)}{\mathrm{d}\zeta^n}. \tag{17.35}$$

Writing z_0 for z and z for ζ in Cauchy's integral (17.22), we have

$$w(z) = \frac{1}{2\pi i}\oint_C \frac{w(z_0)}{z_0 - z}\,\mathrm{d}z_0. \tag{17.36}$$

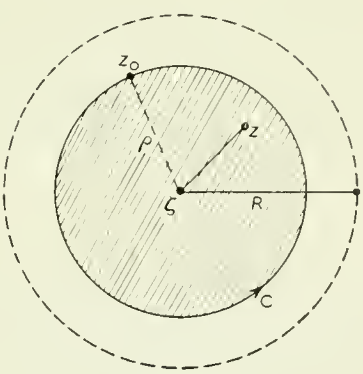

Fig. 17.11

As shown in fig. 17.11, $|z_0-\zeta|=\rho$ for all z_0 on C, while $|z-\zeta|<\rho$ for any point z inside C. So using the binomial expansion, we have

$$\frac{1}{z_0 - z} = \sum_{n=0}^{\infty} \frac{(z-\zeta)^n}{(z_0-\zeta)^{n+1}}, \tag{17.37}$$

which is absolutely convergent for all z_0 on C. Substituting in (17.36), we have

$$w(z) = \sum_{n=0}^{\infty} \frac{(z-\zeta)^n}{2\pi i}\oint_C \frac{w(z_0)}{(z_0-\zeta)^{n+1}}\,\mathrm{d}z_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{\mathrm{d}^n w(\zeta)}{\mathrm{d}\zeta^n}(z-\zeta)^n,$$

using (17.25). So the formula (17.35) for the coefficients in (17.34) is established. Further, the convergence of (17.37) ensures the convergence of (17.34) when z is inside C. Therefore the radius of convergence R of (17.34) is not smaller than ρ; it may exceed ρ, as shown in fig. 17.11.

We may therefore use the Taylor expansion of any function within the circle C; this justifies the statement that the series in Examples 12 and 13 represent the functions $\exp(\lambda z)$ and $\log(1+\lambda z)$ within the circles of convergence. Frequently the series expansion is convergent in domains outside C, and is a valid expression for $w(z)$ there. It also often happens that a function can be represented by one power series expansion in one domain, and by a different power series in another domain.

§ 6. The calculus of residues

§ 6.1. The evaluation of residues

Equation (17.26) tells us that if $w(z)$ is regular in and on a s.c.c. C, and ζ is inside C, then the integral of

$$\frac{w(z)}{(z-\zeta)^{n+1}} \tag{17.38}$$

round C equals $2\pi i\mathscr{R}(\zeta)$, where the residue $\mathscr{R}(\zeta)$ at $z=\zeta$ is given by (17.27).

Poles do not always occur in the simple form (17.26). Suppose that $g(z)$ and $h(z)$ are regular functions in some domain, and $h(z)=0$ at a point $z=\zeta$ within the domain. Then, assuming $g(\zeta)\neq 0$, the function

$$f(z) = \frac{g(z)}{h(z)} \tag{17.39}$$

is infinite at $z=\zeta$, and so is not regular there. Since $h(z)$ is regular in a domain around $z=\zeta$, we may expand it in a Taylor series about ζ; remembering that $h(\zeta)=0$,

$$h(z) = (z-\zeta)h'(\zeta) + \tfrac{1}{2}(z-\zeta)^2h''(\zeta) + \dots . \tag{17.40}$$

So, provided $h'(\zeta)\neq 0$, the denominator in (17.39) will contain a factor $z-\zeta$. We therefore write $f(z)$ as

$$f(z) = \frac{g(z)}{z-\zeta}\left[\frac{z-\zeta}{h(z)}\right]; \tag{17.41}$$

we shall now show that the term in square brackets here is a regular function at $z=\zeta$, so that $f(z)$ has a first order pole at $z=\zeta$. From (17.40),

$$\frac{z-\zeta}{h(z)} = \frac{1}{h'(\zeta) + \frac{1}{2}(z-\zeta)h''(\zeta) + \dots}$$

so that

$$\frac{z-\zeta}{h(z)} \to \frac{1}{h'(\zeta)} \tag{17.42}$$

a finite value, as $z\to\zeta$. This is the value given by l'Hospital's rule (6.20). Further, using the series (17.40) and its derivative with respect to z,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z-\zeta}{h(z)}\right] = \frac{h(z)-(z-\zeta)h'(z)}{[h(z)]^2}$$

$$= \frac{-\frac{1}{2}(z-\zeta)^2h''(z) - \dots}{[(z-\zeta)h'(\zeta) + \frac{1}{2}(z-\zeta)^2h''(\zeta) + \dots]^2},$$

which tends to the finite and unique limit $-h''(\zeta)/2[h'(\zeta)]^2$ as $z \to \zeta$. Therefore $(z-\zeta)/h(z)$ is regular at $z=\zeta$, and $f(z)$ has a first order pole there. Now the residue of $f(z)$ at the first order pole ζ is given by (17.28), and is simply

$$\mathscr{R}(\zeta) = [(z-\zeta)f(z)]_{z=\zeta}.$$

Using (17.41) and (17.42), we have

$$\mathscr{R}(\zeta) = \frac{g(\zeta)}{h'(\zeta)}. \tag{17.43}$$

Example 15

Find the residues of

$$f(z) = \frac{\sinh^n z}{\cosh z - \cos a}$$

at the poles of $f(z)$, assuming that a is not of the form $n\pi$ (n integral).

Now $f(z)$ is regular except when $\cosh z = \cos a$, or $z = \pm ia + 2k\pi i$ ($k=0, \pm 1, \pm 2, \ldots$). The function is of the form (17.39) with $g(z) = \sinh^n z$ and $h(z) = \cosh z - \cos a$, Thus $h'(z) = \sinh z$; at a singularity $z = \pm ia + 2k\pi i$, $h'(z)$ takes the value $+i \sin a$. which is non-zero since $a \neq n\pi$. The residues are therefore given by (17.43) with $\zeta = +ia \pm 2k\pi i$:

$$\mathscr{R}(\pm ia + 2k\pi i) = \sinh^{n-1}(\pm ia + 2k\pi i) = (\pm i \sin a)^{n-1}.$$

When a function $f(z)$ of the form (17.39) has a pole of higher order than first at $z=\zeta$, the residue is in general more difficult to calculate. Let us suppose that $h(z)$ and its first $n-1$ derivatives vanish at $z=\zeta$, while $g(z)$ and its first $m-1$ derivatives vanish where $m<n$. We can expand $g(z)$ and $h(z)$ in Taylor series to give an expression of the form

$$f(z) = \frac{g(z)}{h(z)} = \frac{c_m(z-\zeta)^m + c_{m+1}(z-\zeta)^{m+1} + \ldots}{d_n(z-\zeta)^n + d_{n+1}(z-\zeta)^{n+1} + \ldots}$$

$$= \frac{1}{(z-\zeta)^{n-m}} \frac{c_m + c_{m+1}(z-\zeta) + \ldots}{d_n + d_{n+1}(z-\zeta) + \ldots},$$

exhibiting a pole of order $p=n-m$ at $z=\zeta$ if $n>m$. Since $d_n \neq 0$, we can expand the denominator series by the binomial theorem; this results in an expansion of the form

$$f(z) = \frac{1}{(z-\zeta)^p} \sum_{r=0}^{\infty} a_{r-p}(z-\zeta)^r, \tag{17.44}$$

where a_q is the coefficient of $(z-\zeta)^q$. If we now integrate round a small circle γ with centre at $z=\zeta$, we have

$$\mathscr{R}(\zeta) = \frac{1}{2\pi i} \oint_\gamma f(z)\,\mathrm{d}z$$

$$= \sum_{r=0}^{\infty} a_{r-p} \frac{1}{2\pi i} \oint_\gamma (z-\zeta)^{r-p}\,\mathrm{d}z. \tag{17.45}$$

As in the derivation of (17.22), we put $z-\zeta=\delta e^{i\theta}$; then

$$\oint_\gamma (z-\zeta)^{r-p}\,\mathrm{d}z = i\delta^{r-p+1} \int_{\theta=0}^{2\pi} e^{i\theta(r-p-1)}\,\mathrm{d}\theta,$$

which is zero unless $r=p-1$, when it equals $2\pi i$. Thus (17.45) reduces to

$$\mathscr{R}(\zeta) = a_{-1}. \tag{17.46}$$

This means that the residue of $f(z)$ at $z=\zeta$ is found by expanding it in the form (17.44), and picking out the coefficient a_{-1} of $(z-\zeta)^{-1}$.

Example 16

Find the residue of

$$f(z) = \cot^2 z \operatorname{cosec} z$$

at the origin $z=0$.

Using the usual Taylor series expansions for $\cos z$ and $\sin z$, we have

$$f(z) = \frac{\cos^2 z}{\sin^3 z} = \frac{(1-\frac{1}{2}z^2+\ldots)^2}{z^3(1-\frac{1}{6}z^2+\ldots)^3}$$

$$= z^{-3}(1-z^2+\ldots)(1+\tfrac{1}{2}z^2+\ldots) = z^{-3}[1-z^2(1-\tfrac{1}{2})+\ldots].$$

By (17.46), the residue of $f(z)$ at $z=0$ is the coefficient of z^{-1} in this expansion; that is, $\mathscr{R}(0)=-\frac{1}{2}$.

Example 17

Find the residue at $z=\frac{1}{4}\pi$ of

$$f(z) = \frac{z^2}{(\tan^2 z - 1)^2}.$$

We must expand $f(z)$ in a series in $z_1=(z-\frac{1}{4}\pi)$. We have

$$\tan z = \tan \tfrac{1}{4}\pi + z_1 \sec^2 \tfrac{1}{4}\pi + \tfrac{1}{2}z_1^2 \cdot 2 \sec^2 \tfrac{1}{4}\pi \tan \tfrac{1}{4}\pi + \ldots$$

$$= 1 + 2z_1 + 2z_1^2 + \ldots.$$

So $\tan^2 z - 1 = (1+2z_1+2z_1^2+\ldots)^2 - 1 = 4z_1(1+2z_1+\ldots)$.

Thus

$$\begin{aligned} f(z) &= \tfrac{1}{16} z_1^{-2} (\tfrac{1}{4}\pi + z_1)^2 (1 + 2z_1 + \dots)^{-2} \\ &= \frac{\pi^2}{256} z_1^{-2} \left(1 + \frac{8z_1}{\pi} + \dots\right)(1 - 4z_1 + \dots) \\ &= \frac{\pi^2}{256} z_1^{-2} \left[1 + 4z_1 \left(\frac{2}{\pi} - 1\right) + \dots\right] \end{aligned}$$

So by (17.46), $\mathscr{R}(\tfrac{1}{4}\pi) = \dfrac{\pi^2}{64}\left(\dfrac{2}{\pi} - 1\right) = \dfrac{\pi}{32}(1 - \tfrac{1}{2}\pi)$.

EXERCISE 17.4

Find the residues of the following functions at the given poles.

1. $\cos^2 z \cot z$ at $z = 0$.
2. $\dfrac{\pi + z}{z - \frac{1}{3}\pi \sin z}$ at $z = \frac{1}{6}\pi$.
3. $\operatorname{cosech}^2 z$ at $z = 0$.
4. $e^{2z} \tan^3 z$ at $z = \frac{1}{2}\pi$ and at $z = -\frac{1}{2}\pi$.
5. $\dfrac{z^3}{(\sin z - 1)^2}$ at $z = (2n + \frac{1}{2})\pi$.

§ 6.2. The residue theorem

When a function $f(z)$ is regular in and on a s.c.c. C except at a single pole ζ, the integral of $f(z)$ round C is given by (17.26), the residue $\mathscr{R}(\zeta)$ being calculated by using (17.27) or by the methods of § 6.1. The deformation property, enunciated in § 4.1, allows us to extend the result (17.26) to a contour containing any number of poles.

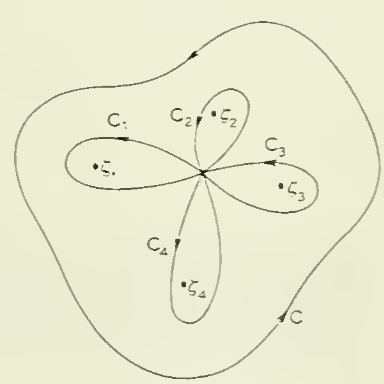

Fig. 17.12

Suppose that a s.c.c. C encloses poles $\zeta_1, \zeta_2, \dots$ of $f(z)$, but no branch points, so that $f(z)$ is regular at all points of C. Then as shown in fig. 17.12 for four poles, we deform the contour C into a 'clover-leaf' contour which consists of s.c.c.'s $C_1, C_2, \dots$, each enclosing one pole only. Then the deformation property tells us that

$$\oint_C f(z)\,dz = \sum_{r=1,2,\dots} \oint_{C_r} f(z)\,dz. \tag{17.47}$$

A particular s.c.c. C_r contains only one singularity at $z=\zeta_r$; so by (17.26),

$$\oint_{C_r} f(z)\,dz = 2\pi i\mathscr{R}(\zeta_r).$$

Thus (17.47) becomes

$$\oint_C f(z)\,dz = 2\pi i \sum_C \mathscr{R}(\zeta_r), \tag{17.48}$$

where $\sum_C$ denotes 'the sum of the residues at the poles ζ_r inside C'. This is the *Residue theorem*, and is a fundamental result in the complex integral calculus. It enables us to perform contour integrals by simply evaluating residues at all the poles inside a contour. It is important to remember that (17.48) applies only when $f(z)$ has no singularities inside C other than poles.

We shall now show how the Residue theorem can be used to evaluate certain types of integrals.

§ 6.3. Trigonometric integrals over range $(0, 2\pi)$

If $\psi(\cos\theta, \sin\theta)$ is a function of $\cos\theta$ and $\sin\theta$ and we write $z=e^{i\theta}$, then integration over the range $(0, 2\pi)$ of θ corresponds to integration in the z-plane round the circle of unit radius with centre at $z=0$, known as the *unit circle*. Further

$$\cos\theta = \tfrac{1}{2}(z+z^{-1}), \qquad \sin\theta = \frac{1}{2i}(z-z^{-1})$$

and $d\theta=-iz^{-1}dz$. So the integral

$$\int_0^{2\pi} \psi(\cos\theta, \sin\theta)\,d\theta = \oint_C f(z)\,dz, \tag{17.49}$$

where

$$f(z) = -iz^{-1}\psi[\tfrac{1}{2}(z+z^{-1}), \tfrac{1}{2i}(z-z^{-1})] \tag{17.50}$$

and C is the unit circle. Provided that $\psi=f(z)$ has no singularities on C, and no singularities other than poles within C, we can immediately apply the Residue theorem (17.48) to evaluate (17.49).

Example 18

Evaluate

$$\int_0^{\pi} \frac{\cos\theta\,d\theta}{a+\cos\theta},$$

where $a > 1$. The integral is clearly half the integral over the range $(0, 2\pi)$, and by (17.49) and (17.50) equals

$$\tfrac{1}{2}\oint_C \frac{-i(z^2+1)}{z(z^2+2az+1)}\,dz = \tfrac{1}{2}\oint_C \frac{-i(z^2+1)}{z(z-\zeta_2)(z-\zeta_3)}\,dz,$$

say, the integral being round the unit circle. The poles of the integrand are at the origin, which we denote by ζ_1, and at

$$\zeta_2 = -a + (a^2-1)^{\frac{1}{2}} \quad \text{and} \quad \zeta_3 = -a - (a^2-1)^{\frac{1}{2}}.$$

Since $a > 1$, ζ_2 and ζ_3 are real. Further, $|\zeta_2| < 1$ and $|\zeta_3| > 1$, so that ζ_2 lies inside, and ζ_3 outside, the unit circle. Equation (17.48) tells us that the integral is

$$\pi i[\mathscr{R}(\zeta_1) + \mathscr{R}(\zeta_2)].$$

The integrand

$$f(z) = \frac{-i(z^2+1)}{z(z-\zeta_2)(z-\zeta_3)}$$

has only first order poles at $z = \zeta_1 \equiv 0$ and $z = \zeta_2$, so the residues at these poles are, using (17.27),

$$\mathscr{R}(\zeta_1) = \frac{-i(\zeta_1^2+1)}{(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)} = \frac{-i}{\zeta_2\zeta_3} = -i$$

$$\mathscr{R}(\zeta_2) = \frac{-i(\zeta_2^2+1)}{\zeta_2(\zeta_2-\zeta_3)} = \frac{2ia\zeta_2}{\zeta_2(\zeta_2-\zeta_3)},$$

since $\zeta_2^2 + 2a\zeta_2 + 1 = 0$. So

$$\mathscr{R}(\zeta_2) = \frac{ia}{(a^2-1)^{\frac{1}{2}}}.$$

Thus the integral is

$$\pi i\left[-i + \frac{ia}{(a^2-1)^{\frac{1}{2}}}\right] = \pi\left[1 - \frac{a}{(a^2-1)^{\frac{1}{2}}}\right].$$

The result can be verified by the method of Ch. 5, § 5.2.

§ 6.4. Integrals over the range $(-\infty, \infty)$

First we give an example of an integral over this range whose evaluation depends upon the periodicity properties of $e^{\pm z}$ discussed in Example 5.

Example 19

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{\cosh x - \cos\alpha} \qquad (0 < \alpha < \tfrac{1}{2}\pi).$$

Consider the integral

$$\oint_C \frac{z\,\mathrm{d}z}{\cosh z - \cos\alpha}$$

round the s.c.c. C shown in fig. 17.13, the rectangle with corners at $\pm R$ and $\pm R+2\pi i$ $(R>0)$. On the sections of C parallel to the y-axis, $z=\pm R+iy$ and $\mathrm{d}z=\pm i\,\mathrm{d}y$, so that the integrals along these sections are

$$\int_{y=0}^{2\pi} \pm \frac{(\pm R+iy)i\,\mathrm{d}y}{\frac{1}{2}(\mathrm{e}^{\pm R+iy}+\mathrm{e}^{\mp R-iy})-\cos\alpha}$$
$$= \frac{R}{\mathrm{e}^R}\int_{y=0}^{2\pi} \frac{1\pm(iy/R)}{\frac{1}{2}[\mathrm{e}^{\pm iy}+\mathrm{e}^{-2R}\,\mathrm{e}^{\mp iy}]-\mathrm{e}^{-R}\cos\alpha}\,i\mathrm{d}y.$$

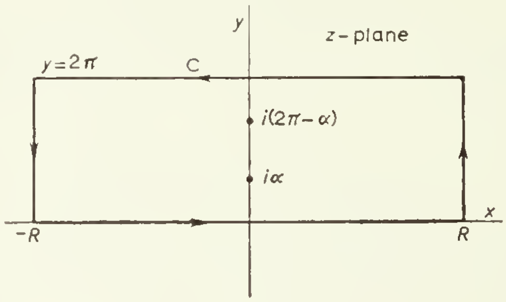

Fig. 17.13

As $R\to\infty$, the integrand in the last integral remains finite, tending to $2i\mathrm{e}^{\mp iy}$. Since the integration is over a finite range, the integral is finite. Since $R\mathrm{e}^{-R}\to 0$ as $R\to\infty$, the contributions from these sections tend to zero.

On the x-axis, $z=x$ and $\mathrm{d}z=\mathrm{d}x$, while on the section parallel to it, $z=x+2\pi i$ and $\mathrm{d}z=-\mathrm{d}x$. Further, $\cosh(x+2\pi i)=\cosh x$, so the contribution of these two sections to the contour integral is

$$\int_{x=-R}^{R} \frac{x\,\mathrm{d}x}{\cosh x-\cos\alpha} - \int_{x=-R}^{R} \frac{(x+2\pi i)\,\mathrm{d}x}{\cosh x-\cos\alpha}$$
$$= -2\pi i \int_{-R}^{R} \frac{\mathrm{d}x}{\cosh x-\cos\alpha}.$$

Therefore as $R\to\infty$, we find using the residue theorem,

$$I = -\frac{1}{2\pi i}\oint_C \frac{z\,\mathrm{d}z}{\cosh z-\cos\alpha}$$
$$= -\sum_C \mathscr{R}(\zeta_r).$$

The poles of the integrand occur when $\cosh z = \cos\alpha$, as in Example 15; the only poles inside C are at $\zeta_1 \equiv i\alpha$ and $\zeta_2 \equiv i(2\pi - \alpha)$, as shown in fig. 17.13, and these are first order poles. Using (17.43), the residue at a pole ζ is

$$\frac{\zeta}{\mathrm{d}(\cosh\zeta)/\mathrm{d}\zeta} = \frac{\zeta}{\sinh\zeta}.$$

So $\mathscr{R}(\zeta_1) = \dfrac{i\alpha}{\sinh i\alpha} = \dfrac{\alpha}{\sin\alpha}$ and $\mathscr{R}(\zeta_2) = \dfrac{i(2\pi-\alpha)}{\sinh i(2\pi-\alpha)} = \dfrac{\alpha - 2\pi}{\sin\alpha}$.

Hence

$$I = -[\mathscr{R}(\zeta_1) + \mathscr{R}(\zeta_2)] = \frac{2(\pi-\alpha)}{\sin\alpha}.$$

Next we discuss the integrals of the form

$$\int_{-\infty}^{\infty} Q(x)\,\mathrm{d}x \tag{17.51}$$

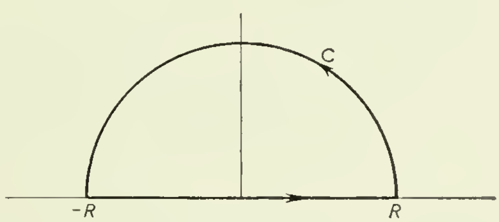

Fig. 17.14

which can be evaluated by calculating $\oint_C Q(z)\,\mathrm{d}z$ round the s.c.c. shown in fig. 17.14, consisting of the range $(-R, R)$ of the real axis and the semicircle (radius R, centre at the origin) in the upper half-plane.

Provided that the integral round the semicircle tends to zero as $R \to \infty$, the integral (17.51) is equal to

$$\lim_{R\to\infty} \oint_C Q(z)\,\mathrm{d}z = 2\pi i \sum_{(+)} \mathscr{R}, \tag{17.52}$$

where $\sum_{(+)} \mathscr{R}$ means the sum of residues at poles in the upper half-plane. There must be no singularities on C of course, so that

(i) $Q(z)$ must be regular at all points on the real axis,

(ii) the only singularities of $Q(z)$ in the upper half-plane must be poles (we then say that Q is *meromorphic* in the upper half-plane).

In addition to (i) and (ii), a condition must be imposed to ensure the vanishing of the integral round the semicircle. It is not difficult to show that this happens if

(iii$_1$) *either* $zQ(z) \to 0$ at all points on the semicircle as $R \to \infty$, and the integral $\int_{-R}^{R} Q(z)\,\mathrm{d}z$ tends to a unique finite limit as $R \to \infty$, equal to the integral (17.51)
(iii$_2$) *or* $Q(z) = \mathrm{e}^{imz} N(z)/D(z)$, where $m > 0$, $N(z)$ and $D(z)$ are polynomials in z, and the degree of D is greater than that of N.

If $Q(z)$ satisfies conditions (i), (ii) and either (iii$_1$) or (iii$_2$) then the integral (17.51) is given by (17.52). The validity of (iii$_2$) depends upon *Jordan's lemma*, which uses the fact that the factor $\mathrm{e}^{imz} = \mathrm{e}^{imx}\,\mathrm{e}^{-my}$ tends to zero exponentially at all points on the semicircle (except the end points) as $R \to \infty$. Condition (iii$_2$) is particularly useful in the evaluation of Fourier transforms.

Example 20

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^6\,\mathrm{d}x}{(x^4 + a^4)^2}, \quad \text{where} \quad a > 0.$$

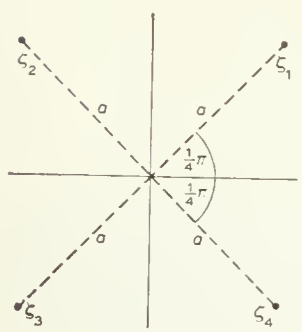

Fig. 17.15

The function $Q(z) = z^6/(a^4 + z^4)^2$ satisfies conditions (i), (ii) and (iii$_1$), so that the integral is given by (17.52). The poles of $Q(z)$ are given by $a^4 + z^4 = 0$, and are at

$$\zeta_1 = a\mathrm{e}^{\frac{1}{4}i\pi}, \qquad \zeta_2 = a\mathrm{e}^{\frac{3}{4}i\pi},$$
$$\zeta_3 = a\mathrm{e}^{\frac{5}{4}i\pi}, \qquad \zeta_4 = a\mathrm{e}^{\frac{7}{4}i\pi},$$

as shown in fig. 17.15. The points ζ_1 and ζ_2 are in the upper half-plane, so we evaluate the residues at these points.

Since $a^4 + z^4 = (z - \zeta_1)(z - \zeta_2)(z - \zeta_3)(z - \zeta_4)$, the integrand can be written as $w(z)/(z - \zeta_1)^2$, where

$$w(z) = \frac{z^6}{(z - \zeta_2)^2(z - \zeta_3)^2(z - \zeta_4)^2}.$$

Using (17.27) with $n = 1$,

$$\mathscr{R}(\zeta_1) = w'(\zeta_1)$$

$$= \frac{1}{\zeta_1\left(1 - \dfrac{\zeta_2}{\zeta_1}\right)^2\left(1 - \dfrac{\zeta_3}{\zeta_1}\right)^2\left(1 - \dfrac{\zeta_4}{\zeta_1}\right)^2}\left[6 - \frac{2}{1 - \dfrac{\zeta_2}{\zeta_1}} - \frac{2}{1 - \dfrac{\zeta_3}{\zeta_1}} - \frac{2}{1 - \dfrac{\zeta_4}{\zeta_1}}\right]$$

$$= \frac{\mathrm{e}^{-\frac{1}{4}i\pi}}{a(1 - i)^2(2)^2(1 + i)^2}\left[6 - \frac{2}{1 - i} - \frac{2}{2} - \frac{2}{1 + i}\right]$$

$$= \frac{3\sqrt{2}(1 - i)}{32a}.$$

Similarly $\mathscr{R}(\zeta_2) = -\dfrac{3\sqrt{2}(1+i)}{32a}$.

By (17.52), the integral is

$$2\pi i[\mathscr{R}(\zeta_1) + \mathscr{R}(\zeta_2)] = \frac{3\sqrt{2}\pi}{8a}.$$

Example 21

Evaluate

$$\int_0^\infty \frac{\cos kx}{(a^2+x^2)^2}\,\mathrm{d}x$$

where k, a are real and positive.

The integral is equal to

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx}}{(a^2+x^2)^2}\,\mathrm{d}x.$$

Now $Q(z) = \mathrm{e}^{ikz}/(a^2+z^2)^2$ satisfies conditions (i), (ii) and (iii$_2$), so we can apply (17.52). The only singularity of $Q(a)$ in the upper half-plane is at $z = ia$. We can evaluate the residue either by using (17.27), or as in § 6.1 by expanding $Q(z)$ in powers of $z_0 = z - ia$. Using the expansion method,

$$\begin{aligned} Q(z) &= \frac{\exp[ik(z_0+ia)]}{z_0^2(z_0+2ia)^2} \\ &= -\frac{\exp(-ka)}{4a^2}\, z_0^{-2}(1 + ikz_0 + \ldots)\left(1 - \frac{z_0}{ia} + \ldots\right). \end{aligned}$$

Picking out the coefficients of z_0^{-1}, we have by (17.47),

$$\mathscr{R}(ia) = \frac{-i\mathrm{e}^{-ka}(1+ka)}{4a^3}.$$

By (17.52), the integral is

$$\pi i\mathscr{R}(ia) = \frac{\pi\mathrm{e}^{-ka}(1+ka)}{4a^3}.$$

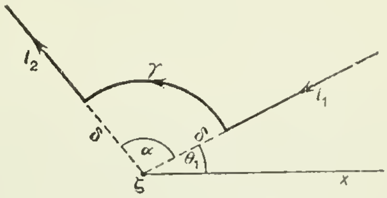

Fig. 17.16

§ 6.5. Principal values and similar integrals

★ It is sometimes necessary to perform integrals along contours which pass very near to poles of a function $f(z)$. For first order poles, we can perform these integrals conveniently by choosing a contour of the kind shown in fig. 17.16, approaching and leaving a pole ζ along lines l_1 and l_2 making an angle α and completed by an arc of a circle (radius δ) whose centre is at ζ.

It is not difficult to see that the integral round γ for first order poles is, as $\delta \to 0$, simply

$$\frac{\alpha}{2\pi} \cdot 2\pi i \mathscr{R}(\zeta) = i\alpha \mathscr{R}(\zeta), \tag{17.53}$$

using the proof given in § 4.2 of Cauchy's integral with the upper limit 2π in (17.23) replaced by α. The integral along the lines l_1 and l_2 is

$$\lim_{\delta \to 0} \left[\int_{l_1} f(z)\,\mathrm{d}z + \int_{l_2} f(z)\,\mathrm{d}z \right]. \tag{17.54}$$

Near to ζ on the line l_1,

$$z \approx \zeta + r\mathrm{e}^{i\theta_1} \tag{17.55}$$

where θ_1 is the angle between l_1 and the x-axis, and $\mathrm{d}z = \mathrm{e}^{i\theta_1}\mathrm{d}r$. Since $f(z)$ has a simple pole at ζ, it has the form

$$f(z) \approx \frac{w(\zeta)}{z - \zeta} \approx r^{-1} w(\zeta) \mathrm{e}^{-i\theta_1}$$

on l_1 near ζ by (17.55). So the integral in (17.54) along a portion of l_1 near to ζ is approximately

$$\int^{r=\delta} r^{-1} w(\zeta) \mathrm{e}^{-i\theta_1}\, \mathrm{e}^{i\theta_1} \mathrm{d}r = w(\zeta) [\log r]_{r=\delta}$$

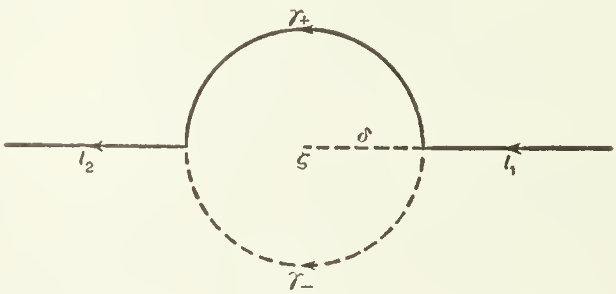

Fig. 17.17

so in the limit $\delta \to 0$, the integral along l_1 will contain the logarithmic infinity

$$w(\zeta) \lim_{\delta \to 0} \log \delta.$$

However, the integral along l_2 will contain a compensating logarithmic infinity as $\delta \to 0$, and (17.54) will give a definite finite contribution to the contour integral.

The integrals (17.53) and (17.54) are particularly important when $\alpha = \pm\pi$, so that l_2 leaves the point ζ in the direction that l_1 enters, as in fig. 17.17. Putting $\alpha = \pi$, γ becomes the semicircle γ_+, anti-clockwise about ζ; the contribution (17.53) from γ_+ is $i\pi\mathscr{R}(\zeta)$.

For the semicircle γ_-, clockwise about ζ, $\alpha = -\pi$ and the contribution (17.53) is $-i\pi\mathscr{R}(\zeta)$. If we omit the contribution from either semicircle, and simply evaluate the limit (17.54), we obtain the *principal value* of the integral, denoted by

$$\mathrm{P}\int f(z)\,\mathrm{d}z.$$

Denoting the integrals round paths including γ_+ and γ_- by $\int_{\gamma_+}$ and $\int_{\gamma}$ respectively, we therefore have

$$\int_{\gamma_\pm} f(z)\,\mathrm{d}z = \mathrm{P}\int f(z)\,\mathrm{d}z \pm i\pi\mathscr{R}(\zeta). \tag{17.56}$$

Example 22

If a, b are real and $a<0<b$, find the values of

$$\int_a^b \frac{\mathrm{d}z}{z}$$

along contours passing above and below $z=0$.

The residue at $z=0$ is $\mathscr{R}(0)=1$. The principal value, defined by (17.54) with l_1 and l_2 along the x-axis, is

$$\int_a^{-\delta} \frac{\mathrm{d}x}{x} + \int_\delta^b \frac{\mathrm{d}x}{x} = \log\frac{\delta}{|a|} + \log\frac{b}{\delta}$$

or

$$\mathrm{P}\int_a^b \frac{\mathrm{d}z}{z} = \log\frac{b}{|a|}.$$

The integrals round contours above and below $z=0$ can be deformed into paths containing γ_- and γ_+ respectively, so by (17.56) the contour above $z=0$ gives $\log b/|a| - i\pi$, while that below gives $\log b/|a| + i\pi$.

Example 23

Evaluate

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

We can write the integral as

$$\lim_{\delta\to 0}\frac{1}{2i}\int_{\delta}^{\infty}\frac{e^{ix}-e^{-ix}}{x}\,dx = \frac{1}{2i}\,P\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz.$$

Consider the integral $\oint z^{-1}e^{iz}\,dz$ round the contour shown in fig. 17.18, where $\delta\to 0$ and $R\to\infty$. The contour contains no singularities, so the integral is zero. Since the integrand satisfies conditions (i), (ii) and (iii$_2$) of § 6.4, the integral round C is zero in the limit $R\to\infty$. We therefore have, using the notation of (17.56),

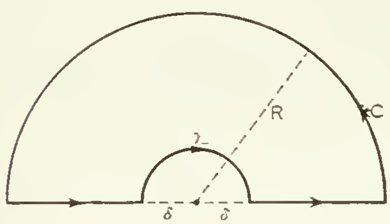

Fig. 17.18

$$\int_{\gamma-}\frac{e^{iz}}{z}\,dz = 0,$$

so that by (17.56),

$$P\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz = i\pi\mathscr{R}(0).$$

But the residue of $z^{-1}e^{iz}$ at $z=0$ is $\mathscr{R}(0)=1$; thus

$$\int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{1}{2i}\,P\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz = \tfrac{1}{2}\pi.$$

Example 24

Show that if $0<a<2$,

$$\int_{0}^{\infty}\frac{x^{a-1}}{1+x^2}\,dx = \tfrac{1}{2}\pi\operatorname{cosec}\tfrac{1}{2}\pi a.$$

Consider the integral

$$\oint_{C}\frac{z^{a-1}}{1+z^2}\,dz,$$

where C is the contour of fig. 17.18. For non-integral a, the integrand has a branch point at $z=0$; but provided we take the branch cut from $z=0$ to be in the lower half-plane, the integrand will be regular in C except at the pole $z=i$. The integral is therefore equal to $2\pi i\mathscr{R}(i)$, where

$$\mathscr{R}(i) = \left[\frac{z^{a-1}}{z+i}\right]_{z=i} = \tfrac{1}{2}i^{a-2}.$$

Since conditions (i), (ii) and (iii$_1$) of § 6.4 are satisfied since $a<2$, the integral round the large semicircle vanishes in the limit $R\to\infty$. The integral round the

small semicircle is

$$i\delta^a \int_{\theta=0}^{\pi} \frac{e^{ia\theta}}{1+\delta^2 e^{2i\theta}}\, d\theta,$$

which tends to zero as $\delta \to 0$, since $a > 0$. On the positive and negative real axes, we put $z = x$ and $z = xe^{i\pi}$ respectively, since the branch line is in the lower half-plane. Then these integrals sum to

$$\int_{x=\delta}^{R} \frac{x^{a-1}(1-e^{ia\pi})}{1+x^2}\, dx.$$

When $\delta \to 0$, $R \to \infty$, this integral is equal to $2\pi i \mathscr{R}(i) = \pi i^{a-1} = -i\pi e^{\frac{1}{2}i\pi a}$. Therefore

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}\, dx = \frac{i\pi}{e^{\frac{1}{2}i\pi a} - e^{-\frac{1}{2}i\pi a}} = \tfrac{1}{2}\pi \operatorname{cosec} \tfrac{1}{2}\pi a. \qquad \bigstar$$

EXERCISE 17.5

1. Use contour integration to prove that $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{(2n-1)!}{(n!)^2}\, 2\pi$, when n is a positive integer.

2. Prove that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \frac{2\pi}{b}[a-(a^2-b^2)^{\frac{1}{2}}], \qquad a > b > 0.$

3. By integrating $\exp(-z^2)$ round a rectangle with sides $x = -R$, $x = R$, $y = 0$, $y = a > 0$, show that

$$\int_0^\infty \exp(-x^2)\cos 2ax\, dx = \tfrac{1}{2}\pi^{\frac{1}{2}}\exp(-a^2).$$

4. Prove that $\displaystyle\int_0^\infty \frac{x}{\sinh x}\, dx = \tfrac{1}{4}\pi^2.$

5. Prove that if m, n are integers and $n > m > 0$,

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\, dx = \frac{\pi}{2n} \operatorname{cosec}\left(\frac{2m+1}{2n}\pi\right).$$

6. Prove that if a, b are real and positive,

$$\int_0^\infty \frac{x^3 \sin bx}{x^4 + 4a^4}\,\mathrm{d}x = \tfrac{1}{2}\pi \mathrm{e}^{-ab} \cos ab.$$

7. Prove that if a, b are real and positive,

$$\mathrm{P}\int_0^\infty \frac{\cos bx}{x^2 - a^2}\,\mathrm{d}x = -\frac{\pi}{2a} \sin ab, \qquad \mathrm{P}\int_0^\infty \frac{x \sin bx}{x^2 - a^2}\,\mathrm{d}x = \tfrac{1}{2}\pi \cos ab.$$

8. By integrating $\dfrac{\log(z + i)}{z^2 + 1}$ round the contour of fig. 17.14, show that

$$\int_0^\infty \frac{\log(1 + x^2)}{1 + x^2}\,\mathrm{d}x = \pi \log 2.$$

9. Prove that $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh^2 \pi x}\,\mathrm{d}x = \frac{a}{2\pi} \operatorname{cosec} \tfrac{1}{2}a, \qquad |a| < 2\pi.$

10. By integrating

$$\frac{2zr \log(1 - iz)}{[z^2(1 + r^2) + (1 - r)^2](1 + z^2)}$$

round the contour of fig. 17.18 show that when r is real,

$$\int_0^{\frac{1}{2}\pi} \frac{r\theta \sin 2\theta}{1 - 2r \cos 2\theta + r^2}\,\mathrm{d}\theta \begin{cases} = \tfrac{1}{4}\pi \log(1 + r) & \text{if} \quad |r| < 1, \\ = \tfrac{1}{4}\pi \log(1 + r^{-1}) & \text{if} \quad |r| > 1. \end{cases}$$

11. By integrating

$$\frac{\mathrm{e}^{imz} - 1}{(z^2 + 1) \sin z}$$

round a suitable contour, show that when m is an even positive integer,

$$\int_0^\infty \frac{\sin mx}{(x^2 + 1) \sin x}\,\mathrm{d}x = \frac{\pi(\mathrm{e}^m - 1)}{(\mathrm{e}^2 - 1)\mathrm{e}^{m-1}}.$$

12. By integrating $(\cot z)/(z - \zeta)$ round a rectangle in the z-plane with corners at $[\pm(N + \tfrac{1}{2})\pi i \pm R]$, show that

$$\cot \zeta = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{2\zeta}{\zeta^2 - n^2\pi^2}.$$

Find a series expansion for $\operatorname{cosec} \zeta$.

13. Prove that $\operatorname{cosec}^2 \zeta = \sum_{n=-\infty}^{\infty} (\zeta - n\pi)^{-2}$.

14. By integrating $(\log z)^2/(1+z^2)$ round a contour C consisting of two circles, radii δ and R, with centres at the origin, connected by straight lines just above and just below the positive real axis, show that

$$\int_0^\infty \frac{\log x}{1 + x^2}\,\mathrm{d}x = 0.$$

15. By integrating $(\log z)^3/(1 + z + z^2)$ round the contour C of Question 14, prove that

$$\int_0^\infty \frac{(\log x)^2}{1 + x + x^2}\,\mathrm{d}x = \frac{16\pi^3}{81\sqrt{3}}.$$

16. Prove that $\int_0^\infty x^{-\frac{1}{2}} \cos x\,\mathrm{d}x = \int_0^\infty x^{-\frac{1}{2}} \sin x\,\mathrm{d}x = (2\pi)^{\frac{1}{2}}$.

17. By using the contour C of Question 14, show that

$$\int_0^\infty \frac{x^{\frac{1}{2}} \log x}{(1 + x)^2}\,\mathrm{d}x = \pi, \qquad \int_0^\infty \frac{x^{\frac{1}{2}}}{(1 + x)^2}\,\mathrm{d}x = \tfrac{1}{2}\pi.$$

THE DIRAC δ-FUNCTION. FOURIER SERIES AND INTEGRALS

§ 1. The Dirac δ-function

The main results established in this chapter will be derived by using the properties of a mathematical object known as the *Dirac δ-function.* The δ-function is of very general interest, and is frequently encountered in highly abstract mathematics, in theoretical physics, and in practical subjects such as electrical engineering. The δ-function corresponds to the physical ideal of something concentrated into a point in space or an instant in time; for example, the δ-function can be used to represent a point mass or a point charge, an impulse given to a dynamical system, or a pulse produced by a sudden electrical discharge in some circuit. Suppose for example, we think of a given amount e of charge spread along a line; if we concentrate this charge into smaller and smaller regions, we shall ideally end up with a point charge e. If x is the coordinate measuring distance along the line, and $E(x)$ is the charge density, then

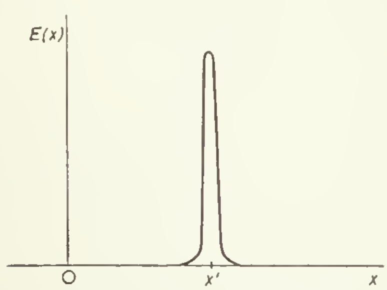

Fig. 18.1

$$\int_{-\infty}^{\infty} E(x)\,\mathrm{d}x = e. \qquad (18.1)$$

As the charge becomes concentrated about one point, say x', $E(x)\to 0$ except when $x=x'$, as shown in fig. 18.1. This peak supplies the main contribution to the integral (18.1). In the limit of a point charge, $E(x)=0$ except at x', but equation (18.1) still holds. This means that $E(x)$ has a sufficiently large infinity at x' for (18.1) to hold. This limit of $E(x)$ is called $e\delta(x-x')$, defining the δ-function. The δ-function is not a function in the usual mathematical sense since its value at $x=x'$ cannot be stated; it is more properly termed a *generalised function.* The equations defining the δ-function are

$$\delta(x-x') = 0 \qquad (x' \neq x) \qquad (18.2)$$

and from (18.1),

$$\int_{-\infty}^{\infty} \delta(x - x')\, \mathrm{d}x = 1. \tag{18.3}$$

We can generalise equation (18.3) to

$$\int_{a}^{b} \delta(x - x')\, \mathrm{d}x = 1 \tag{18.4}$$

where $a<x'<b$, since $\delta(x-x')=0$ outside the range (a, b).

Note that by writing $x-x'=y$, the δ-function can be defined by

$$\delta(y) = 0 \qquad (y \neq 0) \tag{18.5}$$

and

$$\int_{-\infty}^{\infty} \delta(y)\, \mathrm{d}y = \int_{a}^{b} \delta(y)\, \mathrm{d}y = 1 \qquad (a < 0 < b). \tag{18.6}$$

§ 1.1. Properties of the δ-function

The most useful property of the δ-function is that

$$\int_{a}^{b} g(x')\,\delta(x' - x)\, \mathrm{d}x' = \begin{cases} g(x) & \text{if} \quad a < x < b, \\ 0 & \text{if} \quad x < a \quad \text{or} \quad x > b. \end{cases} \tag{18.7}$$

for any function $g(x)$ which is continuous at x. The proof of (18.7) is simple: since the integrand is zero except when $x'=x$, we can replace $g(x')$ by $g(x)$ everywhere; this factor $g(x)$ can then be taken outside the integral over x', giving

$$g(x) \int_{a}^{b} \delta(x' - x)\, \mathrm{d}x'.$$

Using (18.2) and (18.4) with x and x' interchanged, (18.7) follows immediately.

Two other properties of the δ-function can be expressed formally by the equations

$$\delta(\lambda y) = \lambda^{-1}\delta(y) \qquad (\lambda > 0), \tag{18.8}$$

and

$$\delta(-y) = \delta(y). \tag{18.9}$$

To establish (18.8), we must show that the basic properties (18.5) and (18.6) are satisfied by $\lambda\delta(\lambda y)$. Clearly $\lambda\delta(\lambda y)=0$ when $y\neq 0$, using (18.5)

itself; further, putting $\lambda y = u$,

$$\int_{y=-\infty}^{\infty} \lambda\delta(\lambda y)\,\mathrm{d}y = \int_{u=-\infty}^{\infty} \delta(u)\,\mathrm{d}u = 1,$$

so that (18.6) holds. Equation (18.9) is also true, since if $y = -v$,

$$\int_{y=-\infty}^{\infty} \delta(-y)\,\mathrm{d}y = -\int_{v=\infty}^{-\infty} \delta(v)\,\mathrm{d}v = \int_{v=-\infty}^{\infty} \delta(v)\,\mathrm{d}v = 1.$$

§ 1.2. Representations of the δ-function

Equation (18.9) tells us that $\delta(y)$ is an *even function* of y. If we look upon $\delta(y)$ as the limit of a function $D(y)$ which is peaked about $y=0$, and split $D(y)$ into symmetric and anti-symmetric parts

$$D(y) = \tfrac{1}{2}[D(y) + D(-y)] + \tfrac{1}{2}[D(y) - D(-y)],$$

then the result (18.9) reflects the fact that as $D(y)$ tends to the limit $\delta(y)$, only the symmetric part $\frac{1}{2}[D(y)+D(-y)]$ contributes to the integral (18.6). It is therefore reasonable to restrict the choice of $D(y)$ to symmetrical functions.

So long as a function $D(y)$ satisfies (18.5) and (18.6) in some limit, it is said to represent the δ-function in that limit. One such function is

$$D_\beta(y) = \frac{\beta}{\pi(\beta^2 + y^2)}. \tag{18.10}$$

As $\beta \to 0$, $D_\beta(y) \to 0$ when $y \neq 0$, but $D_\beta(0) = (\pi\beta)^{-1} \to \infty$. Further,

$$\int_{-\infty}^{\infty} D_\beta(y)\,\mathrm{d}y = \frac{\beta}{\pi}\left[\frac{1}{\beta}\tan^{-1}\frac{y}{\beta}\right]_{-\infty}^{\infty} = 1;$$

so (18.5) and (18.6) are satisfied if we write

$$\delta(y) = \lim_{\beta\to 0} D_\beta(y). \tag{18.11}$$

Using the representation (18.10) and (18.11) of the δ-function, we can establish a very important property of the δ-function. First we show that

$$D_\beta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{iky}\,\mathrm{e}^{-\beta|k|}\,\mathrm{d}k; \tag{18.12}$$

the integral on the right is

$$\frac{1}{2\pi}\int_0^\infty e^{k(iy-\beta)}\,dk + \frac{1}{2\pi}\int_{-\infty}^0 e^{k(iy+\beta)}\,dk$$

$$= \frac{1}{2\pi}\left[\frac{e^{k(iy-\beta)}}{iy-\beta}\right]_0^\infty + \frac{1}{2\pi}\left[\frac{e^{k(iy+\beta)}}{iy+\beta}\right]_{-\infty}^0.$$

The factors $e^{-k\beta}$ and $e^{k\beta}$ ensure that the limits $k=\infty$ and $k=-\infty$ give no contribution, so the integral equals

$$\frac{1}{2\pi}\left[-\frac{1}{iy-\beta}+\frac{1}{iy+\beta}\right]=\frac{\beta}{\pi(\beta^2+y^2)};$$

comparing with (18.10), we see that (18.12) is true. If we let $\beta\to 0$ in (18.12), remembering (18.11), we find

$$\delta(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iky}\,dk. \tag{18.13}$$

Equation (18.13) is the *integral representation* of the δ-function. The integral in (18.13) is an integral to infinite limits of an oscillatory function e^{iky}, and so does not converge; the integral has meaning, however, as the limit of the convergent integral (18.12). The non-convergence of (18.13) reflects the fact that the δ-function is an 'improper function' which can be interpreted only when it appears under the integral sign, as in (18.3) and (18.7). In these circumstances equation (18.13) is a valid limit of (18.12), and we shall use (18.13) under the integral sign in the study of Fourier integrals.

★ Another representation of the δ-function which we shall use involves the limit as $\gamma\to 1$ of the function

$$D_\gamma(y)=\frac{1-\gamma^2}{2\pi(1-2\gamma\cos y+\gamma^2)}. \tag{18.14}$$

This is a function of $\cos y$, and so is periodic in y with period 2π; if we examine $D_\gamma(y)$ in the range $(-\pi,\pi)$, then the periodicity tells us how it behaves for all other values of y. In the range $(-\pi,\pi)$, if $y\neq 0$ then

$\cos y \neq 1$, and $D_\gamma(y) \to 0$ as $\gamma \to 1$; but

$$D_\gamma(0) = \frac{1+\gamma}{2\pi(1-\gamma)}$$

which becomes infinite as $\gamma \to 1$. So $D_\gamma(y)$ obeys condition (18.5) as $\gamma \to 1$. Further, it was shown in Ch. 5 Example 41 that

$$\int_{-\pi}^{\pi} D_\gamma(y)\,\mathrm{d}y = 1.$$

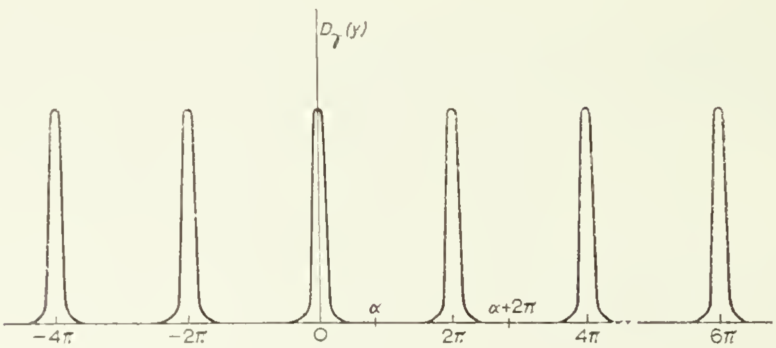

Fig. 18.2

independently of γ. So in the range $(-\pi, \pi)$, $\lim_{\gamma\to 1} D_\gamma(y)$ represents $\delta(y)$; since $D_\gamma(y)$ is periodic, however, it must give rise to a row of δ-functions at $y = \pm 2\pi$, $\pm 4\pi$, and so on, as shown in fig. 18.2. That is to say,

$$\lim_{\gamma\to 1} D_\gamma(y) = \sum_{n=-\infty}^{\infty} \delta(y - 2n\pi), \tag{18.15}$$

when y takes values in the range $(-\infty, \infty)$. For our purposes, it is important to note that in any range $(\alpha, \alpha+2\pi)$ of length 2π there is exactly one δ-function, as shown. ★

§ 2. General properties of Fourier series

A function $f(x)$ is said to be 'expanded as a Fourier series' when it is expressed as a series $S(x)$ of sine and cosine terms:

$$f(x) = S(x) \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{18.16}$$

a_n $(n=0, 1, 2,\ldots)$ and b_n $(n=1, 2,\ldots)$ being constants known as the *Fourier coefficients* of $f(x)$; the appearance of the factor $\frac{1}{2}$ in the constant term will be explained later. The most obvious property of the series $S(x)$ occurring in (18.16) is that each term is periodic of period 2π, so that the

series itself has this period. Suppose that a function $f(x)$ is given, and that equation (18.16) is satisfied for values of x in the range $(\alpha, \alpha+2\pi)$ where α is some constant; then the equation cannot be satisfied outside this range unless $f(x)$ is also periodic of period 2π. To make this point clear we have drawn in solid lines in fig. 18.3 a function $f(x)$ which is not periodic. We have also drawn in dotted lines the graph of the series $S(x)$ which coincides with $f(x)$ in the range $(\alpha, \alpha+2\pi)$. Since $S(x)$ is periodic, it is determined for all values of x by its value in the range $(\alpha, \alpha+2\pi)$. In other words, we can only hope to represent an arbitrary function $f(x)$ by a series of the form (18.16) over a range of length 2π. This does not

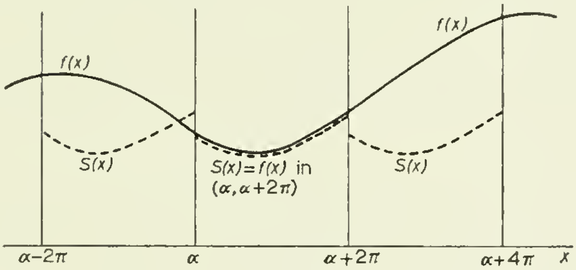

Fig. 18.3

of course prove that a function $f(x)$ *can* be represented by a convergent Fourier series in a range of length 2π; but before we tackle this problem, we shall find the coefficients a_n and b_n in the Fourier series on the assumption that equation (18.16) does hold over the range $(\alpha, \alpha+2\pi)$.

§ 2.1. Calculation of Fourier coefficients

In order to calculate a_n and b_n, we note first that if m and n are positive integers, and if δ_{mn} is the Kronecker delta defined by equation (12.17), then

$$\int_{\alpha}^{\alpha+2\pi} \cos(m-n)x \, \mathrm{d}x = 2\pi\delta_{mn},$$

$$\int_{\alpha}^{\alpha+2\pi} \cos(m+n)x \, \mathrm{d}x = 0;$$

adding and subtracting these equations gives

$$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, \mathrm{d}x = \pi\delta_{mn}, \tag{18.17}$$

$$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, \mathrm{d}x = \pi\delta_{mn}. \tag{18.18}$$

We can likewise show that

$$\int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx \, dx = 0, \tag{18.19}$$

and it is clear that

$$\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = 0. \tag{18.20}$$

Since we are assuming (18.16) to hold over the range $(\alpha, \alpha+2\pi)$, we can multiply this equation by $\cos mx$ $(m>0)$ and integrate over the range. By (18.19) and (18.20) the terms depending on a_0 and all the coefficients b_r vanish, and we have

$$\begin{aligned}\int_{\alpha}^{\alpha+2\pi} f(x) \cos mx \, dx &= \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \cos mx \, dx, \\ &= \pi \sum_{n=1}^{\infty} a_n \delta_{mn} = \pi a_m,\end{aligned}$$

using (18.17). So the cosine term coefficients are

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx. \tag{18.21}$$

Similarly, by multiplying equation (18.16) by $\sin mx$ $(m>0)$ and integrating over the range $(\alpha, \alpha+2\pi)$ we find that

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx. \tag{18.22}$$

The coefficient a_0 is given simply by integrating equation (18.16) over the range. Using (18.20) we have

$$\pi a_0 = \int_{\alpha}^{\alpha+2\pi} f(x) \, dx,$$

so that a_0 is also given by equation (18.21); this is the reason why the constant term in (18.16) is written as $\frac{1}{2}a_0$ instead of a_0.

§ 3. Fourier's theorem

We have shown that if equation (18.16) holds, then the Fourier coefficients must be given by (18.21) and (18.22). We shall now answer the

converse question: if a_n and b_n are defined by (18.21) and (18.22), is equation (18.16) satisfied in the range $(\alpha, \alpha+2\pi)$? Fourier's theorem states that, with a slight modification, this is true under fairly broad conditions which we shall state in § 3.2; for the present we shall prove the theorem for functions $f(x)$ which are continuous in the range $(\alpha, \alpha+2\pi)$ and which are not infinite at the end points of the range.

§ 3.1. Proof of Fourier's theorem

★ In proving Fourier's theorem, we shall take $\alpha=-\pi$, so that the range $(\alpha, \alpha+2\pi)$ is $(-\pi, \pi)$; this simplifies the notation, but does not affect the proof otherwise. Assuming (18.21) and (18.22), we shall establish (18.16) by summing the series $S(x)$ and proving it equal to $f(x)$; substituting from (18.21) and (18.22), the series in (18.16) becomes

$$S(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')[\tfrac{1}{2} + \sum_{n=1}^{\infty} (\cos nx' \cos nx + \sin nx' \sin nx)]\,\mathrm{d}x'$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')[\tfrac{1}{2} + \sum_{n=1}^{\infty} \cos n(x-x')]\,\mathrm{d}x'. \tag{18.23}$$

The series

$$\frac{1}{\pi}[\tfrac{1}{2} + \sum_{n=1}^{\infty} \cos ny],$$

appearing in (18.23) with $y=x-x'$, consists of an infinite number of terms whose values may lie anywhere between -1 and $+1$; so the series is not convergent. We therefore regard it as the limit as $\gamma\to 1-$ of the series

$$D_\gamma(y) = \frac{1}{\pi}[\tfrac{1}{2} + \sum_{n=1}^{\infty} \gamma^n \cos ny], \tag{18.24}$$

which is convergent for $\gamma<1$, being dominated by the series $\pi^{-1}\sum_{n=0}^{\infty}\gamma^n$. Thus (18.23) can be written

$$S(x) = \lim_{\gamma\to 1-} \int_{-\pi}^{\pi} f(x')\, D_\gamma(x-x')\,\mathrm{d}x'. \tag{18.25}$$

The series (18.24) can be summed by putting $\cos ny=\mathrm{Re}(\mathrm{e}^{iny})$; then

$\pi D_\gamma(y)$ is the real part of

$$\tfrac{1}{2} + \sum_{n=1}^{\infty} \gamma^n e^{iny} = \frac{1}{2} + \frac{\gamma e^{iy}}{1 - \gamma e^{iy}}$$

$$= \frac{1}{2} + \frac{\gamma e^{iy}(1 - \gamma e^{-iy})}{1 - 2\gamma \cos y + \gamma^2}.$$

Thus

$$D_\gamma(y) = \frac{1}{\pi}\left[\frac{1}{2} + \frac{\gamma \cos y - \gamma^2}{1 - 2\gamma \cos y + \gamma^2}\right]$$

$$= \frac{1 - \gamma^2}{2\pi(1 - 2\gamma \cos y + \gamma^2)}. \tag{18.26}$$

Hence $D_\gamma(y)$ is identical with the function defined by (18.14); its limit when $\gamma \to 1$ is given by (18.15) so that equation (18.25) becomes

$$S(x) = \int_{-\pi}^{\pi} f(x') \sum_{n=-\infty}^{\infty} \delta(x' - x - 2n\pi)\, dx'. \tag{18.27}$$

We have already noted that just one of the δ-functions in (18.15) lies in any range of length 2π. The range of x' in (18.27) is $(-\pi, \pi)$; if we want to evaluate $S(x)$ at a point in the range $-\pi < x < \pi$, only $\delta(x' - x)$ in the sum in (18.27) gives a non-zero contribution, the other δ-functions having their peaks at $x' = x + 2n\pi$ $(n \neq 0)$, outside the range $-\pi < x < \pi$. Hence for $-\pi < x < \pi$,

$$S(x) = \int_{-\pi}^{\pi} f(x')\, \delta(x' - x)\, dx' = f(x), \tag{18.28}$$

proving Fourier's theorem.

We know from its definition that $S(x)$ is periodic of period 2π. This also follows from (18.27): for if $(2l-1)\pi < x < (2l+1)\pi$, the only δ-function which contributes to (18.27) is then $\delta(x' - x - 2l\pi)$ and we find

$$S(x) = \int_{-\pi}^{\pi} f(x')\, \delta(x - x' - 2l\pi)\, dx' = f(x - 2l\pi) = S(x - 2l\pi).$$

This result is of course a direct reflection of the fact that $D_\gamma(y)$ and its limit (18.15) are periodic. ★

§ 3.2. End points and discontinuities

★ The result (18.28) follows from (18.27) whenever $-\pi<x<\pi$. We must be careful when $x=\pm\pi$, when (18.27) reduces to

$$S(\pm\pi) = \int_{-\pi}^{\pi} f(x') \sum_{n=-\infty}^{\infty} \delta[x' - (2n + 1)\pi] \,\mathrm{d}x'. \tag{18.29}$$

The peaks of the δ-functions are now at $x=(2n+1)\pi$, and two of these are at the end points of the range $(-\pi, \pi)$, as in fig. 18.4; so we must be careful when we evaluate their contributions to the integral (18.29).

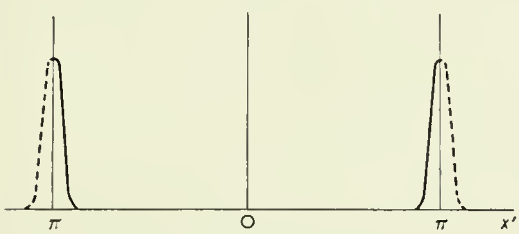

Fig. 18.4

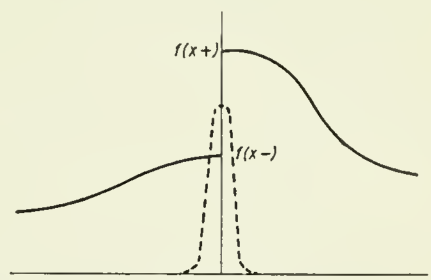

Fig. 18.5

Since the δ-function is an even function, half of each peak is inside the range $(-\pi, \pi)$, and half outside. Therefore these two δ-functions each behave like half a δ-function, and (18.29) becomes

$$S(\pm\pi) = \tfrac{1}{2}[f(\pi) + f(-\pi)]. \tag{18.30}$$

For the general range $(\alpha, \alpha+2\pi)$, equation (18.30) becomes

$$S(\alpha) = S(\alpha + 2\pi) = \tfrac{1}{2}[f(\alpha) + f(\alpha + 2\pi)]. \tag{18.31}$$

Reference to fig. 18.3 shows us that $S(x)$ is in general discontinuous at the end points of the range. Equation (18.31) tells us that at these discontinuities the series has value equal to the mean of its values just to the left and just to the right.

We have so far assumed that $f(x)$ is continuous in the range $(\alpha, \alpha+2\pi)$. Fourier series can however represent functions with a finite number of discontinuities if equation (18.16) is slightly modified. Suppose that $f(x)$ is discontinuous at some point x in the range, and that $f(x+)$ and $f(x-)$ are the limiting values as we approach x from above and from below, as in fig. 18.5. Then in evaluating the integral in (18.28), half the δ-function is multiplied by $f(x+)$ and half by $f(x-)$. Hence (18.28) is replaced by

$$S(x) = \tfrac{1}{2}[f(x+) + f(x-)]. \tag{18.32}$$ ★

So to include discontinuities of the function $f(x)$, we must modify (18.16) to

$$\tfrac{1}{2}[f(x+) + f(x-)] = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad (18.33)$$

At points where $f(x)$ is continuous, (18.33) is identical with (18.16). At points where $f(x)$ is discontinuous, the Fourier series gives the mean of the values to the right and left of the discontinuity; similarly, at the end points of the range $(\alpha, \alpha+2\pi)$, the series gives the mean value (18.32) of $f(x)$ at these points. Formula (18.33) can be shown to be valid when certain conditions, known as *Dirichlet's conditions*, are satisfied. These conditions are:

(i) $f(x)$ is infinite at only a finite number of points in the range $(\alpha, \alpha+2\pi)$.

(ii) The integral

$$\int_{\alpha}^{\alpha+2\pi} |f(x)| \, \mathrm{d}x$$

is convergent; that is, the integral of $f(x)$ is absolutely convergent. This condition ensures that all of the integrals (18.21) and (18.22) converge.

(iii) The interval $\alpha < x < \alpha+2\pi$ can be divided into a finite number of sub-intervals, in each of which $f(x)$ is monotonic. In other words, $f(x)$ oscillates only a finite number of times in the range.

Example 1

Expand the function $f(x) = x$ as a Fourier series (i) in the range $(0, 2\pi)$, (ii) in the range $(-\pi, \pi)$.

To evaluate the coefficients by (18.21) and (18.22), we need the integrals

$$\int x \cos nx \, \mathrm{d}x = \begin{cases} \dfrac{1}{n} x \sin nx + \dfrac{1}{n^2} \cos nx & (n \neq 0) \\ \tfrac{1}{2}x^2 & (n = 0); \end{cases}$$

$$\int x \sin nx \, \mathrm{d}x = -\frac{1}{n} x \cos nx + \frac{1}{n^2} \sin nx \qquad (n \neq 0).$$

(i) Putting in limits 0 and 2π we find

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx \, \mathrm{d}x = \begin{cases} 0 & (n \neq 0) \\ 2\pi & (n = 0); \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx \, \mathrm{d}x = -\frac{2}{n} \qquad (n \neq 0).$$

Hence the Fourier expansion of x in the range $(0, 2\pi)$ is

$$x = S(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

The (periodic) function that this series represents is drawn in fig. 18.6; when $x=0$ or $x=2\pi$, the series has value π, in agreement with equation (18.31). The values at these end points have been indicated by ⊙ in fig. 18.6.

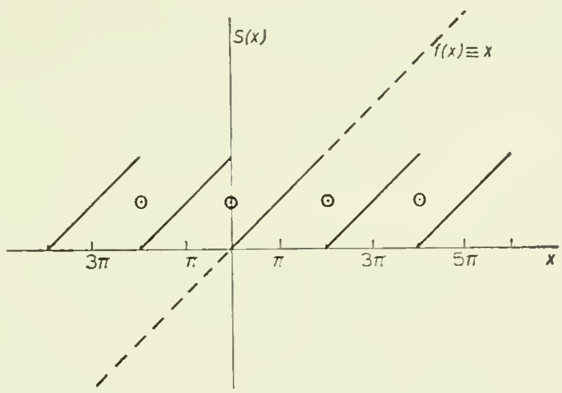

Fig. 18.6

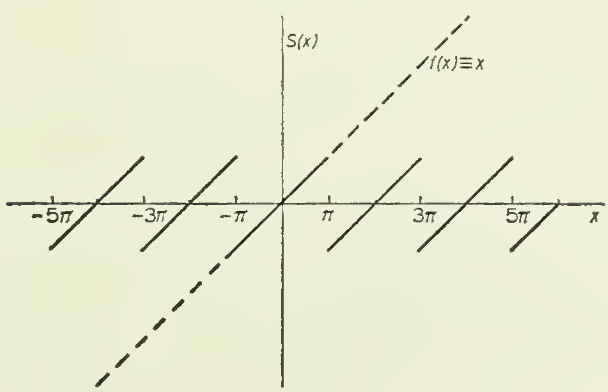

Fig. 18.7

It is of interest to note that putting $x=\frac{1}{2}\pi$ in this Fourier series gives a series for π:

$$\begin{aligned}\pi &= 4 \sum_{n=1}^{\infty} \frac{1}{n} \sin \tfrac{1}{2}n\pi \\ &= 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] = 8\left[\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \dots\right].\end{aligned}$$

This series is not rapidly convergent, the first four terms giving the very poor approximation $\pi=3.017\dots$. More rapidly convergent series give better results.

(ii) Putting in limits $-\pi$ and π, we find that x is represented in range $(-\pi, \pi)$ by the series

$$S(x) \equiv 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx$$

consisting only of sine terms. The function represented by this series is shown in fig. 18.7.

Example 2

Expand as a Fourier series in range $(0, 2\pi)$ the function $f(x)$ defined by

$$f(x) = 0 \quad \text{in} \quad (0, \pi),$$
$$f(x) = \pi \quad \text{in} \quad (\pi, 2\pi).$$

From (18.21) and (18.22),

$$a_n = \frac{1}{\pi} \int_{\pi}^{2\pi} \pi \cos nx \, dx = \begin{cases} 0 & (n \neq 0) \\ \pi & (n = 0), \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{\pi}^{2\pi} \pi \sin nx \, dx = -\frac{1}{n} [1 - (-1)^n] \qquad (n \neq 0).$$

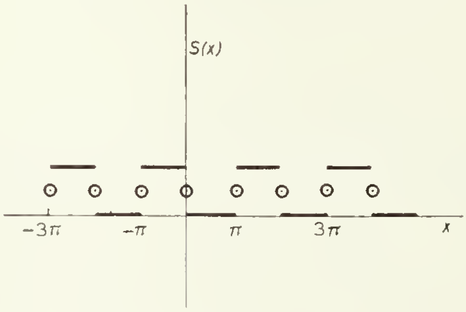

Fig. 18.8

Hence the Fourier series is

$$S(x) \equiv \tfrac{1}{2}\pi - \sum_{n=1}^{\infty} \frac{1}{n} [1 - (-1)^n] \sin nx$$

$$= \tfrac{1}{2}\pi - 2 \sum_{l=0}^{\infty} \frac{\sin(2l+1)x}{2l+1}.$$

At the end points $x=0$, 2π, and at the discontinuity $x=\pi$, we see that $S(x)=\frac{1}{2}\pi$, in agreement with (18.31) and (18.33). The function represented by $S(x)$ is shown in fig. 18.8.

EXERCISE 18.1

Find the Fourier series (i) in the range $(0, 2\pi)$, and (ii) in the range $(-\pi, \pi)$, of the following functions, a and b being constants, not necessarily integral:

1. x^2. **2.** e^{bx}. **3.** $\sin ax$.

4. $\cos ax$. **5.** $e^{bx} \cos ax$. **6.** $x^2 \sin ax$.

7. $f(x) = \begin{cases} x^2 & \text{for} \quad 0 < x < \frac{1}{2}\pi \\ 0 & \text{elsewhere.} \end{cases}$

Draw diagrams to show what functions these Fourier series represent in the range $(-\pi, 4\pi)$.

§ 4. Properties of Fourier series

§ 4.1. Sine and cosine series

In Example 1, we found that the Fourier series for x itself in the range $(-\pi, \pi)$ contained only sine terms. This is not surprising, for x is an odd function in the range $(-\pi, \pi)$, and it would be peculiar if an odd function were represented by a series containing even functions of the form $\cos nx$. It is easy to show that when $S(x)$ represents any odd function $f(x)$, for which

$$f(-x) = -f(x) \tag{18.34}$$

then all the cosine coefficients are zero. In fact, the formula (18.21) for the coefficient a_n, with $\alpha = -\pi$, gives

$$\begin{aligned} a_n &= \frac{1}{\pi} \Big[\int_{-\pi}^{0} f(x) \cos nx \,\mathrm{d}x + \int_{0}^{\pi} f(x) \cos nx \,\mathrm{d}x \Big] \\ &= \frac{1}{\pi} \Big[\int_{0}^{\pi} f(-y) \cos ny \,\mathrm{d}y + \int_{0}^{\pi} f(x) \cos nx \,\mathrm{d}x \Big], \end{aligned} \tag{18.35}$$

putting $x = -y$ in the first integral. Using (18.34), we see that all $a_n = 0$. So quite generally, odd functions of x are represented by Fourier series in the range $(-\pi, \pi)$ consisting only of sine terms, known as *sine series*. Likewise an even function $f(x)$, for which

$$f(-x) = f(x) \tag{18.36}$$

is represented by a series in the range $(-\pi, \pi)$ with all sine coefficients b_n zero, known as a *cosine series*. We note that a cosine series can contain a constant term $\frac{1}{2}a_0$, whereas a sine series cannot.

If one of the restrictions (18.34) and (18.36) is imposed on a function, we are no longer free to choose $f(x)$ in the whole range $(-\pi, \pi)$. In fact, if $f(x)$ is given in the half-range $(0, \pi)$, it is then determined in $(-\pi, 0)$ by (18.34) or (18.36). Thus a sine or cosine series can be chosen to represent any given function in the range $(0, \pi)$ only; it is therefore desirable to use only this range in the definition of the Fourier coefficients. The coefficients a_n in a cosine series are defined by (18.35); condition (18.36) immediately gives

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx, \tag{18.37}$$

which uses the values of $f(x)$ in the range $(0, \pi)$ only. In a very similar way we can show that the coefficients in a sine series are

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx. \tag{18.38}$$

Since sine and cosine series are particular types of Fourier series, their properties at discontinuities and end points are the same as for more general Fourier series.

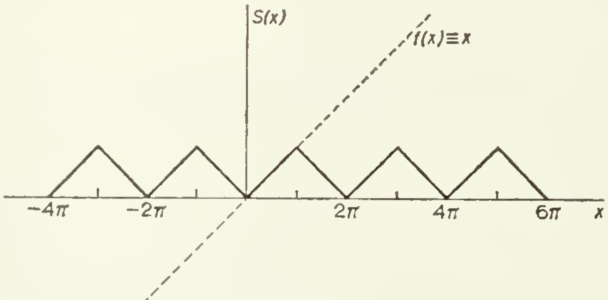

Fig. 18.9

Example 3

Expand $f(x)=x$ as (i) a cosine series, and (ii) a sine series in the range $(0, \pi)$.

Using integrals given in Example 1, the cosine coefficients (18.37) become

$$a_n = \frac{2}{\pi} \left[\frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx \right]_0^\pi = \frac{2}{\pi n^2} [(-1)^n - 1] \qquad (n \neq 0)$$

and $a_0 = \pi$. Thus the cosine series is

$$\tfrac{1}{2}\pi - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [1 - (-1)^n] \cos nx$$

$$= \tfrac{1}{2}\pi - \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{\cos(2l+1)x}{(2l+1)^2}.$$

The even function represented by this series is the 'saw-tooth' function as shown in fig. 18.9, and can be compared with the Fourier series in Example 1.

The sine series coefficients, given by (18.38), are $b_n = 2/n$; as expected, the series is identical with the Fourier series in the range $(-\pi, \pi)$, worked out in Example 1, and with graph given in fig. 18.7.

EXERCISE 18.2

Expand as sine and cosine series in $(0, \pi)$:

1. x^2.

2. $\sin ax$.

3. $e^{bx} \cos ax$.

4. $f(x) = \begin{cases} x^2 & \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}\pi \\ 0 & \text{elsewhere.} \end{cases}$

Draw diagrams to show what functions these series represent in the range $(-\pi, 4\pi)$.

§ 4.2. Differentiation of Fourier series

If a differentiable function $f(x)$ has a continuous first derivative $f'(x)$ in the range $(\alpha, \alpha + 2\pi)$, and a_n and b_n are the Fourier coefficients of $f(x)$ in this range, then the Fourier coefficients of $f'(x)$ bear a simple relation to a_n and b_n. Let us assume that

$$f'(x) = \tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx). \tag{18.39}$$

Then the coefficients a_n' are given by

$$a_n' = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f'(x) \cos nx \,\mathrm{d}x$$

$$= \frac{1}{\pi} \Big[f(x) \cos nx \Big]_{\alpha}^{\alpha+2\pi} + \frac{n}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \,\mathrm{d}x$$

$$= \frac{1}{\pi} [f(\alpha + 2\pi) - f(\alpha)] \cos n\alpha + nb_n, \tag{18.40}$$

using (18.38). Similarly

$$b_n' = \frac{1}{\pi}\,[f(\alpha + 2\pi) - f(\alpha)]\,\sin n\alpha - na_n. \tag{18.41}$$

We see that the Fourier series is not in general given by differentiating equation (18.16) term by term. This would give coefficients

$$a_n' = nb_n, \qquad b_n' = -na_n. \tag{18.42}$$

simply. The following example shows how wrong it would be to assume that (18.42) was true generally.

Example 4

Use the Fourier series for $f(x)=x$ in $(0, 2\pi)$ to check that the derivative of x is 1.

The Fourier series for x in $(0, 2\pi)$ evaluated in Example 1, has $a_n=0$ and $b_n=-2/n$ for $n\neq 0$. For this range, $\alpha=0$, so that

$$[f(\alpha + 2\pi) - f(\alpha)]\cos n\alpha = f(2\pi) - f(0) = 2\pi,$$

while $[f(\alpha+2\pi)-f(\alpha)]\sin n\alpha=0$ for all n. So (18.40) and (18.41) give

$$a_n' = \frac{1}{\pi}\,2\pi + n\left(-\frac{2}{n}\right) = 0 \qquad (n \neq 0)$$

$$a_0' = \frac{1}{\pi}\,2\pi = 2$$

and $b_n' = -na_n = 0$.

Thus the Fourier series for $f'(x)$ is simply $\frac{1}{2}a_0'=1$, as expected. Note however that the coefficients a_n' would be quite wrong if we merely differentiated the series for x term by term, equivalent to omitting $[f(\alpha+2\pi)-f(\alpha)]\cos n\alpha$ from (18.40); we would then have $a_0'=0$ and $a_n'=-2$, giving the series

$$-2\sum_{n=1}^{\infty}\cos nx,$$

which is certainly not equal to unity!

There are two important situations when term by term differentiation of a Fourier series is correct:

(i) If $f(\alpha)=f(\alpha+2\pi)$, then formulae (18.40) and (18.41) reduce to (18.42). A function obeying these conditions is known as a *fully periodic function.* For example $\sin\frac{1}{2}x$ is fully periodic in the range $(0, 2\pi)$, so that its Fourier series can be differentiated term by term.

(ii) If $f(x)$ is an even function, represented by a cosine series, then the coefficients a_n and b_n are given by (18.40) and (18.41) with $\alpha=-\pi$. Since

$f(x)$ is even, $f(\pi)=f(-\pi)$, so that equations (18.42) are once again valid. Since every $b_n=0$, every $a_n'=0$, and $f'(x)$ is represented by a sine series. In other words, the cosine series representing any function $f(x)$ in $(0, \pi)$ may be differentiated term by term, giving the sine series for $f'(x)$. For example, the Fourier series for cosh ax in $(-\pi, \pi)$ is a cosine series, and so can be differentiated term by term.

EXERCISE 18.3

By using the Fourier series expansions of the following functions in the range $(-\pi, \pi)$ find the Fourier series for their first derivatives, and show that the series for the derivatives are correct:

1. x^2. **2.** e^{bx}.

3. $\cosh bx$. **4.** $\sin \frac{1}{2}x$.

5. $\sin ax$.

§ 4.3. Fourier series expansions in the range $(-l, l)$

So far we have discussed the Fourier series representing a function in a range of length 2π. We can easily find the formula for a Fourier series representing a function in a range of length $2l$. Suppose the function $g(y)$ is defined in the range $(-l, l)$. Put $y=lx/\pi$; then if we define $f(x)=g(lx/\pi)$, $f(x)$ is given in the range $(-\pi, \pi)$ of x. So we can expand $f(x)$ by (18.16) as

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with a_n and b_n given by (18.21) and (18.22) with $\alpha=-\pi$. Hence $g(y) =g(lx/\pi)$ is represented in $(-l, l)$ by the Fourier series

$$g(y) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi y}{l} + b_n \sin \frac{n\pi y}{l}\right), \tag{18.43}$$

with

$$a_n = \frac{1}{l} \int_{-l}^{l} g(y) \cos \frac{n\pi y}{l}\, dy, \tag{18.44}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} g(y) \sin \frac{n\pi y}{l}\, dy. \tag{18.45}$$

At discontinuities, of course, $g(y)$ in (18.43) must be replaced by $\frac{1}{2}[g(y+)+g(y-)]$.

§ 5. The Fourier integral theorem

Equation (18.43) enables us to expand a function $g(y)$ defined over any finite interval. The coefficients $(n\pi/l)$ of y in the terms of this expansion are spaced out at intervals π/l, which is very small if the range $2l$ of the expansion is very large. If we let $l \to \infty$ in order to represent $g(y)$ over the infinite range $(-\infty, \infty)$, the spacing π/l between the coefficients tends to zero; in this limit, therefore the Fourier series summed over the points $n\pi/l$ becomes an integral over a continuous variable. This expansion of $g(y)$ as an integral over sines and cosines is the *Fourier integral* of $g(y)$. The proof of the Fourier integral theorem by letting $l \to \infty$ in (18.43) will be left as an exercise to the reader; we shall give a simpler proof depending on the integral representation (18.13) of the δ-function.

Putting $y = x' - x$ in (18.13) gives

$$\delta(x' - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x'-x)}\, dk. \tag{18.46}$$

Now suppose that a function $f(x)$ is defined in the range $(-\infty, \infty)$; then multiplying (18.46) by $f(x')$ and integrating over the whole range, we find using (18.7)

$$f(x) = \int_{-\infty}^{\infty} f(x')\,\delta(x' - x)\, dx' = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') \int_{-\infty}^{\infty} e^{ik(x'-x)}\, dk.$$

Assuming the order of integration over x' and k can be changed, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{-ikx}\, dk \tag{18.47}$$

where

$$F(k) = \int_{-\infty}^{\infty} f(x')\, e^{ikx'}\, dx'. \tag{18.48}$$

Equations (18.47) and (18.48) express the Fourier integral theorem in a form which exhibits the remarkable symmetry between $f(x)$ and the

function $F(k)$, which is called the *Fourier transform* of $f(x)$: we see at once that $2\pi f(-k)$ is the Fourier transform of $F(x)$. If we write $e^{ikx} = \cos kx + i \sin kx$, equation (18.47) becomes

$$f(x) = \int_0^\infty [a(k) \cos kx + b(k) \sin kx] \, dk \tag{18.49}$$

where

$$a(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \cos kx' \, dx' \tag{18.50}$$

$$b(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x') \sin kx' \, dx'. \tag{18.51}$$

If the function $f(x)$ is discontinuous at any point, then in (18.47) we must replace $f(x)$ by $\frac{1}{2}[f(x+)+f(x-)]$, as in (18.33). The Fourier integral theorem in this form can be shown to be true provided Dirichlet's conditions, stated in § 3.2, are satisfied, the range of integration in condition (ii) being $(-\infty, \infty)$ instead of $(\alpha, \alpha+2\pi)$.

If $f(x)$ is an odd function with $f(-x) = -f(x)$, then (18.50) gives $a(k) = 0$, so the Fourier integral (18.49) reduces to a sine integral. So if $f(x)$ is defined in the range $(0, \infty)$, we can express it as a sine integral

$$f(x) = \int_0^\infty b(k) \sin kx \, dx \tag{18.52}$$

by choosing $f(-x) = -f(x)$. Then (18.51) gives

$$b(k) = \frac{2}{\pi} \int_0^\infty f(x') \sin kx' \, dx'. \tag{18.53}$$

Here $b(k)$ is called the *sine transform* of $f(x)$. Formulae analogous to (18.52) and (18.53) hold for the cosine integral of $f(x)$ in $(0, \infty)$, defining the *cosine transform*.

In § 1.2 we have already evaluated two Fourier transforms. Equation (18.12) tells us that $e^{-\beta|k|}$ is the transform of $D_\beta(x) = \beta/\pi(\beta^2 + x^2)$, while (18.13) tells us that the transform of $\delta(x)$ is simply 1. Many Fourier transforms can be evaluated as complex integrals by calculus of residues, discussed in Ch. 17 § 6.4, in particular by applying Jordans' lemma.

Example 5

Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{for} \quad a \leqslant x \leqslant b \\ 0 & \text{elsewhere.} \end{cases}$$

From (18.48), the transform is

$$F(k) = \int_a^b e^{ikx'}\, dx' = \frac{e^{ikb} - e^{ika}}{ik}.$$

Example 6

Find the Fourier transform of $(a^2+x^2)^{-2}$, where $a>0$. It was shown in Ch. 17 Example 21, that

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{(a^2+x^2)^2}\, dx = \frac{\pi e^{-ka}(1+ka)}{2a^3}.$$

Comparing this equation with (18.48), we see that if $f(x)=(a^2+x^2)^{-2}$, its Fourier transform is

$$F(k) = \frac{\pi e^{-ka}(1+ka)}{2a^3}.$$

EXERCISE 18.4

1. By writing $k=n\pi/l$, $a(k)=la_n/\pi$ and $b(k)=lb_n/\pi$ in equations (18.43), (18.44) and (18.45), and letting $l\to\infty$, establish the Fourier integral theorem in the form (18.49), (18.50) and (18.51).

2. Find the Fourier, sine and cosine transforms of

(i) $\sin x/x$,

(ii) $(x^6 + a^6)^{-1}$,

(iii) $\exp(-x^2/a^2)$,

(iv) $f(x) = \begin{cases} e^{-ax} \quad (a>0) & \text{when} \quad x>0, \\ 0 & \text{when} \quad x<0. \end{cases}$

FACTORIAL, LEGENDRE AND BESSEL FUNCTIONS

§ 1. The factorial functions

§ 1.1. Definition of $u!$ for $u>-1$ and $u<-1$

The factorial $n!=n(n-1)\cdots 2\cdot 1$ of an integer n is well known; it can be expressed as an integral by considering the Laplace transform of x^n, evaluated in Ch. 16 § 4; putting $p=1$, we have

$$\int_0^\infty e^{-x}x^n \, dx = n!. \tag{19.1}$$

We can extend the idea of a factorial by defining the *factorial function*

$$u! = \int_0^\infty e^{-x}x^u \, dx \tag{19.2}$$

for all real values of u for which this integral converges. For all values of u, the factor e^{-x} ensures convergence of the integral at the infinite end of the range; near $x=0$ the integral behaves like $\int_0 x^u \, du$, which converges for $u>-1$. So for $u>-1$, $u!$ is defined by (19.2); this equation also defines the factorial function for complex values of u provided that the convergence condition $\mathrm{Re}(u)>-1$ is satisfied, but we shall only discuss the factorial function of a real variable. In older text-books $u!$ is written as $\Gamma(u+1)$, and is called the *gamma function.*

On integration by parts, the integral (19.2) gives

$$\left[\frac{e^{-x}x^{u+1}}{u+1}\right]_0^\infty + \int_0^\infty \frac{e^{-x}x^{u+1}}{u+1} \, dx.$$

For $u>-1$, the first term is zero, leaving simply $(u+1)!/(u+1)$. Hence the factorial function (19.2) satisfies the relation

$$(u+1)! = (u+1)(u!) \tag{19.3}$$

for non-integral as well as for integral values of u. From (19.3) it follows that for $u > -1$ and for any positive integer m,

$$(u+m)! = (u+m)(u+m-1)\cdots(u+2)(u+1)(u!). \qquad (19.4)$$

It is natural to extend the definition of $u!$ to the range $u < -1$ by assuming (19.4) to be true also in this range. More precisely, if $u > -(m+1)$ where m is a positive integer, then $(u+m) > -1$ and so $(u+m)!$ is defined by (19.2) with u replaced by $u+m$. Then equation (19.4) is taken as the definition of $u!$; it is clear that $u!$ is finite everywhere except when $u = -1, -2, -3, \ldots$, where it is infinite.

§ 1.2. Factorials of half odd integers

In § 7.2 we shall encounter factorials $(l-\frac{1}{2})!$ where l is integral. By using (19.4) these can all be expressed in terms of $(-\frac{1}{2})!$; now putting $u = -\frac{1}{2}$ and $x = r^2$ in (19.2), we find

$$(-\tfrac{1}{2})! = \int_0^\infty \mathrm{e}^{-x} x^{-\frac{1}{2}}\,\mathrm{d}x$$

$$= 2\int_0^\infty \mathrm{e}^{-r^2}\,\mathrm{d}r = \pi^{\frac{1}{2}}, \qquad (19.5)$$

using the result of Ch. 14 Example 15. From (19.4) and (19.5) we have for $l \geqslant 1$,

$$(l-\tfrac{1}{2})! = (l-\tfrac{1}{2})(l-\tfrac{3}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\pi^{\frac{1}{2}}; \qquad (19.6)$$

for $l \leqslant -1$,

$$(l-\tfrac{1}{2})! = \frac{\pi^{\frac{1}{2}}}{(l+\frac{1}{2})(l+\frac{3}{2})\cdots(-\frac{3}{2})(-\frac{1}{2})}. \qquad (19.7)$$

§ 1.3. The beta function

We shall now establish an important result involving the factorial function. If $u, v > -1$, then from (19.2),

$$u!\,v! = \lim_{A\to\infty} \int_0^A \mathrm{d}x \int_0^A \mathrm{d}y\; \mathrm{e}^{-(x+y)} x^u y^v. \qquad (19.8)$$

The integral in (19.8) is over the square in the positive quadrant shown in fig. 19.1. As $A \to \infty$, the factor $\exp -(x+y)$ in the integrand ensures the convergence of (19.8) as a double integral, since it is a negative ex-

ponential everywhere in the positive quadrant. We divide the region of integration into two (equal) triangles T_1 and T_2 by the line $x+y=A$. Then since $x+y \geqslant A$, $x \leqslant A$ and $y \leqslant A$ in T_2,

$$\int_{T_2} dx\, dy\, e^{-(x+y)} x^u y^v \leqslant e^{-A} \int_0^A dx \int_0^A dy\, x^u y^v = e^{-A} \frac{A^{u+v+2}}{(u+1)(v+1)},$$

which tends to zero as $A \to \infty$. Hence (19.8) reduces to

$$u!\, v! = \lim_{A\to\infty} \int_{T_1} dx\, dy\, e^{-(x+y)} x^u y^v. \tag{19.9}$$

Now let us change variables from x, y to z, t, where

$$x = zt, \qquad y = z(1-t). \tag{19.10}$$

If we allow z and t to vary independently over the ranges $0<t<1$, $0<z<A$, the point (x, y) ranges over the triangle T_1. The Jacobian of the transformation (19.10) is, by (14.25),

$$\frac{\partial(x, y)}{\partial(t, z)} = \begin{vmatrix} z & t \\ -z & 1-t \end{vmatrix} = z.$$

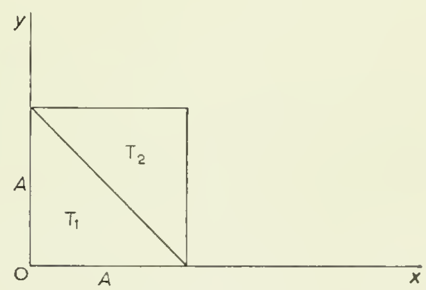

Fig. 19.1

So, by (14.26), equation (19.9) becomes

$$u!\, v! = \lim_{A\to\infty} \int_0^1 dt \int_0^A dz\, z e^{-z} \cdot z^{u+v}\, t^u (1-t)^v$$

$$= \lim_{A\to\infty} \int_0^A dz\, e^{-z}\, z^{u+v+1} \int_0^1 dt\, t^u (1-t)^v.$$

The z-integral here is, by (19.2), just $(u+v+1)!$. So finally we have

$$\frac{u!\, v!}{(u+v+1)!} = \int_0^1 dt\, t^u (1-t)^v. \tag{19.11}$$

The integral in (19.11) is of course symmetrical between u and v; it is known as the *beta function*, and is denoted by $B(u+1, v+1)$.

§ 1.4. Gauss's definition of $u!$

An important formula for $u!$ can be deduced from (19.11). If we choose v to be an integer n and write $t=p/n$, we find

$$\frac{u!\,n!}{(u+n+1)!} = \frac{1}{n^{u+1}} \int_0^n \mathrm{d}p\, p^u(1-p/n)^n. \tag{19.12}$$

If we now let $n\to\infty$, then using (4.34) the integral in (19.12) becomes

$$\int_0^\infty \mathrm{d}p\, p^u \mathrm{e}^{-p} = u!.$$

Hence if we replace $u+1$ by u, (19.12) gives

$$\lim_{n\to\infty} \frac{n!\,n^u}{(u+n)!} = 1 \tag{19.13}$$

for $u>0$. Using (19.4), the equation gives

$$u! = \lim_{n\to\infty} \frac{n!\,n^u}{(u+1)(u+2)\cdots(u+n)}, \tag{19.14}$$

defining $u!$ for $u>0$ as a limit containing only simple algebraic quantities. This limit exists and is finite for all real values of u except the negative integers, and we can regard (19.14) as a general definition of $u!$. We know that it coincides with our earlier definition (19.2) when $u>0$; it will coincide when $u<0$ also if the expression (19.14) satisfies (19.3), and hence (19.4): using (19.14),

$$\begin{aligned}(u+1)u! &= \lim_{n\to\infty} \frac{n!\,n^u}{(u+2)(u+3)\cdots(u+n)} \\ &= \lim_{n\to\infty} \frac{(n-1)!(n-1)^{u+1}}{[(u+1)+1][(u+1)+2]\cdots[(u+1)+(n-1)]},\end{aligned}$$

since $\lim_{n\to\infty}[n/(n-1)]^{u+1}=1$. Replacing $n-1$ by n, we see that this limit is just $(u+1)!$ as defined by (19.14), so that equation (19.4) is satisfied. Thus (19.14) defines $u!$ for all values of u; it is *Gauss's definition* of the factorial function.

§ 1.5. The digamma function

The derivative with respect to u of $\log u!$ is called the *digamma function*, and we denote it by $G(u)$; from (19.14),

$$G(u) = \frac{\mathrm{d}}{\mathrm{d}u}\log u!$$

$$= \lim_{n\to\infty}\left[\log n - \frac{1}{u+1} - \frac{1}{u+2} - \dots - \frac{1}{u+n}\right]. \qquad (19.15)$$

It follows at once that if m is an integer, $G(u)$ satisfies the relation

$$G(u+m) = G(u) + \frac{1}{u+1} + \frac{1}{u+2} + \dots + \frac{1}{u+m}. \qquad (19.16)$$

When $u=0$, we have

$$G(0) = \lim_{n\to\infty}[\log n - 1 - \tfrac{1}{2} - \dots - \tfrac{1}{n}] = -\gamma, \qquad (19.17)$$

say; it is not difficult to show that this limit exists and is finite (see Exercise 6.2, No. 15); the constant γ is known as *Euler's constant*, and is given approximately by $\gamma \approx 0.5772$. Putting $u=0$, $m=1$ in (19.16), we have

$$G(1) = G(0) + 1 \approx 0.4228.$$

It is clear that the function

$$\log n - \frac{1}{u+1} - \frac{1}{u+2} - \dots - \frac{1}{u+n}$$

is a strictly increasing function of u for $u>-1$; hence by (19.15) its limit $G(u)$ is an increasing function of u. So, since $G(1)>0$, $G(u)>0$ for $u>1$; from the definition of $G(u)$ it follows that $\log u!$, and hence $u!$, is an increasing function of u for $u \geqslant 1$; thus

$$u! \geqslant 1 \quad \text{for} \quad u \geqslant 1, \qquad (19.18)$$

giving a very crude lower bound for $u!$.

EXERCISE 19.1

1. Using Gauss's definition of $u!$, prove that, for any positive integer n,

$$u!\left(u-\frac{1}{n}\right)!\left(u-\frac{2}{n}\right)!\dots\left(u-\frac{n-1}{n}\right)! = (2\pi)^{\frac{1}{2}(n-1)}n^{-nu-\frac{1}{2}}(nu)!.$$

2. Show that if $n>1$,

$$\int_0^\infty y^{-n} \exp(-a^2/y^2)\,\mathrm{d}y = \frac{(\frac{1}{2}n-\frac{3}{2})!}{2a^{n-1}}.$$

3. By putting $t=x/(1+x)$ in (19.11), show that

$$u!(-u)! = \int_0^\infty \frac{x^u}{(1+x)^2}\,\mathrm{d}x.$$

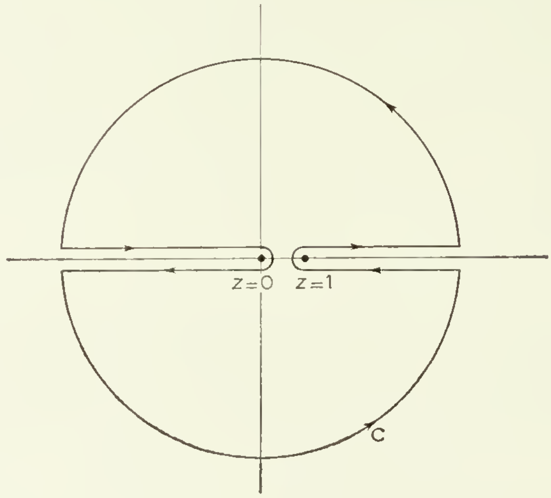

Fig. 19.2

By evaluating a complex integral round the contour shown in fig. 19.2, show that

$$u!(-u)! = \frac{\pi u}{\sin \pi u}.$$

§ 2. Laplace's equation in spherical polar coordinates

As we have mentioned in Ch. 15 § 6, the Laplace equation

$$\nabla^2\psi \equiv \operatorname{div}\operatorname{grad}\psi = 0 \tag{19.19}$$

is probably the most important partial differential equation in physics. It is therefore of interest to find solutions of this equation in various systems of orthogonal coordinates. We shall study the solutions of (19.19) in both cylindrical polar and spherical polar coordinates, which are of very great intrinsic importance; further, this work will exemplify the method generally used to find solutions of Laplace's equation and other linear partial differential equations in various coordinate systems.

§ 2.1. Separation of variables; Legendre's equation

The Laplacian in spherical polar coordinates is given by (15.63). Hence Laplace's equation (19.19) is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = 0. \tag{19.20}$$

This equation is homogeneous in r of order -2, which suggests that there will be solutions of the form

$$\psi = r^n S(\theta, \varphi). \tag{19.21}$$

Substituting (19.21) into (19.20) and performing the differentiations with respect to r, we find that $S(\theta, \varphi)$ satisfies

$$n(n+1)S + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2} = 0. \tag{19.22}$$

Note that this equation is unchanged if we replace n by $-(n+1)$; so if we find a solution of the form (19.21), we know that $r^{-(n+1)}S(\theta, \varphi)$ is also a solution. The standard method of finding solutions of linear equations such as (19.22) is to look for *separable solutions* of the form

$$S(\theta, \varphi) = -\Theta(\theta)\Phi(\varphi). \tag{19.23}$$

Substituting (19.23) into (19.22) gives

$$n(n+1)\Theta(\theta)\Phi(\varphi) + \frac{\Phi(\varphi)}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right] + \frac{\Theta(\theta)}{\sin^2\theta}\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} = 0,$$

the partial derivatives now being identical with ordinary derivatives. Multiplying by $\sin^2\theta/\Theta(\theta)\Phi(\varphi)$ and omitting the arguments of Θ and Φ, we have

$$\frac{\sin^2\theta}{\Theta}\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + n(n+1)\Theta\right] = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}. \tag{19.24}$$

Since Θ is a function of θ only, by hypothesis, the left-hand member of equation (19.24) is independent of φ. Likewise the right-hand member is independent of θ, so both sides must equal a constant, which we denote by m^2. Thus Φ satisfies

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -m^2,$$

the equation of simple harmonic motion, and hence is of the form

$$\Phi(\varphi) = A \cos(m\varphi + \epsilon). \tag{19.25}$$

Since φ is the azimuthal angle, we shall in most physical problems be looking for solutions of (19.19) which are unchanged when φ is increased by 2π. This means that we take m to be a real integer. Then $\Theta(\theta)$ will satisfy the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0. \tag{19.26}$$

This equation simplifies if we put

$$\cos\theta = \mu, \tag{19.27}$$

so that $\sin^2\theta = 1 - \mu^2$ and

$$-\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\mu};$$

making these substitutions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}\Theta}{\mathrm{d}\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0. \tag{19.28}$$

For any given values of n and the integer m, there will be two independent solutions of the second order equation (19.28). We shall only consider in detail the equation with $m=0$,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}\Theta}{\mathrm{d}\mu}\right] + n(n+1)\Theta = 0. \tag{19.29}$$

The corresponding solutions of Laplace's equation are then independent of φ, and thus have azimuthal symmetry. Equation (19.29) is *Legendre's equation*; for a given value of n, there are two independent solutions of (19.29), known as Legendre functions.

★ In general, equation (19.28) can be further simplified if we write

$$\Theta = (1-\mu^2)^{\frac{1}{2}m} f(\mu); \tag{19.30}$$

substituting in (19.28) and dividing by $(1-\mu^2)^{\frac{1}{2}m}$, we have

$$(1-\mu^2)\frac{\mathrm{d}^2 f}{\mathrm{d}\mu^2} - 2(m+1)\mu\frac{\mathrm{d}f}{\mathrm{d}\mu} + [n(n+1) - m(m+1)]f = 0. \tag{19.31}$$

Equation (19.28) is the *generalised Legendre equation*, and its two independent solutions denoted by $P_n^m(\mu)$ and $Q_n^m(\mu)$, are known as *generalised Legendre functions*. If we differentiate equation (19.29) m times, using Leibnitz' theorem we find immediately that $f(\mu)=\mathrm{d}^m\Theta/\mathrm{d}\mu^m$ satisfies (19.31). So using (19.30), we know that any solution $\Theta=P_n(\mu)$ of (19.29) gives us a solution

$$\Theta = (1-\mu^2)^{\frac{1}{2}m} \frac{\mathrm{d}^m P_n(\mu)}{\mathrm{d}\mu^m} \tag{19.32}$$

of the more general equation (19.28). ★

§ 3. Solutions of Legendre's equation

§ 3.1. Series solutions

We shall look for a series solution of (19.29) of the form

$$\Theta = \sum_{l=0} c_l \mu^l \tag{19.33}$$

the upper limit of the summation being unspecified. Substituting (19.33) into (19.29) and performing the differentiations, we find

$$\sum_{l=2} l(l-1)c_l\mu^{l-2} - \sum_{l=1} l(l+1)c_l\mu^l + n(n+1)\sum_{l=0} c_l\mu^l = 0.$$

In the first term we make the replacement $l \to l+2$, giving

$$\sum_{l=0} [(l+2)(l+1)c_{l+2} + \{n(n+1) - l(l+1)\}c_l]\mu^l = 0.$$

Thus the series (19.33) gives a solution provided the coefficients in the series are related by

$$c_{l+2} = -\frac{n(n+1)-l(l+1)}{(l+2)(l+1)}\, c_l; \tag{19.34}$$

the coefficients c_0 and c_1 can be chosen arbitrarily, and are the two arbitrary constants in the solution of the second order equation (19.29). If we choose $c_1=0$, then all odd coefficients $c_3, c_5, \ldots$ are zero by (19.34). The solution is then an even series, c_0 being the only arbitrary coefficient. Likewise the choice $c_0=0$ gives an odd series solution, with c_1 arbitrary.

§ 3.2. Legendre polynomials

For general values of n, none of the coefficients given by (19.34) are zero, and for large values of l this relation gives $c_{l+2} \approx c_l$. Hence both even and

odd series solutions will behave like the series

$$\mu^r \sum_s (\mu^2)^s = \frac{\mu^r}{1 - \mu^2}$$

which is infinite when $\cos\theta \equiv \mu = \pm 1$, which is along the z-axis. It sometimes happens that we want a solution which is infinite along this axis, but in most physical problems we are seeking solutions which are finite at all angles θ in the range $(0, \pi)$. We can only find such a solution if the series (19.33) terminates, one of the coefficients being zero. This will happen if n is integral, for then $c_{n+2}=0$ by (19.34). If n is even, the even series terminates, so we choose $c_1=0$; if n is odd, the odd series terminates, and we choose $c_0=0$. These polynomial solutions are known as *Legendre polynomials*.

In these polynomials the summation suffix l takes the values $0, 2, 4, \ldots, n$ for n even and $1, 3, 5, \ldots, n$ when n is odd. Both summations can be written in the same form if we write

$$l = n - 2k; \tag{19.35}$$

then, for n both even and odd, the above values of l (in the reverse order) are given by putting $k=0, 1, 2, \ldots, [\frac{1}{2}n]$, where $[\frac{1}{2}n]$ is the integral part of $\frac{1}{2}n$, equal to $\frac{1}{2}n$ when n is even and $\frac{1}{2}(n-1)$ when n is odd. The ratio of the terms labelled by $k-1$ and k is, from (19.34) and (19.35),

$$\frac{c_{l+2}}{c_l} = -\frac{2k(2n-2k+1)}{(n-2k+2)(n-2k+1)}. \tag{19.36}$$

It can easily be checked that the polynomial

$$P_n(\mu) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, \mu^{n-2k} \tag{19.37}$$

has successive terms in the ratio (19.36); the coefficients c_0 and c_1 in even and odd polynomials have been given certain values which, we shall see, simplify their properties. The functions $P_n(\mu)$ are the *Legendre polynomials*. The first six Legendre polynomials are

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu,$$

$$P_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1), \qquad P_3(\mu) = \tfrac{1}{2}(5\mu^3 - 3\mu),$$

$$P_4(\mu) = \tfrac{1}{8}(35\mu^4 - 30\mu^2 + 3), \qquad P_5(\mu) = \tfrac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu).$$

From (19.21), $r^n P_n(\mu)$ ($n=0, 1, 2, \ldots$) are solutions of Laplace's equation: $P_0(\mu)$ is the trivial constant solution; if (x, y, z) are rectangular coordinates with the z-axis along $\theta=0$, then $rP_1(\mu)=r\cos\theta=z$ and $r^2P_2(\mu)=\frac{1}{2}r^2(3\cos^2\theta-1)=\frac{1}{2}(2z^2-x^2-y^2)$ clearly satisfy Laplace's equation. Remembering that if $r^nS(\theta, \phi)$ is a solution of Laplace's equation, then so is $r^{-(n+1)}S(\theta, \phi)$, we know that $r^{-(n+1)}P_n(\mu)$ are also solutions. The solution $r^{-1}P_0(\mu)$ is simply r^{-1}, representing the potential due to a point charge or mass; in § 3.5 we shall see that the solutions $r^{-2}P_1(\mu)=r^{-2}\cos\theta$, $r^{-3}P_2(\mu)$, ..., represent the potentials due to dipoles and higher order multipoles.

§ 3.3. Second definition of Legendre polynomials

As in fig. 19.3, suppose that Q is a fixed point distant a from the origin O along the axis $\theta=0$. If R is the distance of an arbitrary field point P from

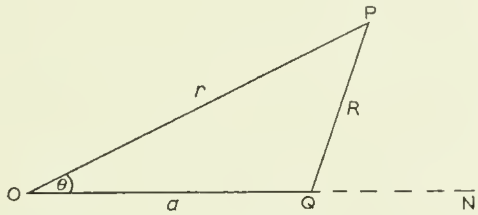

Fig. 19.3

Q then $\psi=R^{-1}$ is a solution of Laplace's equation. Expressed in terms of the spherical polar coordinates r and θ

$$\psi = \frac{1}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{1}{2}}} = \frac{1}{r(1 - 2h\mu + h^2)^{\frac{1}{2}}} \tag{19.38}$$

writing $\mu=\cos\theta$ and $h=a/r$. Provided h is small enough, we can expand the factor $(1-2h\mu+h^2)^{-\frac{1}{2}}$ in powers of h by the binomial theorem, the coefficient of h^n being a polynomial in μ. We call these polynomials $P_n(\mu)$, so that

$$(1 - 2h\mu + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(\mu). \tag{19.39}$$

We shall discuss the convergence of this series later; for the present we assume the expansion (19.39) valid for sufficiently small values of h. We shall now show that the coefficients $P_n(\mu)$ in (19.39) are identical with

the polynomials (19.37). Performing the expansion,

$$(1 - 2h\mu + h^2)^{-\frac{1}{2}} = [1 + h(h - 2\mu)]^{-\frac{1}{2}}$$

$$= \sum_{m=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2} - m)}{m!} h^m(h - 2\mu)^m. \tag{19.40}$$

Now $(h-2\mu)^m = \sum_{k=0}^{m} \binom{m}{k} h^k(-2\mu)^{m-k}$, so the coefficient of h^n in (19.40) is

$$\sum_{\substack{m+k=n \\ 0 \leqslant k \leqslant m}} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2} - m)}{m!} \frac{m!}{k!(m - k)!} (-2\mu)^{m-k};$$

putting $m=n-k$, this becomes

$$\sum_{k=0}^{[\frac{1}{2}n]} \left[(-\tfrac{1}{2})^{n-k} \frac{1 \cdot 3 \cdots (2n - 2k - 3)(2n - 2k - 1)}{k!(n - 2k)!} (-2)^{n-2k} \right] \mu^{n-2k}.$$

Simplifying, this gives for the coefficient of h^n in (19.39)

$$P_n(\mu) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n - 2k)!}{2^n\, k!(n - k)!\,(n - 2k)!} \mu^{n-2k}. \tag{19.41}$$

This is identical with the polynomials $P_n(\mu)$ defined by (19.37), so the equivalence of the two definitions of $P_n(\mu)$ is established.

§ 3.4. Rodrigues' formula

This series (19.41) can be expressed in the concise form

$$P_n(\mu) = \frac{1}{2^n\, n!} \frac{\mathrm{d}^n}{\mathrm{d}\mu^n} (\mu^2 - 1)^n, \tag{19.42}$$

known as *Rodrigues' formula*. The right-hand member of (19.42) is

$$\sum_{k=0}^{n} \frac{1}{2^n\, n!} \frac{\mathrm{d}^n}{\mathrm{d}\mu^n} \left[\binom{n}{k} \mu^{2(n-k)}(-1)^k \right]$$

$$= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{2^n\, k!(n - k)!} \cdot \frac{(2n - 2k)!}{(n - 2k)!} \mu^{n-2k},$$

the nth derivatives of $u^{2(n-k)}$ with $k > [\frac{1}{2}n]$ being zero. This proves Rodrigues' formula.

§ 3.5. Physical significance of Legendre polynomials

From (19.38) and (19.39), we have

$$\psi = \frac{1}{R} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n P_n(\cos\theta) \tag{19.43}$$

for small values of a. Differentiating equation (19.43) m times partially with respect to a, and then putting $a=0$, we find

$$\left[\frac{\partial^m}{\partial a^m}\left(\frac{1}{R}\right)\right]_{a=0} = \frac{m!\, P_m(\cos\theta)}{r^{m+1}}. \tag{19.44}$$

The right-hand member of (19.44) is $m!$ times the coefficient of a^m in (19.43), and is known to be a solution of the Laplace equation. The physical significance of the solution can be seen by examining the left-hand member of (19.44). Writing $R=R(a)$ to exhibit the dependence on a, the solution (19.44) with $m=0$ is $1/R(0)=r^{-1}$, the potential due to a unit charge or mass at O. For $m=1$, the left-hand member of (19.44) is

$$\lim_{a\to 0} \frac{1}{a}\left[\frac{1}{R(a)} - \frac{1}{R(0)}\right]$$

by the definition of a partial derivitive. Referring to fig. 19.2, we see that this solution is the potential due to poles of strength $\pm a^{-1}$ at Q and O, with $OQ=a\to 0$ in the limit. Thus $m=1$ gives a dipole potential.

When $m=2$, the right-hand member of (19.44) is the second derivative of R^{-1} at O; again returning to the definition of partial derivatives, this is

$$\lim_{a\to 0} \frac{1}{a^2}\left[\frac{1}{R(2a)} - \frac{2}{R(a)} + \frac{1}{R(0)}\right]$$

and is just the limit as $a\to 0$ of the potential produced by charges of magnitude a^{-2}, $-2a^{-2}$, a^{-2} at O, Q and N in fig. 19.3. Thus the potential (19.44) is called the *quadrupole potential.* In the same way (19.44) with $m=3$ gives the *octopole potential*; in general the potential (19.44) is called the 2^m-pole potential.

§ 4. Properties of Legendre polynomials

§ 4.1. The orthogonality property

One of the most important properties of Legendre polynomials is the *orthogonality property*, which is expressed as

$$\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, \mathrm{d}\mu = 0 \tag{19.45}$$

when $m \neq n$. This property is analogous to the orthogonality properties of sines and cosines expressed by the vanishing of the integrals (18.17) and (18.18) when $m \neq n$ and of the integrals (18.19). These equations enabled us to find the simple formulae (18.21) and (18.22) for the Fourier coefficients a_n and b_n; likewise the orthogonality property (19.45) allows us to expand functions of μ in terms of $P_n(\mu)$. In addition to proving (19.45) we shall also evaluate the integral when $m=n$. For definiteness we shall assume that $m>n$.

If we write $\mathrm{d}/\mathrm{d}\mu = D$, then Rodrigues' formula gives

$$\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, \mathrm{d}\mu = \frac{1}{2^{m+n}\, m!\, n!} \int_{-1}^{1} \mathrm{d}\mu \{D^m(\mu^2-1)^m\}\{D^n(\mu^2-1)^n\}. \tag{19.46}$$

Now by Leibnitz' theorem

$$\begin{aligned} D^t(\mu^2-1)^m &= D^t[(\mu-1)^m(\mu+1)^m] \\ &= \sum_{s=0}^{t} \binom{t}{s} [D^s(\mu-1)^m][D^{t-s}(\mu+1)^m]; \end{aligned}$$

if $t<m$, each term here contains at least one factor $\mu-1$ and one factor $\mu+1$; hence for $t<m$,

$$D^t(\mu^2-1)^m = 0 \quad \text{when} \quad \mu = \pm 1. \tag{19.47}$$

Integrating the integral in (19.46) by parts gives

$$\Big[\{D^{m-1}(\mu^2-1)^m\}\{D^n(\mu^2-1)^n\}\Big]_{-1}^{1} - \int_{-1}^{1} \mathrm{d}\mu \{D^{m-1}(\mu^2-1)^m\}\{D^{n+1}(\mu^2-1)^n\}.$$

The first term vanishes by (19.47), so we are left with an integral similar to that in (19.46). We can repeat the process of integrating by parts a

further $m-1$ times, the integrated term always vanishing at $\mu=\pm 1$ by (19.47). Carrying out the integrations, (19.46) becomes

$$\frac{(-1)^m}{2^{m+n}\, m!\, n!} \int_{-1}^{1} (\mu^2 - 1)^m \{D^{m+n}(\mu^2 - 1)^n\}\, \mathrm{d}\mu. \tag{19.48}$$

If $m>n$, $D^{m+n}(\mu^2-1)^n=0$, so the integral (19.46) is zero, proving (19.45).

If $m=n$, (19.48) is not zero, and we have

$$\int_{-1}^{1} [P_n(\mu)]^2\, \mathrm{d}\mu = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - \mu^2)^n \{D^{2n}(\mu^2 - 1)^n\}\, \mathrm{d}\mu. \tag{19.49}$$

The only non-zero term in $D^{2n}(\mu^2-1)^n$ is $D^{2n}(\mu^{2n})=2n!$, so (19.49) becomes

$$\frac{2n!}{2^{2n}(n!)^2} \int_{-1}^{1} (1 - \mu^2)^n\, \mathrm{d}\mu.$$

The integral here has been evaluated in Ch. 5 Example 17, and equals $2^{2n+1}(n!)^2/(2n+1)!$. So (19.49) becomes simply

$$\int_{-1}^{1} [P_n(\mu)]^2\, \mathrm{d}\mu = \frac{2}{2n + 1}.$$

This equation and (19.45) can be combined, using the Kronecker delta, into

$$\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, \mathrm{d}\mu = \frac{2\delta_{mn}}{2n + 1}. \tag{19.50}$$

for all m and n.

★ Solutions $P_n^m(\mu)$ of the generalised Legendre equation (19.28) are defined within a constant factor by (19.32), $P_n(\mu)$ being the nth Legendre polynomial. It can be shown that these functions satisfy the orthogonality relations

$$\int_{-1}^{1} P_n^m(\mu)\, P_l^m(\mu)\, \mathrm{d}\mu = 0 \qquad (n \neq l). \qquad (19.51) \bigstar$$

§ 4.2. Expansion of polynomials in terms of Legendre polynomials

If $f_n(\mu)$ is any nth order polynomial in μ, we can express it in terms of the first n Legendre polynomials,

$$f_n(\mu) = \sum_{l=0}^{n} a_l P_l(\mu); \tag{19.52}$$

this is possible because we can in principle solve successively for the constants $a_n, a_{n-1}, \ldots, a_1, a_0$, by equating in turn the coefficients of $\mu^n, \mu^{n-1}, \ldots, \mu, 1$. We can however find a formula for the a_l; multiply (19.52) by $P_m(\mu)$ and integrate over the range $-1 \leqslant \mu \leqslant 1$, giving

$$\int_{-1}^{1} P_m(\mu) f_n(\mu)\,\mathrm{d}\mu = \sum_{l=0}^{n} a_l \int_{-1}^{1} P_m(\mu) P_l(\mu)\,\mathrm{d}\mu$$

which by (19.50) equals $2a_m/(2m+1)$.

Hence the constants in (19.52) are given by

$$a_m = (m + \tfrac{1}{2}) \int_{-1}^{1} P_m(\mu) f_n(\mu)\,\mathrm{d}\mu. \tag{19.53}$$

The integral (19.53) involves terms of the form

$$\int_{-1}^{1} P_m(\mu) \mu^k\,\mathrm{d}\mu \quad \text{with} \quad k \leqslant n. \tag{19.54}$$

Using Rodrigues' formula, and integrating by part m times as in the proof of the orthonality property, we obtain

$$\frac{1}{2^m\, m!} \int_{-1}^{1} \{D^m(\mu^2 - 1)^m\} \mu^k\,\mathrm{d}\mu = \frac{(-1)^m}{2^m\, m!} \int_{-1}^{1} (\mu^2 - 1)^m\, D^m(\mu^k)\,\mathrm{d}\mu. \tag{19.55}$$

The integral (19.55) is clearly zero if $m > k$, and this tells us once more that the expansion of a kth degree polynomial involves only the polynomials $P_0(\mu), \ldots, P_k(\mu)$. For $m < k$, $D^m(\mu^k) = \mu^{k-m}\, k!/(k-m)!$, so that (19.55) becomes

$$\frac{k!}{2^m\,(k-m)!\; m!} \int_{-1}^{1} (1 - \mu^2)^m\, \mu^{k-m}\,\mathrm{d}\mu. \tag{19.56}$$

The integral in (19.56) has been evaluated in Ch. 5 Example 18. If $k - m$ is odd, the integral is zero; so an odd power μ^k is expanded in terms of

odd polynomials $P_m(\mu)$ only, and likewise for even powers. Taking the value for $k-m$ even from Ch. 5, Example 18, we find that the integral (19.54) is

$$\frac{2^m\, k!\, (\tfrac{1}{2}m + \tfrac{1}{2}k)!}{(\tfrac{1}{2}k - \tfrac{1}{2}m)!\, (m + k + 1)!} \tag{19.57}$$

for $m<k$ and $k-m$ even, and zero otherwise. Formulae (19.53) and (19.57) enable us to find the coefficients a_l in the expansion (19.52).

§ 4.3. Infinite series of Legendre polynomials, convergence

Provided that suitable convergence conditions are satisfied, we can let $n\to\infty$ in (19.52), and obtain an expansion of a function $f(\mu)$ as an infinite series in $P_n(\mu)$,

$$f(\mu) = \sum_{l=0}^{\infty} a_l\, P_l(\mu);$$

the coefficients are again given by (19.53), but this integral cannot in general be reduced to integrals of the type (19.54).

★ If we put $\mu=\pm 1$ in equation (19.39), we find

$$(1 \mp h)^{-1} = \sum_{n=0}^{\infty} h^n\, P_n(\pm 1).$$

Expanding $(1\mp h)^{-1}$ in powers of h by the binomial theorem and equating coefficients of h^n gives

$$P_n(1) = 1 \quad \text{and} \quad P_n(-1) = (-1)^n \tag{19.58}$$

for all n. Putting $\mu=\cos\theta$, the left-hand member of (19.39) can be written

$$(1 - he^{i\theta})^{-\frac{1}{2}}\, (1 - he^{-i\theta})^{-\frac{1}{2}} = \left[\sum_{r=0}^{\infty} \alpha_r (he^{i\theta})^r\right]\left[\sum_{s=0}^{\infty} \alpha_s (he^{-i\theta})^s\right], \tag{19.59}$$

where the coefficients $\alpha_r = 1\cdot 3\cdots(2r-1)/2^{r-1}r!$ are all positive. The coefficient of h^n in (19.59) is $P_n(\cos\theta)$, which therefore equals

$$\sum_{\substack{r+s=n\\ r,s\geqslant 0}} \alpha_r\alpha_s e^{i\theta(r-s)} = \sum_{\substack{r+s=n\\ r,s\geqslant 0}} \alpha_r\alpha_s \cos(r-s)\theta.$$

Since all the terms in this summation have positive coefficients, $P_n(\cos\theta)$ has its maxima for θ in $(0,\pi)$ when $\theta=0$ or π, since then the terms all have the same sign and each take their maximum value. We therefore

know that, for all μ in $(-1, 1)$,

$$|P_n(\mu)| \leqslant |P_n(\pm 1)| = 1. \tag{19.60}$$

A further consequence of equation (19.60) is that the series in (19.39) is convergent for $h<1$, by comparison with the geometric progression $\sum_{n=0}^{\infty} h^n$. Thus ψ, given by (19.38), has the expansion

$$\psi = \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos\theta) \quad \text{for} \quad a < r. \tag{19.61}$$

Since ψ is originally symmetrical between r and a, we can likewise expand ψ when $a>r$ by interchanging a and r; thus

$$\psi = \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(\cos\theta) \quad \text{for} \quad a > r. \tag{19.62}$$

Equations (19.58) tell us that neither of the expansions (19.61) and (19.62) is convergent when $a=r$ and $\theta=0$ or π. ★

§ 5. Recurrence relations for Legendre polynomials

Several relations between Legendre polynomials of different order and their derivatives known as *recurrence relations*, are of great practical importance. If we differentiate (19.39) partially with respect to h we obtain

$$\frac{\mu - h}{(1 - 2h\mu + h^2)^{\frac{3}{2}}} = \sum_{n=1}^{\infty} nh^{n-1}P_n(\mu)$$

or, using (19.39) itself,

$$(\mu - h)\sum_{n=0}^{\infty} h^n P_n(\mu) = (1 - 2h\mu + h^2)\sum_{n=1}^{\infty} nh^{n-1}P_n(\mu).$$

This equation is true for all h in the range $0\leqslant h<1$, so we can equate coefficients of h^n to obtain the first recurrence relation

$$(n+1)P_{n+1}(\mu) - (2n+1)\mu P_n(\mu) + nP_{n-1}(\mu) = 0. \tag{19.63}$$

If we differentiate (19.39) partially with respect to μ, writing $\mathrm{d}P_n(\mu)/\mathrm{d}\mu = P_n'(\mu)$, we can show similarly that

$$(\mu - h)\sum_{n=0}^{\infty} h^n P_n'(\mu) = h\sum_{n=0}^{\infty} nh^{n-1}P_n(\mu).$$

Equating coefficients of h^n gives the second recurrence relation

$$\mu P_n'(\mu) - P_{n-1}'(\mu) = nP_n(\mu). \tag{19.64}$$

Three other useful relations can be deduced from (19.63) and (19.64) and their derivatives. Differentiating (19.63) and using equation (19.64) gives

$$P_{n+1}'(\mu) - P_{n-1}'(\mu) = (2n+1)P_n(\mu). \tag{19.65}$$

Subtracting (19.64) from this equation gives

$$P_{n+1}'(\mu) - \mu P_n'(\mu) = (n+1)P_n(\mu). \tag{19.66}$$

Then multiplying (19.64) by μ and subtracting (19.66) with n replaced by $n-1$ leads to the fifth relation

$$(\mu^2 - 1)P_n'(\mu) = n\mu P_n(\mu) - nP_{n-1}(\mu). \tag{19.67}$$

★ By differentiating these relations we can obtain relations between the generalised Legendre functions $P_n^m(\mu)$. One of the simplest of these relations is

$$P_{n+1}^m(\mu) = \mu P_n^m(\mu) - \frac{1-\mu^2}{n+1}\frac{\mathrm{d}P_n^m(\mu)}{\mathrm{d}\mu}. \tag{19.68}$$

Another important relation is a generalisation of (19.63):

$$(n-m+2)P_{n+2}^m(\mu) - (2n+3)\mu P_{n+1}^m(\mu) + (n+m+1)P_n^m(\mu) = 0. \quad (19.69) ★$$

EXERCISE 19.2

1. Establish the following formulae for the second derivatives of the Legendre polynomials P_n:

$$P_n''(\mu) = \tfrac{1}{2}\sum_{l=0}^{\frac{1}{2}n-1}(4l+1)(n-2l)(n+2l+1)P_{2l}(\mu) \qquad (n \text{ even}),$$

$$P_n''(\mu) = \tfrac{1}{2}\sum_{l=0}^{\frac{1}{2}(n-3)}(4l+3)(n-2l-1)(n+2l+2)P_{2l+1}(\mu) \qquad (n \text{ odd}).$$

2. Show that $\dfrac{\mathrm{d}}{\mathrm{d}\mu}\{(1-\mu^2)P_n(\mu)P_n'(\mu)\} + n(n+1)[P_n(\mu)]^2 = (1-\mu^2)[P_n'(\mu)]^2$

Hence show that $\int_\alpha^\beta (1-\mu^2)[P_n'(\mu)]^2\,\mathrm{d}\mu = n(n+1)\int_\alpha^\beta [P_n(\mu)]^2\,\mathrm{d}\mu$ where α and β are each equal to 0, 1 or -1.

3. Use the binomial expansion of $(1+\alpha u^2)^{-n-\frac{1}{2}}$ and equation (19.57) to show that when $-1 < \alpha < 1$,

$$\int_{-1}^{1}\frac{P_{2n}(\mu)}{(1+\alpha\mu^2)^{n+\frac{3}{2}}}\,\mathrm{d}\mu = \frac{2(-\alpha)^n}{(2n+1)(1+\alpha)^{n+\frac{1}{2}}}.$$

4. Show that when $|m-n| \neq 1$,

$$\int_{-1}^{1} \mathrm{d}\mu\, P_n(\mu) P_m'(\mu) = 0 \quad \text{and} \quad \int_{-1}^{1} \mathrm{d}\mu\, \mu(1-\mu^2) P_n'(\mu) P_m'(\mu) = 0.$$

Determine the values of the integrals when $m=n+1$ and when $m=n-1$.

5. Show that $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}^r P_m(\mu)}{\mathrm{d}\mu^r} \frac{\mathrm{d}^r P_n(\mu)}{\mathrm{d}\mu^r} (1-\mu^2)^r \,\mathrm{d}\mu = \frac{2(n+r)!\,\delta_{mn}}{(2n+1)(n-r)!}.$

6. Prove that

$$\mu^2 \frac{\mathrm{d}^2 P_n(\mu)}{\mathrm{d}\mu^2} = n(n-1)P_n(\mu) + \sum_r (2n-4r+1)[r(2n-2r-1)-2]P_{n-2r}(\mu),$$

the summation over r being from $r=1$ to $r=\frac{1}{2}n$ when n is even, and from $r=1$ to $r=\frac{1}{2}(n-1)$ when n is odd.

7. Use Rodrigues' formula to show that $P_n(\mu)$ can be expressed as the integral

$$\frac{1}{2^{n+1}\pi i} \oint_C \frac{(z^2-1)^n}{(z-\mu)^{n+1}}\,\mathrm{d}z$$

in the z-plane, provided C is a simple closed contour encircling the point $z=\mu$ once anti-clockwise.

By choosing the contour C to be a circle of radius $|(\mu^2-1)^{\frac{1}{2}}|$, show that

$$P_n(\mu) = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta [\mu + (\mu^2-1)^{\frac{1}{2}} \cos\theta]^n.$$

§ 6. Laplace's equation in cylindrical polar coordinates

The Laplace operator ∇^2 in cylindrical polar coordinates (ρ, φ, z) is given by (15.62), and so the Laplace equation is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

As with the equation in spherical polar coordinates, we look for separable solutions of the form

$$\psi = R(\rho)\,\Phi(\varphi)\,Z(z). \tag{19.70}$$

It follows that

$$\frac{1}{\rho R(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2 \Phi(\varphi)}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = -\frac{1}{Z(z)}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -k^2, \tag{19.71}$$

where k^2 is a constant, so that $Z(z)$ is of the form

$$Z(z) = A\mathrm{e}^{kz} + B\mathrm{e}^{-kz}; \tag{19.72}$$

k may be real, imaginary or even complex, depending on the physical problem to be solved. From (19.72)

$$\frac{\rho}{R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + k^2\rho^2 = -\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = u^2, \tag{19.73}$$

where u^2 is again constant. Thus $\Phi(\varphi)$ is of the form

$$\Phi(\varphi) = \cos(u\varphi + \alpha); \tag{19.74}$$

normally we want $\Phi(\varphi)$ to be periodic in φ with period 2π, so that u must equal some integer n. However, solutions with u equal to half an odd integer are also of considerable importance, and have a specially simple form. If u is real, then for definiteness we take $u>0$.

If we put $x=k\rho$ and put $R(\rho)\equiv R(x/k)=g(x)$, then using (19.72), (19.73) and (19.74), the solution (19.70) becomes

$$\psi = g(x)(A\mathrm{e}^{kz} + B\mathrm{e}^{-kz})\cos(u\varphi + \alpha), \tag{19.75}$$

where $g(x)$ satisfies

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right) + (x^2 - u^2)g(x) = 0. \tag{19.76}$$

Equation (19.76) is *Bessel's equation.*

§ 6.1. Series solutions of Bessel's equation

Let us look for a series solution of (19.76) putting

$$g(x) = x^\lambda \sum_{l=0}^{\infty} c_l x^l. \tag{19.77}$$

Substituting (19.77) in (19.76) and changing the summation index in the term x^2g, we find

$$\sum_{l=0}^{\infty} [(l+\lambda)^2 - u^2]c_l x^{l+\lambda} + \sum_{l=2}^{\infty} c_{l-2}x^{l+\lambda} = 0.$$

This equation will be satisfied if the coefficient of every power of x is

zero. Equating the coefficients of $x^{l+\lambda}$ to zero, we have for $l \geqslant 2$,

$$c_l = -\frac{c_{l-2}}{(l+\lambda)^2 - u^2} = -\frac{c_{l-2}}{(l+\lambda+u)(l+\lambda-u)}, \tag{19.78}$$

and for $l=0$, 1,

$$(\lambda^2 - u^2)c_0 = [(\lambda+1)^2 - u^2]c_1 = 0. \tag{19.79}$$

This equation tells us that λ may be chosen so that either c_0 or c_1 is non-zero. If we put $c_1=0$, then we can take $c_0 \neq 0$ provided that we choose $\lambda = \pm u$. Then by (19.78), all the odd coefficients $c_3, c_5, \ldots$, are zero in general, while the even coefficients $c_2, c_4, \ldots$, are determined successively in terms of c_0.

One solution is given by putting $\lambda = +u$; using $r=\frac{1}{2}l$ as summation suffix instead of l, (19.78) gives for this solution

$$c_{2r} = \frac{(-1)^r c_0}{2^{2r}(r+u)(r+u-1)\cdots(u+1)\cdot r(r-1)\cdots 2\cdot 1}.$$

If we choose $c_0 = (\frac{1}{2})^u/u!$, where $u!$ is the factorial function defined in § 1, then

$$c_{2r} = \frac{(-1)^r(\frac{1}{2})^{u+2r}}{(r+u)!\, r!};$$

then the solution (19.77) with $\lambda=+u$, $l=2r$ takes the form $g(x)=J_u(x)$, where

$$J_u(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{u+2r}}{(r+u)!\, r!}, \tag{19.80}$$

known as the *Bessel function of the first kind*, of order u. When u is an integer n, $(r+n)!$ in (19.80) is just the ordinary factorial. When $u=n+\frac{1}{2}$, n being an integer, then $(r+n+\frac{1}{2})!$ is given by equation (19.7) with $l=r+n+1$.

When u is not an integer, the solution $\lambda=-u$, $c_1=0$ of (19.79) likewise gives a solution $g(x)=J_{-u}(x)$ of (19.76), where

$$J_{-u}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{-u+2r}}{(r-u)!\, r!}. \tag{19.81}$$

When u is an integer n, and we choose $\lambda=-n$, then from (19.78) with $l=2r$,

$$c_{2r} = \frac{-c_{2r-2}}{4r(r-n)},$$

and clearly c_{2n} will be infinite unless $c_{2n-2}=0$. So all the coefficients $c_{2n-2}, c_{2n-4}, \ldots, c_2, c_0$ must be chosen to be zero; then the first non-zero coefficient in the series (19.77) will be c_{2n}, and the first non-zero term will be $c_{2n}x^n$; the solution (19.81) therefore becomes

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{-n+2r}}{(r-n)!\,r!}\,. \tag{19.82}$$

We note that (19.81) is still formally correct when $u=n$, since $(r-n)!$ is infinite when $r-n$ is a negative integer. If we replace r by $r-n$ in (19.82), and compare with (19.80), we see at once that

$$J_{-n}(x) = (-1)^n J_n(x). \tag{19.83}$$

For non-integral u, (19.80) and (19.81) define two independent solutions of the second-order equation (19.76), and every solution is therefore of the form $AJ_u(x)+BJ_{-u}(x)$. But when u is an integer n, (19.80) and (19.81) define only one independent solution, and a further solution of (19.76) is needed.

The series solutions (19.80) and (19.81) are absolutely convergent for all values of x and u, apart from the obvious divergence of (19.81) when $x=0$. For by (19.18), the terms in these series with $r>-u+1$ are in modulus less than those of the series

$$(\tfrac{1}{2}|x|)^{\pm u} \sum_{r=1}^{\infty} (\tfrac{1}{2}|x|)^{2r}/r!,$$

which converges to $(\frac{1}{2}|x|)^{\pm u} \exp(\frac{1}{4}x^2)$.

The values of the Bessel functions $J_{\pm u}(x)$ for various values of x and u, in particular when u is half an integer, can be found from published tables of Bessel functions.

§ 6.2. Bessel functions of the second kind

★ For non-integral values of u, the function

$$Y_u(x) = \frac{J_u(x)\cos \pi u - J_{-u}(x)}{\sin \pi u} \tag{19.84}$$

is a solution of Bessel's equation, since it is a linear combination of the solutions $J_{\pm u}(x)$. As $u \to n$, when n is an integer, both the numerator and

the denominator in (19.84) vanish, by (19.83). In the limit, however,

$$Y_n(x) = \lim_{u \to n} \frac{J_u(x) \cos \pi u - J_{-u}(x)}{\sin \pi u}$$

is still a solution of (19.76), provided this limit exists. The limit does in fact exist, and using l'Hospital's rule, it is

$$Y_u(x) = \frac{(-1)^n}{\pi} \lim_{u \to n} \frac{\partial}{\partial u} [J_u(x) \cos \pi u - J_{-u}(x)]. \tag{19.85}$$

From (19.80) and (19.81) we see that the evaluation of (19.85) involves first derivatives of factorial function, which are given in terms of the digamma function G by (19.15). We shall not evaluate (19.85) in detail, but simply quote the result

$$Y_n(x) = -\frac{1}{\pi} \sum_{r=0}^{n-1} \frac{(\frac{1}{2}x)^{2r-n}(n-r-1)!}{r!}$$

$$+ \frac{2}{\pi} J_u(x) \log(\tfrac{1}{2}x) - \frac{1}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{2r+n}}{r!(u+r)!} [G(r) + G(u+r)]. \tag{19.86}$$

The detailed evaluation of $Y_n(x)$ is given for example, on pp. 577–8 of JEFFREYS and JEFFREYS [1950]. The function $Y_n(x)$ is called a Bessel function of the second kind, and provides a second solution to Bessel's equation when $u=n$; thus any solution to Bessel's equation is expressible as a linear combination of $J_u(x)$ and $Y_u(x)$ for all values of u. Values of $Y_u(x)$ can be found from tables of Bessel functions. ★

§ 6.3. BESSEL FUNCTIONS OF IMAGINARY ARGUMENT

★ We have noted that the constant k in (19.71), and hence $x=k\rho$, may be imaginary; putting $x=iy$ in (19.76) gives

$$y \frac{\mathrm{d}}{\mathrm{d}y}\left(y \frac{\mathrm{d}R}{\mathrm{d}y}\right) - (y^2 + u^2)R = 0$$

which has solutions

$$I_{\pm u}(y) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}y)^{2r \pm u}}{r!(r \pm u)!} \tag{19.87}$$

which are constant multiples of (19.80) and (19.81) with $x = iy$. The functions $I_{\pm u}(y)$ are called *Bessel functions of imaginary argument*, and are tabulated along with $J_{\pm u}(y)$ and $Y_u(x)$. ★

§ 7. Properties of the Bessel functions $J_u(x)$

§ 7.1. Recurrence relations

The Bessel functions $J_u(x)$ satisfy several important recurrence relations, and we shall now establish these. Using the series (19.80) we have

$$\tfrac{1}{2}x[J_{u-1}(x) + J_{u+1}(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{u+2r}}{(r+u-1)!\,r!} + \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{u+2r+2}}{(r+u+1)!\,r!}.$$

Replacing the summation suffix r by $r-1$ in the second term, this becomes

$$\frac{(\frac{1}{2}x)^u}{(u-1)!} + \sum_{r=1}^{\infty} (-1)^r(\tfrac{1}{2}x)^{u+2r}\left[\frac{1}{(r+u-1)!\,r!} - \frac{1}{(r+u)!\,(r-1)!}\right]$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{u+2r}u}{(r+u)!\,r!},$$

since $[(r+u-1)!]^{-1}=(r+u)/(r+u)!$ by (19.3) and $[(r-1)!]^{-1}=r/r!$. This series is simply $uJ_u(x)$ by (19.80), so that the first recurrence relation

$$\tfrac{1}{2}x[J_{u-1}(x) + J_{u+1}(x)] = uJ_u(x) \tag{19.88}$$

is established. Similarly,

$$\tfrac{1}{2}[J_{u-1}(x) - J_{u+1}(x)]$$
$$= \frac{\frac{1}{2}(\frac{1}{2}x)^{u-1}}{(u-1)!} + \sum_{r=1}^{\infty} (-1)^r\,\tfrac{1}{2}(\tfrac{1}{2}x)^{u+2r-1}\left[\frac{1}{(r+u-1)!\,r!} + \frac{1}{(r+u)!\,(r-1)!}\right]$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,\frac{1}{2}(\frac{1}{2}x)^{u+2r-1}(u+2r)}{(r+u)!\,r!}.$$

From (19.80) this series is just $\mathrm{d}J_u(x)/\mathrm{d}x = J_u'(x)$; thus

$$\tfrac{1}{2}[J_{u-1}(x) - J_{u+1}(x)] = J_u'(x). \tag{19.89}$$

Eliminating in turn $J_{u+1}(x)$ and $J_{u-1}(x)$ from (19.88) and (19.89) gives two further recurrence relations

$$J_u'(x) = J_{u-1}(x) - ux^{-1}J_u(x) \tag{19.90}$$

and

$$J_u'(x) = ux^{-1}J_u(x) - J_{u+1}(x). \tag{19.91}$$

The relations (19.88), (19.89), (19.90) and (19.91) hold for all values of u, but are most frequently used when u is an integer n. Because of the identity (19.83), the relations for negative integers n are identical with

those for positive n. For $u=n=0$, however, (19.91) simplifies and becomes

$$J_0'(x) = -J_1(x). \tag{19.92}$$

The function $Y_u(x)$ defined by (19.84) is a linear combination of $J_u(x)$ and $J_{-u}(x)$. Using the recurrence relations for $J_{\pm u}$, it is easy to check that Y_u satisfies these same relations for non-integral values of u. $Y_n(x)$, the limit as $u \to n$ of (19.84), must therefore also satisfy the same recurrence relations.

Another important recurrence relation concerns the function

$$\frac{J_u(x)}{x^u} \equiv \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2})^{u+2r} (x^2)^r}{(r+u)!\, r!}. \tag{19.93}$$

If we regard this as a function of x^2 and differentiate l times with respect to x^2, we obtain

$$\frac{\mathrm{d}^l}{\mathrm{d}(x^2)^l}\left\{\frac{J_u(x)}{x^u}\right\} = \sum_{r=l}^{\infty} \frac{(-1)^r (\frac{1}{2})^{u+2r} (x^2)^{r-l}}{(r+u)!\,(r-l)!}$$

$$= \left(-\frac{1}{2}\right)^l \sum_{s=0}^{\infty} \frac{(-1)^s (\frac{1}{2})^{u+l+2s} (x^2)^s}{(s+l+u)!\, s!}$$

changing the summation suffix to $s=r-l$. Comparing with (19.93),

$$\frac{\mathrm{d}^l}{\mathrm{d}(x^2)^l}\left\{\frac{J_u(x)}{x^u}\right\} = \left(-\frac{1}{2}\right)^l \frac{J_{u+l}(x)}{x^{u+l}}. \qquad (19.94$$

§ 7.2. Bessel functions of half-odd-integral order

The relation (19.94) is particularly useful for evaluating Bessel functions of positive half-odd-integral order. Using (19.80) and (19.6) with $l=r+1$,

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{\frac{1}{2}}\, x^{2r+\frac{1}{2}}}{2^{2r+1}(r+\frac{1}{2})!\, r!}$$

$$= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!},$$

or

$$J_{\frac{1}{2}}(x) = (2/\pi x)^{\frac{1}{2}} \sin x. \tag{19.95}$$

We can likewise show that

$$J_{-\frac{1}{2}}(x) = (2/\pi x)^{\frac{1}{2}} \cos x. \tag{19.96}$$

Hence if l is an integer, (19.94) and (19.95) give

$$J_{l+\frac{1}{2}}(x) = (-2)^l x^{l+\frac{1}{2}} \frac{\mathrm{d}^l}{\mathrm{d}(x^2)^l} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin x}{x}\right];$$

since

$$\frac{\mathrm{d}}{\mathrm{d}(x^2)} = \left[\frac{\mathrm{d}(x^2)}{\mathrm{d}x}\right]^{-1} \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{2x} \frac{\mathrm{d}}{\mathrm{d}x}$$

this equation can be written

$$J_{l+\frac{1}{2}}(x) = (-1)^l (2/\pi)^{\frac{1}{2}} x^{l+\frac{1}{2}} \left[\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sin x}{x}\right]. \tag{19.97}$$

So Bessel functions of positive half-odd-integral order can be expressed in terms of x, $\sin x$, and $\cos x$. For example,

$$J_{\frac{3}{2}}(x) = -(2/\pi)^{\frac{1}{2}} x^{-\frac{3}{2}} (x \cos x - \sin x)$$

and

$$J_{\frac{5}{2}}(x) = -(2/\pi)^{\frac{1}{2}} x^{-\frac{5}{2}} [(x^2 - 3) \sin x + 3x \cos x].$$

For Bessel functions of an imaginary argument, we can obtain $I_{l+\frac{1}{2}}(x)$, apart from a constant factor, by replacing x by ix in (19.97); the function is given by

$$I_{l+\frac{1}{2}}(x) = (2/\pi)^{\frac{1}{2}} x^{l+\frac{1}{2}} \left[\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sinh x}{x}\right]. \tag{19.98}$$

It can also be shown (see Exercise 19.3, No. 3) that for $l \geqslant 0$,

$$Y_{l+\frac{1}{2}}(x) = (-1)^l J_{-l-\frac{1}{2}}(x) = (-1)^{l+1} (2/\pi)^{\frac{1}{2}} x^{l+\frac{1}{2}} \left[\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\cos x}{x}\right]. \tag{19.99}$$

It is quite common to eliminate a factor $x^{\frac{1}{2}}$ from $J_{l+\frac{1}{2}}(x)$ when $l \geqslant 0$, by defining the *spherical Bessel function*

$$\begin{aligned} j_l(x) &= (\pi/2x)^{\frac{1}{2}} J_{l+\frac{1}{2}}(x) \\ &= (-1)^l x^l \left[\left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sin x}{x}\right]. \end{aligned} \tag{19.100}$$

Spherical Bessel functions are well-behaved near $x=0$. In fact, we can

write (19.100) as

$$j_l(x) = (-1)^l x^l \left\{ 2^l \frac{\mathrm{d}^l}{\mathrm{d}(x^2)^l} \sum_{n=0}^{\infty} \frac{(-x^2)^n}{(2n+1)!} \right\}.$$

Near $x=0$ this give the approximate behaviour

$$j_l(x) \approx (-1)^l x^l \, 2^l \frac{\mathrm{d}^l}{\mathrm{d}(x^2)^l} \left[\frac{(-x^2)^l}{(2l+1)!} \right]$$

$$= x^l \frac{2^l \, l!}{(2l+1)!} = \frac{x^l}{1 \cdot 3 \cdot 5 \cdots (2l+1)}. \tag{19.101}$$

The following properties of $j_l(x)$ are quite easily established from the known properties of $J_{l+\frac{1}{2}}(x)$:

$$\frac{2l+1}{x} j_l(x) = j_{l+1}(x) + j_{l-1}(x) \qquad (l > 0), \tag{19.102}$$

$$(2l+1) \frac{\mathrm{d}j_l(x)}{\mathrm{d}x} = l j_{l-1}(x) - (l+1) j_{l+1}(x), \tag{19.103}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [x^{l+1} j_l(x)] = x^{l+1} j_{l-1}(x), \tag{19.104}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [x^{-l} j_l(x)] = -x^{-l} j_{l+1}(x). \tag{19.105}$$

The serious reader is advised to check these formulae.

One can likewise define the *spherical Neumann function*

$$n_l(x) = (-1)^{l+1} (\pi/2x)^{\frac{1}{2}} J_{-l-\frac{1}{2}}(x) \tag{19.106}$$

eliminating the square root from Bessel functions of negative half-odd-integral order. These functions are infinite at $x=0$, and it is easy to show that for small x

$$n_l(x) \approx \frac{1 \cdot 3 \cdot 5 \cdots (2l-1)}{x^{l+1}}. \tag{19.107}$$

The functions $n_l(x)$ satisfy exactly the same relations (19.102)–(19.106) as the spherical Bessel functions.

This account of the definitions and properties of Bessel functions is only intended as an introduction to the subject. A fuller understanding of the

properties and inter-relations of Bessel functions can be gained by expressing them as integrals over a complex variable; this approach is beyond the scope of this book, but is treated fully in more advanced texts.

EXERCISE 19.3

1. Show that the second derivative of $J_u(x)$ is given by

$$4J''_u(x) = J_{u-2}(x) - 2J_u(x) + J_{u+2}(x).$$

Prove by induction that the sth derivative of $J_u(x)$ is given by

$$2^s J_u^{(s)}(x) = J_{u-s} - sJ_{u-s+2} + \binom{s}{2} J_{u-s+4} + \cdots + (-1)^{s-1} s J_{u+s-2} + (-1)^s J_{u+s}.$$

2. Prove that

$$\exp \tfrac{1}{2}x(t - t^{-1}) \equiv \sum_{-\infty}^{\infty} t^n J_n(x),$$

identically in x and t. Putting $t = e^{i\theta}$, show that

$$\cos(x \cos \theta) = J_0(x) + 2 \sum_{l=1}^{\infty} (-1)^l J_{2l}(x) \cos 2l\theta$$

and

$$\sin(x \cos \theta) = 2 \sum_{l=0}^{\infty} (-1)^l J_{2l+1}(x) \cos(2l+1)\theta.$$

By treating these series as cosine expansions of functions of θ, show that

$$J_n(x) = \frac{1}{\pi} \int_0^\pi d\theta \cos(n\theta - x \sin \theta).$$

3. Show that if $x^u J_u(x)$ is regarded as a function of x^2, then

$$\frac{d^l}{d(x^2)^l} \{x^u J_u(x)\} = (\tfrac{1}{2})^l x^{u-l} J_{u-l}(x).$$

Using (19.96), show that

$$J_{-l-\frac{1}{2}} = (2/\pi)^{\frac{1}{2}} x^{l+\frac{1}{2}} \left[\left(\frac{1}{x} \frac{d}{dx} \right)^l \frac{\cos x}{x} \right],$$

and hence establish the formula (19.99) for $Y_{l+\frac{1}{2}}(x)$.

STATISTICS AND PROBABILITY

§ 1. Statistical distributions

In many walks of life, it is useful to classify measurements and other observable numbers in order to reveal 'patterns of behaviour'. For example, insurance companies find it economically necessary to study the expectation of life of various classes of people; professional gamblers find it profitable to know what are the chances of occurrence of particular combinations of cards and dice; experts on transport problems base their suggestions partly on observation of flow of traffic; production in any industry must be based upon a knowledge of the public demand for goods, which may depend upon the economic situation, the time of year, and many other factors; experimental scientists use the analysis of results of a series of similar experiments not only to test or suggest scientific theories, but also to reveal deficiencies in their experimental techniques; and so on.

When a series of measurements is made on a set of objects, the set of objects is called a *population*; a population may, for instance, be a certain group of human beings or animals, the cars produced by a factory in a given year, the set of results achieved by a succession of throws of dice or by dealing successive hands of cards, or by a series of experiments on similar physical or chemical systems. If a single quantity is measured for each member of a population, this quantity is called a *variate*; variates will be denoted by symbols such as x or y. The possible observed values of a variate x will be a finite set $x_1, x_2, \ldots, x_n$; for instance, the possible results of a throw of a die are 1, 2,..., 6. Even when the possible values of a variable are continuous, as in measuring the heights of men, there will be a limit to the accuracy of the measurements, so that the number of possible values is finite. In measuring heights, we might perhaps measure to the nearest millimetre; then the possible values of the variate would be taken at intervals of a millimetre.

In a given population, the number of times a particular value x_r of a variate x is observed is called the *frequency* of x_r, and is denoted by f_r. The *total frequency* is the sum of the individual frequencies, and therefore equals the number N of members of the population:

$$\sum_{r=1}^{n} f_r = N. \tag{20.1}$$

The set of values (f_r) is called the *frequency distribution* or simply the *distribution* of the variate x.

§ 1.1. The mean of a distribution

The most elementary statistical quantity defined by a distribution is the *arithmetic mean* $\bar{x}$ of the variate x, usually known as the *mean*; this is just the sum of the values of x for each member of the population, divided by N. Thus

$$\bar{x} = \frac{1}{N} \sum_{r=1}^{n} f_r x_r. \tag{20.2}$$

If x and y are two variates, it is easy to see that the mean $\overline{x+y}$ of the sum of x and y is the sum of their means:

$$\overline{x+y} = \bar{x} + \bar{y}. \tag{20.3}$$

Suppose that a is any constant, and we define

$$\xi_r = x_r - a \qquad (r = 1, 2, \ldots, n). \tag{20.4}$$

For every member of the population with $x = x_r$, the value of $\xi \equiv x - a$ is ξ_r; so the variate ξ has a distribution (f_r) over the values (ξ_r) given by (20.4). The mean $\bar{\xi}$ of ξ for this distribution is

$$\bar{\xi} = \frac{1}{N} \sum_{r=1}^{n} f_r \xi_r \tag{20.5}$$

and is simply related to $\bar{x}$: using (20.2), (20.4), (20.5) and (20.1), we have

$$\bar{x} = \frac{1}{N} \sum_{r=1}^{n} f_r \xi_r + \frac{1}{N} \sum_{r=1}^{n} f_r a \quad \text{or} \quad \bar{x} = \bar{\xi} + a. \tag{20.6}$$

Combining this result with (20.4), we have

$$x_r - \bar{x} = \xi_r - \bar{\xi} \tag{20.7}$$

for $r = 1, 2, \ldots, n$.

Equation (20.6) helps us to calculate means quickly, provided we choose a to be a simple number close to the mean.

Example 1

The lengths of pieces of wire cut by a machine vary slightly. The lengths x_r to the nearest millimetre and the frequencies f_r of these lengths for a population of 60 pieces are given in the top two rows of Table 20.1.

TABLE 20.1

x_r (cm)	30.6	30.7	30.8	30.9	31.0	31.1	31.2	31.3	31.4	31.5	31.6	31.7	31.8
f_r	2	0	4	2	5	8	9	11	7	6	1	4	1
ξ_r	−0.4	−0.3	−0.2	−0.1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8

In (20.4), choose $a = 31.0$ cm. Then the values of ξ_r are given in the third row of Table 20.1. From (20.5),

$$N\bar{\xi} = \sum_r f_r\xi_r = 2(-0.4) + 4(-0.2) + 2(-0.1) + 8(0.1)$$
$$+ 9(0.2) + 11(0.3) + 7(0.4) + 6(0.5) + 0.6 + 4(0.7) + 0.8 = 12.1.$$

Since $N = \sum_r f_r = 60$, $\bar{\xi} = 0.20$ cm. So from (20.6),

$$\bar{x} = 0.2 \text{ cm} + 31 \text{ cm} = 31.2 \text{ cm}.$$

A second guide to the magnitude of a variate x is the *mode*, which is that value x_r whose frequency f_r is greatest. For instance, the mode in Example 1 is 31.3 cm, whose frequency (11) is greatest. This definition of the mode may not be unique; we must then adopt some rule, such as averaging those values x_r whose frequency is greatest, in order to define a unique mode.

§ 1.2. Second moments and standard deviation

It is clearly of interest to know whether or not a distribution is closely grouped about its mean. The most useful quantity for assessing the spread of a distribution is the *standard deviation* σ. But before we define σ, we must define the *second moment* μ_2' of a distribution about a given value a of the variate: μ_2' is the average over the population of the *squares* of the differences (20.4) between a and the values x_r of the variate:

$$\mu_2' = \frac{1}{N}\sum_{r=1}^{n} f_r\xi_r^2. \tag{20.8}$$

Clearly if most of the measured values of x for the population are near to a, μ_2' will be small, while if $x-a$ is large for most of the population, μ_2 will be large, since all terms in the sum (20.8) are positive. The second moment about the mean $\bar{x}$ of the distribution is called the *variance*, and is denoted by μ_2:

$$\mu_2 = \frac{1}{N} \sum_{r=1}^{n} f_r(x_r - \bar{x})^2. \tag{20.9}$$

The variance gives us an estimate of the spread of values of x about their mean, but is not directly comparable with $\bar{x}$, since it is of the dimension of x^2. The positive square root of μ_2 is the *standard deviation* σ; since it is of the same dimension as x, σ gives us an estimate of the spread of values of x which can be compared directly with $\bar{x}$. If ξ_r are defined by (20.4), then by (20.7), the definition (20.9) becomes

$$\sigma^2 \equiv \mu_2 = \frac{1}{N} \sum_{r=1}^{n} f_r(\xi_r - \bar{\xi})^2. \tag{20.10}$$

Expanding and using (20.8), (20.2) and (20.1),

$$\begin{aligned}\sigma^2 &= \frac{1}{N}\Big[\sum_{r=1}^{n} f_r\xi_r^2 - 2\bar{\xi}\sum_{r=1}^{n} f_r\xi_r + \bar{\xi}^2 \sum_{r=1}^{n} f_r\Big] \\ &= \frac{1}{N}[N\mu_2' - 2N\bar{\xi}^2 + N\bar{\xi}^2].\end{aligned}$$

So the variance μ_2 and the second moment μ_2' about any value a are related by

$$\sigma^2 \equiv \mu_2 = \mu_2' - \bar{\xi}^2. \tag{20.11}$$

In order to calculate σ, it is usually easiest to calculate the second moment μ_2' about a simple value a, and then use (20.11) instead of using (20.9) or (20.10) directly.

Example 2

Calculate the standard deviation of the distribution given in Example 1. First calculate the second moment about $a = 31.0$ cm using (20.8), the values ξ_r being given in Table 20.1:

$$\begin{aligned}\mu_2' = \tfrac{1}{60}[&(2 + 8)(0.1)^2 + (4 + 9)(0.2)^2 + 11(0.3)^2 \\ &+ (2 + 7)(0.4)^2 + 6(0.5)^2 + (0.6)^2 + 4(0.7)^2 + (0.8)^2] = \tfrac{7.51}{60} = 0.1252.\end{aligned}$$

Since $\bar{\xi} = 0.20$, equation (20.11) gives

$$\sigma^2 = 0.1252 - 0.0400 = 0.0852.$$

Thus $\sigma = 0.29$ cm or 0.3 cm to the nearest millimetre. A glance at the frequency distribution shows us that the majority of values of x lie with about 0.3 cm of the mean 31.2 cm, so that σ does in fact give us a good estimate of the spread of the distribution.

EXERCISE 20.1

1. The number of sales x per day of a particular book was recorded for a hundred successive weekdays. The frequencies $f_0, f_1, f_2, \ldots$, of no sale, one sale, 2 sales, ..., are given in the following table.

x_r	0	1	2	3	4	5	6	7
f_r	9	25	26	18	15	5	1	1

find the mean number of books sold per day, and calculate the standard deviation of the distribution.

2. The heights of 500 men were measured to the nearest $\frac{1}{2}$ inch. The frequencies f_r of heights x_r (in inches) are given in the following table:

x_r	f_r	x_r	f_r	x_r	f_r
58	1	64	18	70	16
58.5	0	64.5	23	70.5	15
59	0	65	31	71	10
59.5	1	65.5	34	71.5	9
60	2	66	35	72	7
60.5	0	66.5	36	72.5	2
61	1	67	41	73	3
61.5	4	67.5	41	73.5	1
62	5	68	39	74	0
62.5	8	68.5	35	74.5	1
63	11	69	29	75	0
63.5	17	69.5	23	75.5	1

Find the mean height and the standard deviation.

§ 2. Probability distributions

In § 1, we discussed observed or empirical distributions. The theory of probability is concerned with the *prediction* of distributions when some basic law is assumed to govern the behaviour of a variate. A simple example of a law of probability is that governing the behaviour of a

perfect die; the symmetry of the die with respect to its six faces suggests very strongly that each of the faces is equally likely to end facing upwards if we make an unbiased throw of the die. The probability p of any particular face appearing is thus equal to $\frac{1}{6}$. Similarly, if a pack of 52 cards is thoroughly shuffled (this is very difficult to do, incidentally), then the probability that a particular card is on top is $p=\frac{1}{52}$. Serious players of poker and bridge base their decisions on statistical laws which are derived from this basic law of probability.

When we throw a die, draw a card from a pack or measure the height of a man selected from some group, we are said to be making a *trial.* Suppose we make a series of N completely independent trials on a system, and that there are a finite number n of possible results of a trial, denoted by $x_1, x_2, \ldots, x_n$. If in the series the result x_1 occurs m_1 times, x_2 occurs m_2 times, and so on, then

$$\sum_{r=1}^{n} m_r = N. \tag{20.12}$$

As N becomes large, it is found that the proportion of any particular result x_r, measured by m_r/N, tends to a definite value p_r; this value is known as the *probability* of the result x_r. Since we cannot in practice make an infinite number of trials, p_r cannot be determined exactly by making a series of trials. A *law of probability*, specifying definite values of the probabilities p_r, is therefore never established with certainty by a series of trials, although it may be strongly suggested. The assumption of a law of probability corresponding to a series of trials is therefore a hypothesis, which may or may not be confirmed by further trials.

In a single trial, the observed value of x *must* be one of $x_1, x_2, \ldots, x_n$; so the total probability of one of these values is unity:

$$\sum_{r=1}^{n} p_r = 1. \tag{20.13}$$

This equation follows from (20.12) if we divide by N and then let $N \to \infty$.

Example 3

The possible results of a throw of a die are $x_1=1$, $x_2=2$, ..., $x_6=6$. For an unbiased die, we assume the probability law $p_r=\frac{1}{6}$ $(r=1, 2, \ldots, 6)$. If in a large number of throws, the values of m_r/N $(r=1, 2, \ldots, 6)$ do not tend to the value $\frac{1}{6}$, we conclude that the die is biased.

§ 2.1. Continuous distributions

Very frequently, the possible values of a variate x are not discrete, but lie in a continuous range. For example, the heights of adults normally take values in the continuous range $4\frac{1}{2}$–$6\frac{1}{2}$ feet. Even though observed heights are classified as taking one of a finite discrete set of values, any theoretical probability law should refer to the continuous range; for if a law of probability predicted a distribution of heights at intervals of half an inch, for example, it would be inappropriate and inadequate for comparison with a set of measurements made to the nearest centimetre. A law of probability for a continuous variate is therefore expressed in terms of a *probability function* $\phi(x)$, such that $\phi(x)\,\mathrm{d}x$ is the probability of a trial giving a result in the infinitesimal range $(x-\frac{1}{2}\mathrm{d}x,\ x+\frac{1}{2}\mathrm{d}x)$, for all values of x. The probability of the result of a trial lying between values x_1 and x_2 is then

$$\int_{x_1}^{x_2} \phi(x)\,\mathrm{d}x. \tag{20.14}$$

Since the total probability is unity,

$$\int \phi(x)\,\mathrm{d}x = 1, \tag{20.15}$$

the integral being taken over the complete range of the variate x. Equation (20.15) is the analogue of (20.13) for discrete distributions, integration over the probability function replacing summation over the probabilities.

§ 2.2. Expectation, moments and standard deviation

The results of § 1 apply equally to probability distributions. The mean of a probability distribution for a variate x is called the *expectation*, and is denoted by $\langle x\rangle$; the notations $\bar{x}$ and $E(x)$ are also sometimes used. For a discrete distribution of a variate x,

$$\langle x\rangle = \sum_{r=1}^{n} p_r x_r; \tag{20.16}$$

for continuous distributions, replacing summation by integration over the whole range of x we have

$$\langle x\rangle = \int x\,\phi(x)\,\mathrm{d}x. \tag{20.17}$$

The second moment about the value a is

$$\mu_2' = \sum_{r=1}^{n} (x_r - a)^2 p_r \tag{20.18}$$

or for continuous distributions,

$$\mu_2' = \int (x - a)^2 \phi(x)\,\mathrm{d}x = \langle (x - a)^2 \rangle. \tag{20.19}$$

The variance μ_2 is given by (20.18) or (20.19) with $a = \langle x \rangle$; so the standard deviation σ is given by

$$\sigma^2 \equiv \mu_2 = \langle (x - \langle x \rangle)^2 \rangle. \tag{20.20}$$

The standard deviation and the second moment $\mu_2'(a)$ are related, as in (20.11) by

$$\sigma^2 = \mu_2' - \langle \xi \rangle^2 \tag{20.21}$$

where

$$\xi = x - a, \tag{20.22}$$

so that

$$\langle \xi \rangle = \langle x \rangle - a \tag{20.23}$$

We note that when $a = \langle x \rangle$, (20.23) becomes

$$\langle x - \langle x \rangle \rangle = \langle x \rangle - \langle x \rangle = 0. \tag{20.24}$$

Example 4

In a lottery, a man holding a single ticket has a chance p_1 of winning a sum s_1, p_2 of winning a sum s_2, ..., p_n of winning s_n, and these are the only possible results; then the sum

$$\langle s \rangle = \sum_{r=1}^{n} p_r s_r$$

is his expectation in the lottery. If the man holds single tickets in a large number N of similar lotteries, and m_1 times wins the sum s_1, m_2 times the sum s_2, ..., m_n times the sum s_n, then his average winnings per lottery are

$$\frac{1}{N} \sum_{r=1}^{n} m_r s_r.$$

As $N \to \infty$, m_r/N tend to the values p_r, so that these average winnings tend towards the expectation $\langle s \rangle$.

Example 5

Cards numbered 1, 2,..., n are put in a box and are thoroughly mixed up; then one card is drawn at random. Assuming that each card is equally likely to be drawn, so that the probabilities of drawing the cards are $p_r = n^{-1}$ ($r = 1, 2, \ldots, n$), the expectation of the number x on the card is

$$\langle x \rangle = n^{-1}(1 + 2 + \ldots + n) = \tfrac{1}{2}(n + 1).$$

The second moment about $x=0$ of the probability distribution of x is, by (20.18) with $a=0$,

$$\mu_2' = \langle x^2 \rangle = n^{-1}(1^2 + 2^2 + \dots + n^2) = \tfrac{1}{6}(n+1)(2n+1).$$

Hence by (20.21),

$$\sigma^2 \equiv \mu_2 = \langle x^2 \rangle - \langle x \rangle^2 = \tfrac{1}{12}(n^2 - 1).$$

For large values of n, $\langle x \rangle \approx \frac{1}{2}n$ and $\sigma \approx n/2\sqrt{3}$; it is to be expected that these two values will become proportional to n when n is large.

Example 6

For a variate x which is equally likely to take any value in the range $(0, X)$, the probability function $\phi(x)$ must equal a constant ϕ_0 in this range and zero elsewhere. So satisfy (20.15),

$$\int_0^X \phi_0 \, \mathrm{d}x = 1$$

so that $\phi(x) = X^{-1}$.

The expectation of x is, by (20.17),

$$\langle x \rangle = X^{-1} \int_0^X x \, \mathrm{d}x = \tfrac{1}{2}X.$$

The variance is given by (20.19) with $a = \langle x \rangle = \frac{1}{2}X$:

$$\mu_2 = \int_{x=0}^{X} X^{-1}(x - \tfrac{1}{2}X)^2 \, \mathrm{d}x = \tfrac{1}{12}X^2.$$

So the standard deviation is $\sigma = X/2\sqrt{3}$. Compare these results with those of Example 5 when n is large.

We have seen that the standard deviation σ gives an estimate of the spread of a distribution. *Tchebycheff's theorem* expresses this idea in a definite mathematical form:

'In a distribution of a variate x with standard deviation σ, the probability that a randomly chosen value of x differs from $\langle x \rangle$ by more than $\lambda\sigma$ does not exceed λ^{-2}, for any positive number λ'.

★ We shall prove this theorem by assuming that the probability that $|x - \langle x \rangle| > \lambda\sigma$ *is* greater than λ^{-2}, and deducing that the standard deviation is greater than σ. Making this assumption, the contribution to the variance

$$\int |x - \langle x \rangle|^2 \, \phi(x) \, \mathrm{d}x$$

from the range R of values of x with $|x - \langle x \rangle| > \lambda\sigma$ is at least

$$(\lambda\sigma)^2 \int_R \phi(x) \, \mathrm{d}x > (\lambda\sigma)^2 \cdot \lambda^{-2} = \sigma^2.$$

Thus the variance exceeds σ^2, and the standard deviation exceeds σ, proving the theorem. ★

§ 3. Compounded probabilities

Sometimes an event can be considered as a combination of two or more events, each of which is governed by a simple law of probability; for instance, the probability laws governing the throwing of two dice can be derived from the law governing the throw of each die separately. Two fundamental laws govern the probability of occurrence of a multiple event; we usually take these laws for granted, but we state them here for completeness:

(i) When several events are 'mutually exclusive', meaning that the occurrence of any one event prevents the occurrence of any other, the total probability that one of these events happens is equal to the sum of the probabilities of the individual events.

(ii) The probability of occurrence of an event A followed by an event B equals the probability of A multiplied by the probability of B on the assumption that A has happened.

Example 7

The probability of the results 1, 2,..., 6 of the throw of an unbiased die are each $\frac{1}{6}$. The probability of an even number (2, 4, or 6) resulting is, by law (i) above, $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$.

Example 8

The probability of drawing a black card from a pack of 52 cards is, by law (i), $\frac{26}{52}=\frac{1}{2}$. Assuming that one black card has been drawn, leaving 25 black and 26 red cards, the probability of drawing another black card is $\frac{25}{51}$. Hence the probability of drawing two black cards is, by law (ii), $\frac{1}{2}\cdot\frac{25}{51}=\frac{25}{102}$.

§ 3.1. Sums and products of variates

Suppose that two variates x and y can take, for instance, discrete values $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_l$ respectively. In a trial which measures values of both x and y, the probability of finding the values x_r for x and y_s for y is denoted by p_{rs}; the set of numbers (p_{rs}) define a *bivariate probability distribution*. The probability p_r of finding the value x_r or x, *regardless of the value of* y, is found by summing the probabilities p_{rs} over

all possible values of s:

$$p_r = \sum_{s=1}^{l} p_{rs}. \tag{20.25}$$

Likewise

$$p_s^* = \sum_{r=1}^{n} p_{rs} \tag{20.26}$$

is the probability that y has value y_s, regardless of the value of x.

The expectation of the sum $x+y$ of two variates, is, by (20.16)

$$\begin{aligned}\langle x+y\rangle &= \sum_{r=1}^{n} \sum_{s=1}^{l} p_{rs}(x_r + y_s) \\ &= \sum_{r=1}^{n} x_r \sum_{s=1}^{l} p_{rs} + \sum_{s=1}^{l} y_s \sum_{r=1}^{n} p_{rs} \\ &= \sum_{r=1}^{n} p_r x_r + \sum_{s=1}^{l} p_s^* y_s,\end{aligned}$$

using (20.25) and (20.26). Thus by (20.16),

$$\langle x+y\rangle = \langle x\rangle + \langle y\rangle, \tag{20.27}$$

showing that the expectations of the sum of two variates is the sum of their expectations. This rule can obviously be extended to any finite number of variates.

The expectation $\langle xy\rangle$ of the product xy does not in general obey such a simple law. The result

$$\langle xy\rangle = \langle x\rangle\langle y\rangle \tag{20.28}$$

holds only if the probabilities p_r of the values x_r of x are independent of the value of y, and vice-versa; we then say that the distributions of the variates x and y are *independent*. When the distributions are independent, then by law (ii) of § 2, the probability p_{rs} that $x=x_r$ and $y=y_s$ is given by

$$p_{rs} = p_r p_s^*. \tag{20.29}$$

The probabilities (20.29) clearly satisfy (20.25) and (20.26), since

$$\sum_{r=1}^{n} p_r = \sum_{s=1}^{l} p_s^* = 1.$$

Assuming (20.29),

$$\langle xy\rangle = \sum_{r=1}^{n} \sum_{s=1}^{l} p_{rs}x_r y_s$$

$$= \sum_{r=1}^{n} p_r x_r \sum_{s=1}^{l} p_s^* y_s,$$

establishing (20.28).

The variances of two *independent* variates are additive; for by (20.20), the variance of the sum $x+y$ is

$$\mu_2(x+y) = \langle(x+y-\langle x\rangle - \langle y\rangle)^2\rangle$$
$$= \langle(x-\langle x\rangle)^2\rangle + \langle(y-\langle y\rangle)^2\rangle + 2\langle(x-\langle x\rangle)(y-\langle y\rangle)\rangle.$$

For independent variates, (20.28) and (20.24) give

$$\langle(x-\langle x\rangle)(y-\langle y\rangle)\rangle = \langle x-\langle x\rangle\rangle\langle y-\langle y\rangle\rangle = 0,$$

so that the variance becomes, by (20.20), the sum of the variances of x and y. The same argument can be applied to give the variance of $x-y$; thus for independent variates,

$$\mu_2(x\pm y) = \mu_2(x) + \mu_2(y). \tag{20.30}$$

The results (20.28) and (20.30) can obviously be extended inductively to any finite number of *independent* variates.

Example 9

Two dice are thrown. What is the chance that the numbers on them will total to 9? What is the expectation and the standard deviation of the total?

The combinations totalling to 9 are (3,6), (4,5), (5,4) and (6,3). Assuming the probability for any particular number on one die is $\frac{1}{6}$, independent of the other die, the probability of any particular pair of numbers is $(\frac{1}{6})^2 = \frac{1}{36}$, by (20.29). The probability of occurrence of one of the four combinations is therefore $4\times\frac{1}{36} = \frac{1}{9}$.

The expectation and standard deviation for a single die are given by the results of Example 5 with $n=6$, the six cards being replaced by the six equally probable faces of the die. Thus for one die $\langle x\rangle = 3\frac{1}{2}$ and $\mu_2 = \frac{35}{12}$. Since the probabilities are independent, we can use both (20.27) and (20.30), giving

$$\langle x+y\rangle = 7, \qquad \mu_2(x+y) = \tfrac{35}{6},$$

so that $\sigma = \sqrt{\frac{35}{6}} \approx 2.42$.

EXERCISE 20.2

1. A penny is tossed n times. Find the probability that it comes down heads exactly m times.

2. Four cards are drawn from a complete pack. What is the chance that
(i) there is one card from each suit?
(ii) all four cards belong to the same suit?
(iii) two cards are spades and two are clubs?

3. Cards numbered 1, 4, 9, 16,..., n^2 are placed in a box, and one is drawn at random. What is the expectation of the number x on the card and the standard deviation of x.

4. Assuming that there are 365 days in the year, and that the probability of any man having his birthday on a particular day is $\frac{1}{365}$, find the probability that out of a group of 35 men, two have the same birthday and the others all have different birthdays.

5. Square pegs numbered 1, 2,..., n, are fitted at random into n square holes, also numbered 1, 2,... n. Show that the probability that no peg has its number equal to that of the hole it occupies is $f(n)/n!$, where $f(n)$ is a function obeying the recurrence relation

$$f(n) = (n-1)[f(n-1) + f(n-2)],$$

with $f(1)=0$ and $f(2)=1$.

6. Find the mean and standard deviation of the probability distribution defined by

$$\phi(x) = \begin{cases} \dfrac{2a^3}{\pi(x^2+a^2)^2} & \text{for} \quad x \leqslant 0, \\[2ex] \dfrac{\pi}{a}\exp(-2\pi x/a) & \text{for} \quad x > 0, \end{cases}$$

where $a > 0$.

7. Calculate the expectation of the length x of the chord of a circle of radius a parallel to a given line, assuming that
(i) all perpendicular distances from the centre on to the chord are equally likely.
(ii) all angles subtended by the chord at the centre of the circle are equally likely.
Find the standard deviations of x on these assumptions.

8. A die is thrown repeatedly until a 'six' turns up. What is the expectation of the number of throws required? What is the expectation of the number of throws of two dice required in order to obtain a double six? Find the standard deviations of the distributions of these numbers.

§ 4. The binomial and Poisson distributions

§ 4.1. The binomial distribution

Suppose that a trial can have only two possible results, one called 'a success', with probability p, and the other called 'a failure', with probability $q=1-p$. If we make n independent trials of this type in succession,

then the probability of a *particular sequence* of r successes and $n-r$ failures is $p^r q^{n-r}$. But the number of sequences with r successes and $n-r$ failures is $\binom{n}{r}=n!/[r!(n-r)!]$. So the probability of there being exactly r successes in n trials is

$$p_r = \binom{n}{r} p^r q^{n-r}. \tag{20.31}$$

This is the term containing p^r in the binomial expansion of

$$(p+q)^n. \tag{20.32}$$

The probability distribution (20.31) is therefore known as the *binomial distribution.*

Example 10

A card is drawn at random from each of twelve packs of cards. What is the chance that exactly three of the cards are spades?

The chance of a success in one trial (drawing a spade from one pack) is $p=\frac{13}{52}=\frac{1}{4}$. The chance of a failure is $q=\frac{3}{4}$. The chance of three successes out of twelve is the term containing p^3 in $(p+q)^{12}$, which is (20.31) with $n=12$, $r=3$:

$$\frac{12!}{3!\,9!}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^9 \approx 0.258.$$

The expectation of the number of successes x in a sequence of n independent trials is given by (20.16), the probability for $x=r$ being given by (20.31):

$$\langle x\rangle = \sum_{r=0}^{n} r\binom{n}{r} p^r q^{n-r}.$$

This series is summed by treating p and q as independent variables. The series can then be written

$$p\frac{\partial}{\partial p}\sum_{r=0}^{n}\binom{n}{r} p^r q^{n-r} = p\frac{\partial}{\partial p}(p+q)^n = np(p+q)^{n-1}. \tag{20.33}$$

We can now put $p+q=1$ to give the expected result

$$\langle x\rangle = np. \tag{20.34}$$

The second moment about $x=0$ of the distribution of the number of successes can be found by the same method; we leave the derivation as an exercise for the reader, and simply quote the result:

$$\mu_2' = np[1+p(n-1)]. \tag{20.35}$$

Substituting this result and (20.34) in equation (20.21), we find the variance

$$\mu_2 = np[1 + p(n - 1)] - (np)^2$$
$$= np(1 - p) = npq.$$

Thus the standard deviation of the *total* number of successes in n trials is

$$\sigma = (npq)^{\frac{1}{2}}. \tag{20.36}$$

Hence the standard deviation of the *proportion* of successes, which has expectation p, is

$$\sigma/n = (pq/n)^{\frac{1}{2}}. \tag{20.37}$$

Example 11

When one card is drawn at random from each of twelve packs, the expectation of the number x of spades drawn is given by (20.34) with $n=12$ and $p=\frac{1}{4}$. Thus $\langle x\rangle=3$, as we might expect. The standard deviation of the distribution of x is, by (20.36)

$$\sigma = (12\cdot\tfrac{1}{4}\cdot\tfrac{3}{4})^{\frac{1}{2}} = \tfrac{3}{2}.$$

§ 4.2. The Poisson distribution

The Poisson distribution is derived as the limit of the binomial distribution when the number of trials n is very large and the probability of success is very small. Suppose, for example, we could find probability distribution for the number r of people in the world born in a single minute, by recording this number for every minute of a given week; the probability p of any one person being born in a particular minute is very small, but the number of trials (equal to the world's population) is very large. We would therefore expect the distribution of r to be Poissonian.

We have seen that the mean number of successes for the binomial distribution is np. In going to the limits $n\to\infty$ and $p\to 0$, we keep the quantity $m=np$ constant, so that the distribution will have a definite mean m. We shall see that the distribution is expressible in terms of m. Putting $p=m/n$ and $q=1-p$ in (20.31), the chance of r successes for the binomial distribution is

$$p_r = \frac{n(n-1)\cdots(n-r+1)}{r!}\left(\frac{m}{n}\right)^r\left(1-\frac{m}{n}\right)^{n-r}$$

$$= \frac{m^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-r}. \tag{20.38}$$

For any fixed value of r, as $n \to \infty$,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) \to 1,$$

$$\left(1-\frac{m}{n}\right)^{-r} \to 1,$$

and by (4.45),

$$\left(1-\frac{m}{n}\right)^{n} \to e^{-m}.$$

Hence the distribution (20.38) becomes

$$p_r = \frac{m^r e^{-m}}{r!}. \tag{20.39}$$

The sum of the probabilities (20.39) for $r=0, 1, 2, \ldots$ is $e^{-m} \sum_{r=0}^{\infty} (m^r/r!) = e^{-m} \cdot e^{m} = 1$, which is correct.

The mean of the distribution (20.39) we know to be m. The second moment μ_2' about $r=0$ and the standard deviation σ are easily obtained as limits of (20.35) and (20.36) as $p \to 0$, $n \to \infty$, with $m=pn$. We find

$$\mu_2' = m(m+1) \tag{20.40}$$

and

$$\sigma = m^{\frac{1}{2}}, \tag{20.41}$$

satisfying equation (20.21).

★ One important property of Poisson's distribution is its preservation under the addition of two independent variates. If x_1 and x_2 are the two variates, and their distribution functions are given by (20.39) with $m=m_1$ and $m=m_2$, then the probability that $x=r$ and $y=s$ simultaneously is

$$\frac{m_1^r e^{-m_1} m_2^s e^{-m_2}}{r!\, s!}.$$

So the probability that the sum $x+y$ has value t is

$$\sum_{r+s=t} \frac{m_1^r e^{-m_1} m_2^s e^{-m_2}}{r!\, s!}$$

$$= e^{-(m_1+m_2)} \sum_{r=0}^{t} \frac{m_1^r m_2^{t-r}}{r!(t-r)!} = \frac{e^{-(m_1+m_2)}(m_1+m_2)^t}{t!}.$$

Comparing with (20.39), we see that the distribution of $x+y$ is Poissonian, with the expected mean m_1+m_2.

This result can be generalised by induction to any number of variates; thus the distribution of the sum of any number of Poissonian variates is also Poissonian, with mean equal to the sum of the means of the individual variates. ★

EXERCISE 20.3

1. Twelve dice are thrown simultaneously, the experiment being repeated 1000 times; the number r of sixes in each throw is noted, and the frequencies f_r of various values of r are plotted in the following table:

r	0	1	2	3	4	5	6	7	8	9–12
f_r	116	270	305	189	93	23	3	0	1	0

Show that this distribution is approximately the binomial distribution predicted by assuming that the dice are unbiased. Compare the actual mean and standard deviation with that predicted by the binomial law.

2. Show that the distribution of books sold per day, given in Question 1, Exercise 20.1, is approximately Poissonian with $m=2.3$. Compare the actual mean and standard deviation with that predicted for the Poissonian distribution.

§ 5. The normal distribution

The most important probability distribution is a continuous distribution known as the *normal distribution*, with probability function

$$\phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \exp - \left[\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right]; \tag{20.42}$$

σ is a constant which we shall show (later) to equal the standard deviation. The graph of $\phi(x)$ is shown in fig. 20.1: the distribution is symmetrical about $x=\langle x\rangle$, which is therefore the mean or expectation of x. The maximum value of $\phi(x)$ is $1/(2\pi)^{\frac{1}{2}}\sigma \approx 0.4\sigma^{-1}$. If we write $\xi = x - \langle x\rangle$, measuring the deviation of x from its expectation $\langle x\rangle$, the distribution function becomes

$$\phi(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \exp\left(-\frac{\xi^2}{2\sigma^2}\right). \tag{20.43}$$

This function satisfies (20.15), giving the total probability equal to unity; for, putting $\mu = \xi/\sigma\sqrt{2}$ and using the result of Ch. 14 Example 15,

$$\int_{-\infty}^{\infty} \phi(\xi)\,\mathrm{d}\xi = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu^2}\,\mathrm{d}\mu = 1.$$

It is found that a great many empirical distributions are, more or less accurately, represented by the normal distribution function (20.42). The distribution (20.42) also arises theoretically as limits of a number of other probability distributions. Consider for example the binomial distribution (20.31). By (20.36), its standard deviation is $(npq)^{\frac{1}{2}}$, so if we consider the distribution of the variable $y = r/n^{\frac{1}{2}}$, it will have mean $pn^{\frac{1}{2}}$ and

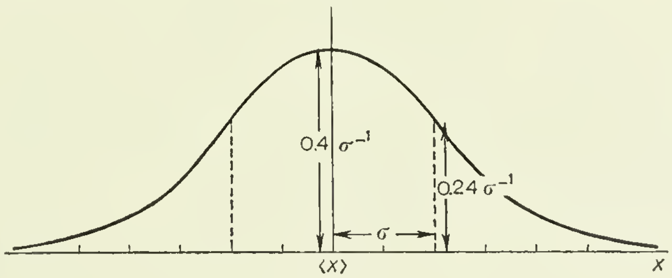

Fig. 20.1

standard deviation $\sigma = (pq)^{\frac{1}{2}}$, independent of n. The possible values of y are

$$0, \quad n^{-\frac{1}{2}}, \quad 2n^{-\frac{1}{2}}, \quad \ldots, \quad (n-1)n^{-\frac{1}{2}}, \quad n^{\frac{1}{2}}.$$

So in the limit $n \to \infty$, y takes values in the continuous range $(0, \infty)$. We shall now show that in this limit the distribution of y is normal.

✶ Since the mean of y is $n^{\frac{1}{2}}p$, the mean of $\xi = y - pn^{\frac{1}{2}}$ is zero. The possible values of ξ are $\xi_r \equiv rn^{-\frac{1}{2}} - pn^{\frac{1}{2}}$, and the probability $\phi(\xi_r)$ that $\xi = \xi_r$ or $y = rn^{-\frac{1}{2}}$ is p_r, given by (20.31):

$$\phi(\xi_r) = \binom{n}{r} p^r q^{n-r}. \tag{20.44}$$

So remembering that $p+q=1$ and $r = pn + \xi_r n^{\frac{1}{2}}$,

$$\begin{aligned}\frac{\phi(\xi_{r+1}) - \phi(\xi_r)}{(\xi_{r+1} - \xi_r)\,\phi(\xi_r)} &= \frac{1}{n^{-\frac{1}{2}}}\left[\frac{(n-r)p}{(r+1)q} - 1\right] \\ &= \frac{np - r - q}{n^{-\frac{1}{2}}(r+1)q} \\ &= -\frac{\xi_r n^{\frac{1}{2}} + q}{q(pn^{\frac{1}{2}} + \xi_r + n^{-\frac{1}{2}})}.\end{aligned} \tag{20.45}$$

Now let $n \to \infty$, so that $\xi_{r+1} - \xi_r = n^{-\frac{1}{2}} \to 0$. Remembering that the standard deviation is $\sigma = (pq)^{\frac{1}{2}}$, (20.45) gives

$$\frac{1}{\phi(\xi_r)} \frac{\mathrm{d}\phi(\xi_r)}{\mathrm{d}\xi_r} = -\frac{\xi_r}{pq} = -\frac{\xi_r}{\sigma^2}.$$

This holds for all ξ_r, so the distribution function $\phi(\xi)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[\log \phi(\xi)] = -\frac{\xi}{\sigma^2},$$

giving

$$\phi(\xi) = A \exp\left(-\frac{\xi^2}{2\sigma^2}\right)$$

where A is a constant. So the distribution function is equal to (20.43) in the limit, apart from a possible constant factor; but since the distribution (20.44) obeys $\sum_r \phi(\xi_r) = 1$, giving $\int_{-\infty}^{\infty} \phi(\xi)\,\mathrm{d}\xi = 1$ in the limit $n \to \infty$, we know that $\phi(\xi)$ is given exactly by (20.43). ★

From (20.43), it is clear that $\sigma\phi(\xi)$ is a function of the variable

$$t = \frac{\xi}{\sigma} \tag{20.46}$$

only. So every normal distribution function is expressible in terms of the function $z(t)$ defined by

$$z(t) \equiv \sigma\phi(\xi) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}t^2}. \tag{20.47}$$

We can look upon z as the probability function ϕ when the standard deviation σ is unity.

The variance of the normal distribution (20.43) is, using integration by parts,

$$\mu_2 = \int_{-\infty}^{\infty} \xi^2 \phi(\xi)\,\mathrm{d}\xi = \frac{\sigma^2}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} t(t\mathrm{e}^{-\frac{1}{2}t^2})\,\mathrm{d}t$$

$$= \frac{\sigma^2}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t = \sigma^2,$$

since the probability function $z(t)$ has integral equal to unity. Thus the symbol σ in (20.43) or (20.42) correctly represents the standard deviation of ξ or of $x = \xi + \langle x \rangle$.

§ 5.1. Integrated probabilities. Significance

The function $z(t)$, from which $\phi(\xi)$ can be found by dividing by σ, is accurately tabulated in many reference books. In Table 20.2 we give the values of $z(t)$ at intervals of 0.1 of $t=\xi/\sigma$ between $t=0$ and $t=2.9$.

TABLE 20.2

$t = \xi/\sigma$	$z(t) = \sigma\phi(\xi)$	t	$z(t)$	t	$z(t)$
0.0	0.3989	1.0	0.2420	2.0	0.0540
0.1	0.3970	1.1	0.2179	2.1	0.0440
0.2	0.3910	1.2	0.1942	2.2	0.0355
0.3	0.3814	1.3	0.1714	2.3	0.0283
0.4	0.3683	1.4	0.1497	2.4	0.0224
0.5	0.3521	1.5	0.1295	2.5	0.0175
0.6	0.3332	1.6	0.1109	2.6	0.0136
0.7	0.3123	1.7	0.0940	2.7	0.0104
0.8	0.2897	1.8	0.0790	2.8	0.0079
0.9	0.2661	1.9	0.0656	2.9	0.0060

Note that the maximum value of $\phi(\xi)$ at $\xi=0$ is approximately $2/5\sigma$, and when $\xi=x-\langle x\rangle=\pm\sigma$, the value of $\phi(\xi)$ is slightly less than $1/4\sigma$, as indicated in fig. 20.1. It is easy to show that the points $\xi=\pm\sigma$ are the points of inflexion of the normal distribution curve.

It is of great interest to know the probability that the deviation ξ from the mean is within certain limits. By (20.14), the probability that ξ lies between limits σt_1 and σt_2 is

$$\int_{\xi=\sigma t_1}^{\sigma t_2} \phi(\xi)\,\mathrm{d}\xi = \int_{t=t_1}^{t_2} z(t)\,\mathrm{d}t \tag{20.48}$$

using (20.46) and (20.47). These integrals are usually expressed as sums and differences of integrals of the type

$$\alpha(\lambda) = \int_{t=-\lambda}^{\lambda} z(t)\,\mathrm{d}t = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{t=0}^{\lambda} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t, \tag{20.49}$$

representing the area under the normal curve between the points $x=\pm\lambda\sigma$. The integrals (20.49) can only be evaluated numerically, and have also been tabulated extensively. In Table 20.3, the values of $\alpha(\lambda)$ are given for values of λ between 0 and 3 at intervals of 0.2.

TABLE 20.3

λ	$\alpha(\lambda)$	λ	$\alpha(\lambda)$	λ	$\alpha(\lambda)$
0.0	0.000	1.0	0.683	2.0	0.954
0.2	0.159	1.2	0.770	2.2	0.972
0.4	0.311	1.4	0.838	2.4	0.984
0.6	0.451	1.6	0.890	2.6	0.991
0.8	0.576	1.8	0.928	2.8	0.995
				3.0	0.997

From (20.48) and (20.49), we know that the probability of observing a value of ξ within the range $(-\lambda\sigma, \lambda\sigma)$ is equal to $\alpha(\lambda)$. But ξ measures the deviation of the variate α from its expectation $\langle x \rangle$; *to the probability that the deviation* $x-\langle x \rangle$ *exceeds* $\lambda\sigma$ *is* $1-\alpha(\lambda)$. The probabilities that the deviation exceeds σ, 2σ and 3σ can therefore be derived simply from Table 20.3:

Deviation	*Probability*
$\lvert x - \langle x \rangle \rvert > \sigma$	$1 - \alpha(1) = 0.317$
$\lvert x - \langle x \rangle \rvert > 2\sigma$	$1 - \alpha(2) = 0.046$
$\lvert x - \langle x \rangle \rvert > 3\sigma$	$1 - \alpha(3) = 0.003.$

So the probability that the deviation exceeds 2σ is less than $\frac{1}{20}$, and the probability that it exceeds 3σ is very small. An observation which is improbable is said to be *significant.* For a normally distributed variate, it is usual to regard a single observation with $|x-\langle x \rangle|>2\sigma$ as significant, and one with $|x-\langle x \rangle|>3\sigma$ as *highly significant.*

Example 12

The expectation and standard deviation of a normal variate are $\langle x \rangle=3.2$ and $\sigma=2$. Find the probabilities that a single value of the variate x will lie in the range (2, 4) and (4, 6).

The corresponding ranges of $\xi=x-\langle x \rangle$ are $(-1.2, 0.8)$ and $(0.8, 2.8)$, so that the ranges of $t=\xi/\sigma=\frac{1}{2}\xi$ are $(-0.6, 0.4)$ and $(0.4, 1.4)$. The probability that t lies in the range $(-0.6, 0.4)$ is, by (20.48) and (20.49),

$$\int_{t=-0.6}^{0.4} z(t)\,\mathrm{d}t = \tfrac{1}{2}[\alpha(0.4) + \alpha(0.6)] = 0.381,$$

using Table 20.3. Likewise the probability that t lies in (0.4, 1.4) is

$$\tfrac{1}{2}[\alpha(1.4) - \alpha(0.4)] = 0.263.$$

EXERCISE 20.4

1. Show that the distribution of heights of 500 men given in Exercise 20.1, Question 2, is approximately normal, with mean 67″ and standard deviation 2.5″.

Assuming this normal distribution of heights, which of the following measurements of heights, considered individually, are significant or highly significant: 63″, 72″, 69.5″, 58.5″, 73″, 75″?

2. A variate x which can take values in a continuous range is assumed to be normally distributed with mean 47 and standard deviation 5. Find the probabilities that an individual measurement lies in each of the following ranges:

(i) 45 to 50,
(ii) 49 to 55,
(iii) 41 to 53,
(iv) 36 to 46.

§ 5.2. Distribution of a sum of normal variates

If two variates x and y are distributed normally, their sum $u=x+y$ is also distributed normally. To prove this, we shall refer each distribution to its mean, and assume that the distribution functions of $\xi=x-\langle x\rangle$ and $\eta=y-\langle y\rangle$ are

$$\phi_1(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_1} \exp\left(-\frac{\xi^2}{2\sigma_1^2}\right)$$

and

$$\phi_2(\eta) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_2} \exp\left(-\frac{\eta^2}{2\sigma_2^2}\right).$$

The probability that $\xi+\eta$ takes a particular value ζ is found by integrating over the probabilities that ξ and η take particular values summing to ζ. This gives

$$\int \phi_1(\xi)\,\phi_2(\zeta-\xi)\,d\xi$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[\frac{-\zeta^2}{2(\sigma_1^2+\sigma_2^2)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\xi-\frac{\zeta\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\right] d\xi, \quad (20.50)$$

on rearranging the exponential factors. If we now put

$$t = \frac{(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}}{\sigma_1\sigma_2}\left(\xi-\frac{\zeta\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right),$$

the integral in (20.50) becomes

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \frac{\sigma_1\sigma_2}{(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}}\, dt = \frac{(2\pi)^{\frac{1}{2}}\sigma_1\sigma_2}{(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}}.$$

Hence the distribution function (20.50) is

$$\frac{1}{(2\pi)^{\frac{1}{2}}(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}} \exp\left[-\frac{\zeta^2}{2(\sigma_1^2+\sigma_2^2)}\right].$$

Comparing with (20.43), we see that $\zeta=\xi+\eta$ is normally distributed about the mean value zero, with standard deviation given by

$$\sigma^2 = \sigma_1^2 + \sigma_2^2. \tag{20.51}$$

The distribution of $u=x+y$ therefore has expectation $\langle u\rangle=\langle x\rangle+\langle y\rangle$, agreeing with (20.27), with standard deviation given by (20.51). As for the Poisson distribution, the inductive extension of this proof and of equation (20.51) to the sum of any number of normally distributed variates is obvious.

§ 6. Sampling. Tests of significance

In many practical applications of statistics, we wish to be able to make predictions about a very large (possibly infinite) population by making observations on a moderately large number n of the population, chosen at random. The observed members are known as a *sample* of the population. In order to make statistical predictions about the distribution of a variate x over the whole population, we must assume a probability distribution for x, and we test the validity of this theoretical distribution by comparing it with the empirical distribution observed in the sample. The empirical and the probability distributions cannot, of course, be expected to be identical – in particular, a probability distribution may be continuous, while an empirical one must be discrete. The two distributions must however, be similar, meaning that quantities such as their means and standard deviations must be nearly equal. Sometimes we wish to know whether two or more empirical distributions can reasonably be fitted by the same probability distribution. We shall indicate how judgments of similarity of distributions are made by considering a few simple types of problem that arise in practice.

§ 6.1. Simple samples of population

Suppose that we make a series of trials on a population, and each trial can result in success or failure. If we assume the probability law that the chance of success is p for each member of the population, the probability distribution of the number of successes is binomial, and is given by (20.31). A sample of such a population is called a *simple sample*; the expectation of the number of successes for a sample of size n is np, and by (20.36) the standard deviation of this number is $\sigma=(npq)^{\frac{1}{2}}$, generally known as the *standard error*. The standard error of the *proportion* of successes is $\epsilon=(pq/n)^{\frac{1}{2}}$, by (20.37). We have seen that for a large number n of trials, the binomial distribution is approximately normal; so for large n, there is less than one chance in twenty that the *observed* number of successes differs from np by more than $2\sigma=2(npq)^{\frac{1}{2}}$. In other words, if p' is the observed proportion of successes in the series of trials, then finding $|p-p'|>2(pq/n)^{\frac{1}{2}}$ is significant, and means that the assumed law of probability may well be wrong. If $|p-p'|>3(pq/n)^{\frac{1}{2}}$, the assumed law is almost certainly wrong.

Example 13

In 6 000 throws of a die, a six resulted 1 112 times. Is it likely that the die is unbiased?

Here $n=6\,000$, and the probability of a six for an unbiased die is $p=\frac{1}{6}$, so that $q=\frac{5}{6}$. So for this size of sample, the standard error of the number of successes is

$$\sigma = (npq)^{\frac{1}{2}} = 50/\sqrt{3} \approx 29.$$

On the hypothesis $p=\frac{1}{6}$, the expectation of the number of successes is 1 000; the observed deviation is 112, which is nearly 4σ. We conclude that the assumption that $p=\frac{1}{6}$ is almost certainly wrong, and that the die is very probably biased.

§ 6.2. Comparison of two large samples

Sometimes we wish to discover by sampling whether the proportion of members in one population possessing a certain attribute is different from the proportion in a second population. Suppose that we take samples of size n_1 and n_2 from two populations, and that the proportions of these samples which are found to possess the attribute are p_1' and p_2'. Now we can look upon p_1' as the number of successes (in finding the attribute) in n_1 trials, and if we were to take many samples of size n_1 from the first population, we should expect the number of successes n_1p_1 to have a binomial distribution. Since n_1 is assumed to be quite large, the expected

distribution of n_1p_1, and hence of p_1, will be approximately normal. Similarly, the expected distribution of n_2p_2 and p_2 for a large number of samples of the second population will be approximately normal.

We tackle the problem by making the hypothesis that the proportions of the populations with this attribute *are* equal. The best estimate that we have of this common proportion p is obtained by combining the two samples. We know that out of n_1+n_2 members of the combined populations, $n_1p_1+n_2p_2$ have the attribute; so the estimate of the common proportion p is

$$p' = \frac{n_1p_1' + n_2p_2'}{n_1 + n_2}. \tag{20.52}$$

Now consider the distribution of $p_1'-p_2'$; on the hypothesis of equal proportions, the expectation for a large number of samples from each population is zero:

$$\langle p_1' - p_2'\rangle = 0.$$

Now the variances of p_1 and p_2 are, by (20.37), pq/n_1 and pq/n_2 where p is the common proportion and $q=1-p$. We can estimate these variances by replacing p by p', given by (20.52). The variance of p_1-p_2 is then given by (20.30) as the sum of the estimated variances $p'q'/n_1$ and $p'q'/n_2$, and is thus

$$\epsilon^2 = \mu_2(p_1' - p_2') = p'q'\left(\frac{1}{n_1} + \frac{1}{n_2}\right). \tag{20.53}$$

Further, since p_1 and p_2 are approximately normally distributed, so is $p_1'-p_2'$; so a deviation of p_1-p_2 from its zero expectation which is more than twice the standard error ϵ, given by (20.53), is significant. An observed value of $|p_1'-p_2'|$ greater than 3ϵ is highly improbable. So if our samples give $|p_1'-p_2'|>3\epsilon$, the hypothesis that the attribute is equally probable in the two populations is very unlikely to be true.

Example 14

In two large cities, 20% and 30% of samples of 300 people were over 45 years of age. Is it unlikely that the proportion of people over 45 is the same in the two cities?

Here the empirical proportions are $p_1=\frac{1}{5}$ and $p_2=\frac{3}{10}$, while $n_1=n_2=300$. If there is a common proportion, the best estimate of it is, by (20.52),

$$p' = \tfrac{1}{2}(p_1' + p_2') = \tfrac{1}{4},$$

so that $q'=\frac{3}{4}$. By (20.53), the variance of $p_1'-p_2'$ is

$$\epsilon^2=\frac{3}{16}\cdot\frac{2}{300}=\frac{1}{800},$$

giving the standard error of $p_1'-p_2'$ as $\epsilon\approx 0.035$. The observed value of $|p_1'-p_2'|$ is 0.10, which is nearly 3ϵ; so it is very unlikely that the proportions over 45 in the two towns are equal.

A slightly different problem is to find out whether a difference between proportions p_1 and p_2 in two populations will be revealed by taking samples of size n_1 and n_2. We are assuming that the actual proportions p_1 and p_2 can be determined by some other method; if we take many samples of size n_1 from the first population, the distribution of the observed proportion p_1' will be binomial, and will have variance p_1q_1/n_1 by (20.37); likewise the distribution of p_2' will have variance p_2q_2/n_2. Thus by (20.30), the variance of $p_1'-p_2'$ is

$$\epsilon^2\equiv\mu_2(p_1'-p_2')=\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}. \tag{20.54}$$

It is quite possible for the difference $p_1'-p_2'$ given by the two samples to differ from the true value p_1-p_2 by 2ϵ, where the standard error ϵ is given by (20.54). So if $|p_1-p_2|<2\epsilon$, $p_1'-p_2'$ may well be zero or have a different sign to p_1-p_2; then the difference in proportions will not be revealed by considering the one pair of samples. If however, $|p_1-p_2|>3\epsilon$, $p_1'-p_2'$ will almost certainly have the same sign as p_1-p_2, revealing the difference of proportions.

Example 15

The percentage of large eggs produced by two farms are known to be 20% and 25%. Is this difference likely to be revealed by taking samples of 40 and 30 eggs respectively from their yield?

We have $p_1=0.2$, $p_2=0.25$, so that $p_1-p_2=0.05$. From (20.54),

$$\epsilon^2=\frac{1}{5}\cdot\frac{4}{5}\cdot\frac{1}{40}+\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{30},$$

giving $\epsilon\approx 0.10$. Since $|p_1-p_2|\approx\frac{1}{2}\epsilon$, $p_1'-p_2'$ may well have opposite sign to p_1-p_2. So the test is not significant.

EXERCISE 20.5

1. A coin is tossed 400 times and comes down 'heads' 187 times. Is it probable that the coin is equally likely to come down 'heads' and 'tails'?

2. Of samples of 500 and 750 people from two towns, 206 and 297 respectively belonged to car-owning families. Is it likely that the proportions of such people in the two towns are equal?

3. If the proportions in two large populations possessing a certain attribute p_1 and $p_2(\neq p_1)$, show that samples of size n_1 and n_2 cannot significantly reveal the difference between p_1 and p_2 unless

$$n_1 > \frac{4p_1q_1}{(p_1 - p_2)^2}$$

and

$$n_2 > \frac{4n_1p_2q_2}{n_1(p_1 - p_2)^2 - 4p_1q_1}.$$

§ 7. Bivariate distributions

If we measure the values of two variates x and y a number of times, we can plot the values of y against those of x giving a set of points in the xy-plane known as a *scatter diagram*. We shall assume that the possible values of x and y are discrete sets (x_r) and (y_s) since we are discussing empirical statistical distributions. The definitions and results can easily be applied to probability distributions; for continuous distributions, summation over r and s must, as in previous sections, be replaced by integrations.

The number of times f_{rs} that a particular pair of values (x_r, y_s) occurs is the frequency of the pair, and the total frequency is $N \equiv \sum_{r,s} f_{rs}$. The empirical distribution specified by the numbers (f_{rs}) is a bivariate distribution, analogous to the bivariate probability distribution specified by (p_{rs}) in § 3.1. The means $\bar{x}$ and $\bar{y}$ for the distribution (f_{rs}) are given by

$$\bar{x} = \frac{1}{N} \sum_{r,s} f_{rs}x_r = \frac{1}{N} \sum_{r} f_r x_r \tag{20.55}$$

and

$$\bar{y} = \frac{1}{N} \sum_{r,s} f_{rs}y_s = \frac{1}{N} \sum_{s} f'_s y_s, \tag{20.56}$$

where

$$f_r = \sum_{s} f_{rs}, \qquad f'_s = \sum_{r} f_{rs} \tag{20.57}$$

are the *total* number of occurrences of x_r and y_s respectively.

The *second moments* of x and y about values a and b are

$$\mu'_{20} = \frac{1}{N} \sum_{r,s} f_{rs}(x_r - a)^2 = \frac{1}{N} \sum_{r} f_r(x_r - a)^2 \tag{20.58}$$

and

$$\mu'_{02} = \frac{1}{N} \sum_{r,s} f_{rs}(y_s - b)^2 = \frac{1}{N} \sum_s f'_s(y_s - b)^2. \tag{20.59}$$

As in § 1.2 for distributions of a single variate, the variances $\mu_{20} \equiv \sigma_x^2$ and $\mu_{02} \equiv \sigma_y^2$ are defined as the second moments of x and y about their means $\bar{x}$ and $\bar{y}$, and are related to μ'_{20} and μ'_{02} by relations analogous to (20.11). A quantity which has no analogue in single variate distributions is the second moment

$$\mu'_{11} \equiv \frac{1}{N} \sum_{r,s} f_{rs}(x_r - a)(y_s - b). \tag{20.60}$$

When a and b are equal to the means $\bar{x}$ and $\bar{y}$, this quantity is called *covariance* μ_{11}. If we write

$$\xi_r = x_r - a, \qquad \eta_s = y_s - b \tag{20.61}$$

so that $\bar{\xi} = \bar{x} - a$, $\bar{\eta} = \bar{y} - b$, then

$$\begin{aligned} \mu_{11} &= \frac{1}{N} \sum_{r,s} f_{rs}(x_r - \bar{x})(y_s - \bar{y}) \\ &= \frac{1}{N} \sum_{r,s} f_{rs}(\xi_r - \bar{\xi})(\eta_s - \bar{\eta}) \\ &= \mu'_{11} - (\bar{\xi}/N) \sum_{r,s} f_{rs}\eta_s - (\bar{\eta}/N) \sum_{r,s} f_{rs}\xi_r + \bar{\xi}\bar{\eta}, \end{aligned}$$

using (20.60). Remembering (20.55) and (20.56), this becomes

$$\mu_{11} = \mu'_{11} - \bar{\xi}\bar{\eta}. \tag{20.62}$$

When we plot the values of a pair of variates in the scatter diagram, we often want to know whether some more or less precise relationship, known as a *correlation*, exists between the values of x and y. If the distribution frequency is of the form $f_{rs} = f_r f'_s$, then the frequencies of the values x_r of x are proportional to f_r, independent of the value of y, and vice-versa; thus the variables are uncorrelated. The covariance is then

$$\begin{aligned} \mu_{11} &= \sum_{r,s} f_r f'_s (x_r - \bar{x})(y_s - \bar{y}) \\ &= \sum_r f_r(x_r - \bar{x}) \sum_s f'_s(y_s - \bar{y}) = 0, \end{aligned}$$

using (20.55) and (20.56). The absence of correlation thus leads to the vanishing of the covariance. Generally, the magnitude of the covariance gives us an estimate of the degree of correlation.

§ 7.1. Lines of regression

The points in a scatter diagram derived from empirical observations in scientific or social studies often nearly lie on a curve in the xy-plane. If we choose to examine curves of some particular class, such as straight lines, parabolae, or exponential curves, then a curve of the class which in some sense gives the best fit, going near to the majority of points in the scatter diagram, is called a *curve of regression*. Quite often some mathematical theory, or the observations themselves, suggest the choice of the type of curve we should try to fit to the data. If the relation between x and y is approximately linear, then we try to find the straight line

$$y = mx + c \tag{20.63}$$

which gives the best fit; a straight line which gives a best fit to the data is called a *line of regression*, and we shall now show how lines of regression are defined.

If (20.63), with particular values of m and c, is assumed to define a precise relationship between the variates x and y, then for any value x_r of x, this equation provides an *estimate*

$$Y_r = mx_r + c \tag{20.64}$$

of the corresponding value of y. The *observed* values of y corresponding to $x = x_s$ are y_s, occurring with frequency f_{rs}; so for given x_r, the deviation of the observed values y_s from the estimate Y_r is measured by their second moment about Y_r, equal to

$$\sum_s f_{rs}(y_s - Y_r)^2.$$

Hence the total deviation of y-values of all points (x_r, y_s) is measured by the sum $G(m, c)$ of these second moments, given by

$$G(m, c) = \sum_{r,s} f_{rs}(y_s - Y_r)^2 = \sum_{r,s} f_{rs}(y_s - mx_r - c)^2, \tag{20.65}$$

using (20.64). If we choose m and c so that $G(m, c)$ is a minimum, then the line (20.63) will give the best set of estimates Y_r of the y-values. The line (20.63) is then *called the line of regression of y on x*. The minimum of $G(m, c)$ as a function of m and c occurs when $\partial G/\partial m = \partial G/\partial c = 0$, or

$$\sum_{r,s} f_{rs}(y_s - mx_r - c) = 0,$$

$$\sum_{r,s} f_{rs}x_r(y_s - mx_r - c) = 0.$$

Using equations (20.55), (20.56), (20.60) and (20.62), these conditions

become

$$\bar{y} - m\bar{x} - c = 0, \qquad (20.66)$$

$$(\mu_{11} + \bar{x}\bar{y}) - m(\sigma_x^2 + \bar{x}^2) - c\bar{x} = 0.$$

Eliminating c we find

$$\mu_{11} = m\sigma_x^2. \qquad (20.67)$$

Equation (20.66) tells us that the line of regression (20.63) passes through the mean $(\bar{x}, \bar{y})$ of the distribution. Equation (20.67) tells us that the slope of the line is $m = \mu_{11}/\sigma_x^2$; this quantity is known as the *coefficient of regression of y on x*. So the line of regression of y on x is

$$y - \bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \bar{x}). \qquad (20.68)$$

By interchanging the roles of x and y, we find that the line of regression of x on y is

$$x - \bar{x} = \frac{\mu_{11}}{\sigma_y^2}(y - \bar{y}), \qquad (20.69)$$

the coefficient of regression of x on y being μ_{11}/σ_y^2.

§ 7.2. The correlation coefficient

It is important to know how well the estimates Y_r, given by (20.64), represent the actual values y_s. We shall therefore establish a relation between the mean square deviation of y-values from their estimates, defined by

$$\begin{aligned} S_y^2 &= \frac{1}{N} \sum_{r,s} f_{rs}(y_s - Y_r)^2 \\ &= \frac{1}{N} \sum_{r,s} f_{rs}(y_s - mx_r - c)^2 \end{aligned} \qquad (20.70)$$

and the standard deviation of all y-values from the mean $\bar{y}$. If we take μ'_{11}, μ'_{20} and μ'_{02} to be the second moments defined by (20.60), (20.58) and (20.59), with $a = -c/m$ and $b = 0$, then (20.70) gives

$$\begin{aligned} S_y^2 &= \frac{1}{N} \sum_{r,s} f_{rs}[y_s^2 - 2my_s(x_r - a) + m^2(x_r - a)^2] \\ &= \mu'_{02} - 2m\mu'_{11} + m^2\mu'_{20} \\ &= [\sigma_y^2 - 2m\mu_{11} + m^2\sigma_x^2] + [\bar{y}^2 - 2m\bar{y}(\bar{x} - a) + m^2(\bar{x} - a)^2], \end{aligned}$$

using (20.11) and (20.62). The terms in the second square bracket sum to

$(\bar{y}-m\bar{x}-c)^2$, which is zero by (20.66). Substituting for m from (20.67),

$$S_y^2 = \sigma_y^2\left[1 - \frac{\mu_{11}^2}{\sigma_x^2\sigma_y^2}\right].$$

If we write

$$r = \left|\frac{\mu_{11}}{\sigma_x\sigma_y}\right|, \tag{20.71}$$

then

$$S_y^2 = \sigma_y^2(1 - r^2). \tag{20.72}$$

The *positive* quantity r defined by (20.71) is the *correlation coefficient*, and it is clear from (20.72) that $r<1$. By definition, S_y measures the deviation of y values from their estimates Y_r, while σ_y measures the deviation from the overall mean $\bar{y}$. If the line of regression is a good fit, we expect the deviations from the estimates Y_r to be on the whole smaller than those from $\bar{y}$, so that S_y will be definitely less than σ_y; then by (20.72), r will be definitely greater than zero. When the line of regression is a perfect fit to the points, every y_s coincides with its estimate Y_r, and S_y, defined by (20.70), is zero; so from (20.72), $r=1$, taking its maximum value; the points are then said to be perfectly correlated. At the other extreme when $r=0$, we have $S_y=\sigma_y$, meaning that $\bar{y}$ is as good an estimate of every y-value as we can obtain; by (20.71), $\mu_{11}=0$, so that the line of regression of y on x is simply $y=\bar{y}$. Likewise, when $r=0$, the line of regression of x on y is $x=\bar{x}$.

The condition $r=1$ for perfect correlation has the further consequence that the two lines of regression (20.68) and (20.69) coincide. As r decreases from the value unity, the angle between the lines of regression increases until when $r=0$ they are at right angles.

Example 16

In examinations at the end of two successive terms, the placings of thirteen boys in a class were (1, 2), (2, 1), (3, 4), (4, 5), (5, 9), (6, 6), (7, 3), (8, 11), (9, 7), (10, 8), (11, 10), (12, 12), (13, 13).

Find the correlation coefficient between the results of the two examinations.

Let x and y be the placings in the first and second examinations respectively. Clearly $\bar{x}=\bar{y}=7$, and by the result of Example 5, $\sigma_x=\sigma_y=[(169-1)/12]^{\frac{1}{2}}=\sqrt{14}$. The covariance μ_{11} is given by (20.60) with $a=b=7$, and is

$$\frac{1}{13}[(1-7)(2-7)+(2-7)(1-7)+(3-7)(4-7)+\ldots]$$

$$=\frac{1}{13}[30+30+12+6-4+1+0+4+0+3+12+25+36]=\frac{155}{13}.$$

Hence by (20.71),

$$r = \frac{\mu_{11}}{\sigma_x \sigma_y} = \frac{155}{182} \approx 0.85.$$

Since r is near to the value unity, the correlation between the sets of results is high.

The lines of regression (20.68) and (20.69) pass through the point (7, 7) and have slopes $\frac{155}{182}$ and $\frac{182}{155}$.

If the correlation coefficient defined by (20.71) is small compared with unity, it means either that there is no reasonably well-defined relation between the variates x and y, or that there is a relation, but that it is not approximately linear. If we suspect for example, that y is roughly proportional to $e^{\kappa x}$, where κ is some constant, we would take $\log y$ and x as the two variates; then the lines of regression should give a high correlation coefficient. It may be desirable, however, to assume a more complicated curve of regression, for example, a polynomial curve

$$y = b_0 + b_1 x + \ldots + b_n x^n, \tag{20.73}$$

where $n > 2$. The coefficients $b_0, b_1, \ldots, b_n$, in (20.73) are again chosen to minimise the mean square deviation of the ordinates y_s from their estimates Y_r, as in (20.66) and (20.67). The correlation coefficient is defined by the equation analogous to (20.71). If a polynomial curve of regression gives a correlation coefficient appreciably higher than that given by the *line* of regression, we know that the line of regression gives a poor fit to the points in the scatter diagram.

EXERCISE 20.6

1. The number N of thousands of spectators watching twenty-one home matches of a certain football team are recorded in the first row of the flowing table; the air temperature readings T(°F) at the ground on the days of the matches are recorded in the second row and the positions P in the League of the opponents (except for the first match) are recorded in the third row.

$N(10^3)$	35	40	28	34	37	29	33	34	22	20	41	24	21	30	33	29	28	39	32	37	32
T(°F)	50	52	47	60	51	55	42	39	44	42	35	31	34	39	32	46	42	51	48	57	47
P	–	1	9	19	3	12	9	4	13	17	7	15	17	5	2	14	20	6	8	16	4

(i) Draw the (N, T) and (N, P) scatter diagrams.

(ii) Find the lines of regression of N on T, T on N, N on P, and P on N; draw these lines on the scatter diagrams.

(iii) Find the correlation coefficients between N and T, and between N and P.

2. In two tests, A and B, of manual skill, a person is graded either 1, 2, 3, 4 or 5. The following table gives the numbers of people out of a group of 650 who gain particular grades in the two tests:

B \ A	1	2	3	4	5
1	38	21	12	8	3
2	9	53	45	17	5
3	4	20	84	63	19
4	1	8	24	112	37
5	0	0	2	18	47

Find the lines of regression and the correlation coefficient for these results.

3. Show that the parabola of form

$$y = b_0 + b_1x + b_2x^2$$

which gives the best 'mean square' fit of y-values to a set of points (x_r, y_s) in the scatter diagram has

$$\sum_{r,s} f_{rs}(y_s - b_0 - b_1x_r - b_2x_r^2) = 0,$$

$$\sum_{r,s} f_{rs}x_r(y_s - b_0 - b_1x_r - b_2x_r^2) = 0$$

and

$$\sum_{r,s} f_{rs}x_r^2(y_s - b_0 - b_1x_r - b_2x_r^2) = 0.$$

Find the equation of the parabola of this form which gives the best fit to the set of values of x and y set out below; draw the scatter diagram and the parabola giving the best fit.

x	1	2	3	4	5	6	7
y	4.3	5.0	5.8	7.1	9.2	11.2	12.5

ANSWERS TO EXERCISES

1.1. **1.** $(2\alpha)^{-1}$. **2.** $-\infty$. **3.** 0.
4. $\frac{1}{2}$. **5.** 1. **6.** $\frac{3}{2}$.
7. $-2b$. **8.** $\frac{5}{2}a$. **9.** 1.
10. $-a^{-2}$. **11.** ∞. **12.** 2.
13. (i) all positive integers n, $N = 1$;
(ii) all even integers, no value of N.
14. (i) $N = 1$, (ii) $N = 8$.
15. (i) $N = 1$, (ii) $N = 2$.

1.2. **1.** $\tan x + x \sec^2 x$. **2.** $\cos 2x$.
3. $(2 \cos x)/(1 - \sin x)^2$.
4. $(4 \cos \alpha)(x^2 - 1)(x^2 + 2x \cos \alpha + 1)^{-2}$.
5. $(3x^2 + 6x) \operatorname{cosec} x - (x^3 + 3x^2 + 1) \operatorname{cosec} x \cot x$.
6. $-\{(x^2 - 1) \operatorname{cosec}^2 x + 2x \cot x\}(x^2 - 1)^{-2}$.
7. $2(6x^2 + 1)/(2x^4 + 3) - 8x^3(4x^3 + 2x + 5)/(2x^4 + 3)^2$.
8. $(ab - a^2x^2 - 2abx - 2bh)(ax^2 + 2hx + b^2)^{-2}$.

1.3. **1.** $-(1 - 3x)^{-\frac{2}{3}}$. **2.** $14x(1 + x^2)^6$.
3. $3 \sin^2 x \cos x$. **4.** $9 \tan^2(3x - 5) \sec^2(3x - 5)$.
5. $(\sin 3x)^{-\frac{2}{3}} \cos 3x$. **6.** $4 \sin 2x - 6 \sin^3 2x$.
7. $-\frac{1}{2}(2x - 5)(x^2 - 5x + 8)^{-\frac{3}{2}}$. **8.** $\frac{1}{2}(1 - x)(x^2 + x + 1)^{-\frac{3}{2}}$.
9. $4x - 2(2x^2 + 1)/\sqrt{(x^2 + 1)}$. **10.** $9 \cos^2(4 - 3x) \sin(4 - 3x)$.
11. $\{(a^2 - ax - x^2)/(a^2 - x^2)\}\sqrt{\{(a - x)/(a + x)\}}$.
12. $\frac{1}{2}(\sin x - x \cos x)x^{-\frac{3}{2}}(x - \sin x)^{-\frac{1}{2}}$.

1.4. **1.** $$-\frac{a}{x^2}\tan^{-1}\frac{x}{a} + \frac{a^2}{x(a^2 + x^2)}.$$
2. $$\frac{x}{1 - x^2} + \frac{\sin^{-1} x}{(1 - x^2)^{\frac{3}{2}}}.$$ **3.** $2(1 + x^2)^{-1}$.
4. $(x^2 - 1)(x^4 + 3x^2 + 1)^{-1}$. **5.** $(1 + x^2)^{-1}$.
6. $-x^{-1}(x^2 - 1)^{-\frac{1}{2}}$. **7.** $x(1 - x^2)^{-\frac{1}{2}}$.
8. $\{\sec^2(\sin^{-1} x)\}(1 - x^2)^{-\frac{1}{2}}$. **9.** t^{-1}.
10. $-\cot \theta$. **14.** $-(x^2 - ay)(y^2 - ax)^{-1}$.

1.5. **1.** $2^n \sin(2x + \frac{1}{2}n\pi)$. **2.** $m^n \cos(mx + \frac{1}{2}n\pi)$.

3. $\frac{1}{2}(-1)^n n![(x-a)^{-(n+1)} + (x+a)^{-(n+1)}]$.

4. $\dfrac{m!}{(m-n)!}\, a^n(ax+b)^{m-n}$, $(n \leqslant m)$; 0 for $n > m$.

5. $(2n)!\, x^n/n!$.

6. $\{x^2 - n(n-1)\}\sin(x + \frac{1}{2}n\pi) - 2nx\cos(x + \frac{1}{2}n\pi)$.

7. $2^{n-2}\{4x^3 - 3n(n-1)x\}\cos(2x + \frac{1}{2}n\pi) + {}$
$+ 2^{n-3}\{12nx^2 - n(n-1)(n-2)\}\sin(2x + \frac{1}{2}n\pi)$.

12. $t(3+2t)/(2+t)$, $2(t+3)(t+1)^3/t(2+t)^3$.

1.6. **1.** (i) $2x\tan^{-1}(y/x) - x^2y/(x^2+y^2)$, (ii) $x^3/(x^2+y^2)$.

2. (i) $y^2/(x+y)^2$, (ii) $x^2/(x+y)^2$.

3. (i) $(x^2 - 2xy - y^2)/(x^2+y^2)^2$, (ii) $(x^2 + 2xy - y^2)/(x^2+y^2)^2$.

4. (i) $(2ax^3y + 3ax^2y^2 + by^4)/(x+y)^2$,
(ii) $(ax^4 + 3bx^2y^2 + 2bxy^3)/(x+y)^2$.

5. (i) $2x\tan^{-1}(x/y) + y$, (ii) $2y\tan^{-1}(x/y) - x$.

12. $\dfrac{\partial z}{\partial x} = (\cos 2y)/\cos^2(x+y)$, $\dfrac{\partial z}{\partial y} = -(\cos 2x)/\cos^2(x+y)$,

$\dfrac{\partial^2 z}{\partial x^2} = \{2\cos 2y\sin(x+y)\}/\cos^3(x+y)$,

$\dfrac{\partial^2 z}{\partial y^2} = -\{2\cos 2x\sin(x+y)\}/\cos^3(x+y)$,

2.1. **1.** $\frac{1}{25}(5x+9)^5$. **2.** $\frac{1}{35}(2-7x)^{-5}$. **3.** $-\frac{1}{2}\cos(2x-5)$.

4. $-\frac{1}{3}\tan(4-3x)$. **5.** $\frac{1}{2}\sin^{-1}\frac{1}{6}(2x-3)$. **6.** $-\frac{1}{20}\tan^{-1}\frac{1}{4}(2-5x)$.

7. $\frac{1}{2}\sec 2x$. **8.** $\frac{1}{11}\cot(3-11x)$. **9.** $-\frac{2}{3}(2-x)^{\frac{3}{2}}$.

10. $\frac{1}{6}\tan^{-1}\frac{1}{3}x^2$. **11.** $\frac{1}{6}\sin^6 x$. **12.** $-(1+\tan x)^{-1}$.

13. $\frac{1}{7}\sin(7x-5)$. **14.** $\sec x$. **15.** $\frac{1}{3}(4+x^2)^{\frac{3}{2}}$.

16. $\frac{2}{3}(3+x^3)^{\frac{1}{2}}$.

2.2. **1.** $\frac{2}{3}[\sin^{-1}\frac{1}{2}(3x-1) + \frac{1}{4}(3x-1)\{4-(3x-1)^2\}^{\frac{1}{2}}]$.

2. $-(5-4x-x^2)^{\frac{1}{2}}$. **3.** $-(9-x^2)^{\frac{1}{2}}$.

4. $\frac{8}{3}(x-2)^{\frac{3}{2}} + \frac{8}{5}(x-2)^{\frac{5}{2}} + \frac{2}{7}(x-2)^{\frac{7}{2}}$.

5. $\frac{1}{2}(x - \sin x)$. **6.** $\frac{1}{8}\cos 4\theta - \frac{1}{12}\cos 6\theta$.

7. $\frac{1}{5}\tan^5 x$. **8.** $\frac{1}{5}\sec^5 x$. **9.** $\theta + \frac{1}{2}\cos 2\theta$.

10. $\frac{1}{4}\sin 2x - \frac{1}{4}x - \frac{1}{16}\sin 4x$.

2.3. **1.** $(x^2-2)\sin x + 2x\cos x$.

2. $\frac{1}{4}(2x^2+1)\cos^{-1}x + \frac{1}{4}x(1-x^2)^{\frac{1}{2}}$.

3. $\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x$.

4. $\frac{1}{3}x^3\sin^{-1}x - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + \frac{1}{3}(1-x^2)^{\frac{1}{2}}$.

5. $(\frac{1}{4} - \frac{1}{2}x^2)\cos 2x + \frac{1}{2}x\sin 2x$.

6. $\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x$.

2.4. **1.** $\frac{1}{4}\pi a^2$. **2.** $-\frac{1}{9}\pi - \frac{2}{27}$. **3.** $\frac{1}{4}(6 - \pi)$.
4. $\frac{25}{3}\sqrt{5} - \frac{19}{5}$. **5.** $\frac{3}{8}\pi$. **6.** $\frac{2}{3}$.
7. $\frac{5}{64}\pi$. **8.** $3 - \sqrt{5} + \sin^{-1}(\frac{2}{3})$. **9.** πab.
10. $2\sqrt{2}$. **11.** $\frac{16}{3}a^2$. **12.** $4a^2$.

2.5. **1.** 1. **2.** divergent to $+\infty$.
3. $\frac{1}{3}\pi$. **4.** $\frac{1}{2}\pi$. **5.** 2.
6. $n > -1$; divergent to $+\infty$; $n < -1$; $-(n+1)^{-1}$.
8. $\frac{1}{2}\pi^2$.

3.2. **6.** (i) convergent for $0 < x < e$, divergent for $x > e$,
(ii) convergent for $0 < x < \frac{1}{4}$, divergent for $x \geqslant \frac{1}{4}$.

3.3. **1.** divergent. **2.** convergent.
3. convergent. **4.** divergent.
5. convergent. **6.** convergent.
7. convergent. **8.** convergent (see ch 6., Example 8).
10. 1.

4.1. **1.** $2x \exp x^2$. **2.** $2x^{-1}(x^2+1)^{-\frac{1}{2}}$.
3. $\{x/(a^2+x^2)\} \sec\{\log\sqrt{(a^2+x^2)}\} \tan\{\log\sqrt{(a^2+x^2)}\}$.
4. $x^{-1}(\log x)^{-1}$.
5. $\exp(\sin 2x) + 2x(\cos 2x)\exp(\sin 2x)$.
6. $(-2x^2+4x+3)\exp(-2x)$.
7. $(a^2+x^2)^{-\frac{1}{2}}$. **8.** $2\sec^3 x$.
9. $-3e^{-3x}\cos(2x - \frac{1}{4}\pi) - 2e^{-3x}\sin(2x - \frac{1}{4}\pi)$.
10. $(b^2-a^2)(a\cos x + b)^{-1}(a + b\cos x)^{-1}\sin x$.
11. $(2ab\sec^2 x)/(a^2 - b^2\tan^2 x)$.
12. $\sec x + \tan x$. **13.** $-(4\tan x)/(1+\sin^2 x)$.
14. $16x(x^4-16)^{-1}$. **15.** $2/(1-x)\sqrt{x}$.
16. $-e^{-2x}(2\cos x + 3\sin x)\cos^2 x$.
17. $2\tan 2x$. **18.** $\sec x$.
19. $(x^2+a^2)^{-\frac{1}{2}}[\log\{x+\sqrt{(x^2+a^2)}\}]^{-1}$.
20. $-\frac{1}{2}x^{\frac{1}{2}}(x^2+6x-6)(x-1)^{-\frac{3}{2}}(x-2)^{-\frac{7}{2}}$.
21. $-2(x^2-x+1)(x-2)^{-3}(x+1)^{-2}$.
22. $x\,e^x(2\sin 3x + x\sin 3x + 3x\cos 3x)$.
23. $\{m\sin x - 2x^2\sin x + 3x\cos x\}x^{m-1}\sin^2 x\exp(-x^2)$.
24. $(x\cos^2 x + \cos x\sin x - \sin x)\exp(x\sin x)$.
25. $(2\log x)x^{\log x - 1}$.
26. $(1+x)^x\log(1+x) + x(1+x)^{x-1}$.
27. $(1+x^2)^x\log(1+x^2) + 2x^2(1+x^2)^{x-1}$.
28. $\log(\log x)(\log x)^x + (\log x)^{x+1}$.
29. $x^{-1}(\sin x)^{\log x}\log(\sin x) + \cos x(\sin x)^{\log x - 1}\log x$.
30. $(\cos x)\{1+\log(\sin x)\}(\sin x)^{\sin x}$.
31. $-(\sin x)^{\cos x+1}\log\sin x + \cos^2 x(\sin x)^{\cos x - 1}$.
32. $\{\sin x\log x + x\cos x\log x + \sin x\}x^{x\sin x}$.
38. $2^n(-1)^{n-1}(n-1)!(2x+3)^{-n}$.

39. $(-1)^{n-1}(n-1)!\{(x-1)^{-n}+(x+1)^{-n}\}$.
40. $(-1)^n(n-1)!\{5^n(5x+2)^{-n}+(x+1)^{-n}\}$.
41. $5^n\,e^{3x}\cos(4x+n\alpha)$, where $\alpha=\cos^{-1}\frac{3}{5}$.
42. $2^{\frac{1}{2}n}\,e^x\sin(x+\frac{1}{4}n\pi)$.
43. $(-1)^n\,13^{\frac{1}{2}n}\,e^{-2x}\{\cos(3x-n\alpha)+\sin(3x-n\alpha)\}$, where $\alpha=\tan^{-1}\frac{3}{2}$.
44. $(-5)^n\,e^{-4x}\cos(3x+2-n\alpha)$, where $\alpha=\cos^{-1}\frac{4}{5}$.
45. $(-1)^{n-1}(n-3)!\{2x^2+4nx+n(n-1)\}(x+1)^{-n}$.
46. $(-5)^{n-3}\,e^{-5x}\{-125x^3+75nx^2-15n(n-1)x+n(n-1)(n-2)\}$.
47. $5^{n-2}\,e^{3x}[25x^2\cos(4x+n\alpha)+10nx\cos\{4x+(n-1)\alpha\}+$
$+\,n(n-1)\cos\{4x+(n-2)\alpha\}]$,
where $\alpha=\cos^{-1}\frac{3}{5}$.
50. $e^y/(1-x\,e^y)$.

4.2. **11.** $(\sinh x+\sin x)^{-2}\,2\sinh x\sin x$.
12. $2\,\mathrm{cosech}\,2x$. **13.** $\mathrm{sech}\,x$.
14. $\sec x$. **15.** $(1-x^2)^{-1}$. **16.** $\pm nx^{n-1}\sec x^n$.
17. $\{\log\sinh x+1\}\cosh x(\sinh x)^{\sinh x}$.
18. $\mathrm{sech}^2 x\exp(\tanh x)\exp\{\exp(\tanh x)\}$.
19. $x^{-1}\,\mathrm{sech}^2(\log x)\{1+\tanh^2(\log x)\}^{-\frac{1}{2}}$ or $x^{-1}\,\mathrm{sech}\,(\log x)\,\{\cosh\,(2\log x)\}^{-\frac{1}{2}}$.

5.1. **1.** $8\sinh^{-1}(\frac{1}{4}x)+\frac{1}{2}x\sqrt{(x^2+16)}$.
2. $\log(e^x+e^{-x})$.
3. $\frac{1}{4}\{(2x^2+1)\sinh^{-1}x-x\sqrt{(x^2+1)}\}$.
4. $\frac{1}{9}x^3(3\log x-1)$. **5.** $-(1+\log x)^{-1}$.
6. $\frac{1}{2}(2x+1)\tanh^{-1}(2x+1)+\frac{1}{4}\log|1-(2x+1)^2|$.
7. $\frac{1}{2}\log(4+\sqrt{15})(2-\sqrt{3})$.
8. $\frac{1}{2}(3\sqrt{5}-4)-2\log\frac{1}{2}(1+\sqrt{5})$.
9. $(\log 2)^{-1}$.
10. $\frac{13}{2}\log 13-4-\frac{5}{2}\log 5$.
11. $1-6e^{-5}$. **12.** 1.
13. $\frac{1}{5}(2e^{\pi}+1)$. **14.** $\frac{3}{8}$.

5.2. **1.** $\frac{7}{10}$. **2.** $\frac{1}{16}\pi+\frac{4}{35}$.
3. $-\frac{1}{8}\pi+\frac{2}{3}$. **4.** 0.
5. $35\pi/256a^9$. **6.** a^{-2}. **8.** $\frac{1}{60}$.
9. $\frac{1}{15}\{3x^4\sqrt{(1+x^2)}-4x^2\sqrt{(1+x^2)}+8\sqrt{(1+x^2)}\}$.
12. $\frac{28}{15}$.
14. $\{-(2\pi)^6n^4+30(2\pi)^4n^2-360(2\pi)^2\}/n^5$.
15. $\frac{1}{26}\{(\sinh\frac{1}{2}\pi)^5-2(\sinh\frac{1}{2}\pi)^3+6\sinh\frac{1}{2}\pi-6\}$.

5.3. **1.** $\frac{1}{4}\log 31$.
2. $\frac{29}{26}\log|2x-3|-\frac{11}{39}\log|3x+2|$.
3. $\frac{1}{3}x^3+x^2+3x-(x-1)^{-1}+4\log|x-1|$.
4. $\frac{1}{4}\log 2+(\tan^{-1}\frac{1}{5}\sqrt{7})/2\sqrt{7}$.
5. $\frac{1}{12}\pi+\frac{1}{3}\log 2$. **6.** $\{6+\log 3\}/4a^2$.
7. $\frac{1}{2}x^2+\frac{8}{3}\log|x^2-4|-\frac{1}{6}\log|x^2-1|$.

8. $\frac{1}{3}(1 - x)^{-1} + \frac{1}{6}\log|x^2 + x + 1| - \frac{1}{3}\log|x - 1| + \dfrac{1}{3\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}}$.

9. $\dfrac{7}{330} + \dfrac{41}{100}\log 11 + \dfrac{8}{50}\log 2 - \dfrac{139}{100}\log 3$.

10. $\frac{1}{2}x^2 + 3\log|x + 3| - \frac{3}{2}\log|x^2 - 3x + 9| - 3\sqrt{3}\tan^{-1}\dfrac{2x - 3}{3\sqrt{3}}$.

5.4. **1.** $\frac{2}{105}(15x^2 + 24x + 32)(x - 2)\sqrt{(2 - x)}$.

2. $\frac{1}{15}(3x^2 + 2x + 2)\sqrt{(2x - 1)}$.

3. $\frac{2}{5}(138 - 34\sqrt{3})$. **4.** $\frac{1}{2}\pi\sqrt{2}$.

5. $\sin^{-1}\frac{1}{2}(x - 1)$, $|x - 1| < 2$.

6. $\frac{1}{8}(4x - 7)\sqrt{(2x^2 - 7x + 5)} - \dfrac{9}{16\sqrt{2}}\cosh^{-1}\dfrac{4x - 7}{3}$.

7. $\frac{1}{2}\{3a^2\sqrt{2} - 2a^2 + a^2\log(1 + \sqrt{2})\}$.

8. $\frac{1}{3}\{x^2 + 6x + 109\}^{\frac{3}{2}} - \frac{3}{2}(x + 3)\{x^2 + 6x + 109\}^{\frac{1}{2}} - 150\sinh^{-1}\dfrac{x + 3}{10}$.

9. $6 - 3\sqrt{5} - 4\sin^{-1}(1/\sqrt{5})$.

10. $\frac{1}{3}\sqrt{3}[\log(3\sqrt{3} + \sqrt{28}) - \log(2\sqrt{3} + \sqrt{13})]$.

13. $\frac{1}{2}a^2\log(2 + \sqrt{3})$. **14.** 2.

5.5. **1.** $-\frac{1}{2}(1 + e^{2x})^{-1}$ or $\frac{1}{4}\tanh x$.

2. $\frac{3}{2}\log|e^x - 1| - \frac{1}{2}\log|e^x + 1|$.

3. $\frac{1}{2}$. **4.** $\dfrac{2}{\sqrt{7}}\tan^{-1}\dfrac{1}{\sqrt{7}}$. **5.** $\frac{1}{3}\pi$.

6. $\frac{1}{2}[2\sqrt{3} - 4 + \log 3]$. **7.** $\log 3 - \log 2$.

8. $\frac{3}{2}\log 3 - \log(3\sqrt{3} - 1)$.

9. $\frac{1}{2}(\log 5 - \log 3)$. **10.** $\frac{1}{2}$.

11. $\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\log|\sec\theta + \tan\theta|$.

12. $\frac{1}{24}\pi\sqrt{3}$. **13.** $\pi + \log 2$.

14. $\log x - \log\{1 + \sqrt{(x^2 + 1)}\} + \frac{1}{2}\sqrt{2}\log(x + 1) +$
$- \frac{1}{2}\sqrt{2}\log\{1 - x + \sqrt{2(1 + x^2)}\}$ $(x > 0)$.

15. $\frac{128}{45}a^7$. **16.** $10 - \frac{7}{2}\log 3$.

6.1. **3.** 2. **4.** m/n. **5.** $-\frac{1}{2}$.

6. $\frac{2}{3}$. **7.** $4/\pi^2$. **8.** 1.

9. $\frac{1}{2}\pi$. **10.** $\frac{1}{3}$. **11.** 1.

12. 1. **13.** 1. **14.** $\frac{2}{3}$.

18. $\frac{1}{6}$. **19.** 1.

6.2. **6.** $x - \frac{1}{3}x^3 + \frac{1}{10}x^5$.

7. $1 + \frac{1}{2}x^2 + \frac{1}{12}x^4$.

8. $-\frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{45}x^6$.

9. $1 - \frac{1}{3}x^2 - \frac{1}{45}x^4$.

10. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n x^n}{n!}\sin(2 + n\alpha)$, where $\alpha = \cos^{-1}\frac{4}{5}$.

12. $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{24}x^5$.
13. $[2^{2n-1}\{(n-1)!\}^2 x^{2n}]/(2n)!$.
14. $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^5$.

6.3. **5.** maximum $\frac{2}{27}$, minimum 0.
6. maximum 54, minimum -71.
7. maximum 2, minimum -2.
8. maximum $2^4a^{-4}5^{-\frac{5}{2}}$, minimum $-2^4a^{-4}5^{-\frac{5}{2}}$.
9. maximum $2e^{-1}$, minimum $-2e^{-1}$.
10. maximum 1, minimum $\frac{2}{3}\sqrt{2}$.
11. $\dfrac{1}{\beta - \alpha}\log(\beta/\alpha)$, provided $\alpha > 0$, $\beta > 0$; maximum value $\frac{2}{9}\sqrt{3}$.
12. maximum ordinate $(3/4e)^3$,
points of inflexion $x = 0$, $x = \frac{1}{4}(3 + \sqrt{3})$, $x = \frac{1}{4}(3 - \sqrt{3})$.
13. maximum value at $x = \cos^{-1}\frac{1}{2}(1 - \sqrt{3}) \doteqdot 111°30'$;
points of inflexion $x = 0$, $\frac{1}{3}\pi$, π.
14. minimum point $x = -\frac{1}{2}$, point of inflexion $x = -\frac{3}{4}$;
maximum point $x = -1$.
15. minimum points $x = 0$, $\pi - \beta$; maximum points $x = \beta$, π;
points of inflexion $x = \alpha$, γ, $\frac{1}{2}\pi$, $\pi - \gamma$, $\pi - \alpha$,
where $\alpha \doteqdot 20°$, $y \doteqdot 39°$, $\gamma \doteqdot 58°$.
16. $0 < x < \frac{1}{4}\pi$ increasing, $\frac{1}{4}\pi < x < \frac{2}{3}\pi$ decreasing,
$\frac{2}{3}\pi < x < \frac{3}{4}\pi$ increasing, $\frac{3}{4}\pi < x < \pi$ increasing;
greatest value in the range is at $x = \frac{1}{4}\pi$ when $f(x) = \frac{1}{2} + \frac{2}{3}\sqrt{2}$.
17. maximum point $x = \tan^{-1}2 - \frac{1}{6}\pi$,
minimum point $x = \tan^{-1}2 + \frac{5}{6}\pi$.
18. $\theta = \frac{1}{6}\pi$, r has a maximum value,
$\theta = \frac{1}{3}\pi$, r has a minimum value,
$\theta = \frac{1}{2}\pi$, r has a maximum value.
19. Sides of square pieces 6 in.
20. $\theta = 2\pi(1 - \frac{1}{3}\sqrt{6})$, maximum volume $\frac{2}{27}\pi a^3\sqrt{3}$.
21. $\theta = \pm\sin^{-1}(1/\sqrt{3})$.
22. (i) $h < a\sqrt{2}$, maximum positions $\theta = \frac{1}{4}\pi$, $\frac{5}{4}\pi$;
minimum positions $\theta = \frac{5}{4}\pi - \alpha$, $\frac{5}{4}\pi + \alpha$,
where $\alpha = \cos^{-1}(h/\sqrt{2}a)$;
(ii) $h > a\sqrt{2}$, maximum position $\theta = \frac{1}{4}\pi$, minimum position $\theta = \frac{5}{4}\pi$.

7.1. **1.** $-3 - 4i$. **2.** $\frac{13}{5} - \frac{16}{5}i$. **3.** $2 - i$.
4. $-\frac{3}{5} - \frac{11}{5}i$. **5.** $-\frac{4}{5} + \frac{7}{5}i$. **6.** $\frac{23}{25} + \frac{89}{25}i$.
In Nos. 7–13, r is used for the moduli, and θ for the arguments.
7. $r = 2\sqrt{2}$, $\theta = -\frac{3}{4}\pi$.
8. $r = 2\sqrt{3}$, $\theta = \frac{2}{3}\pi$.
9. $r = 2$, $\theta = -\frac{1}{6}\pi$.
10. $r = \sqrt{\{4 + 2\sqrt{2}\}}$, $\theta = \tan^{-1}(\sqrt{2} - 1)$.
11. $r = 2$, $\theta = \pi$.
12. $r = 2\cos\frac{1}{2}\alpha$, $\theta = \frac{1}{2}\alpha$.

13. For $\alpha > \beta$, $r = 2\sin\frac{1}{2}(\alpha - \beta)$, $\theta = \frac{1}{2}(\pi + \alpha + \beta)$;
for $\alpha < \beta$, $r = 2\sin\frac{1}{2}(\beta - \alpha)$, $\theta = -\frac{1}{2}(\pi - \alpha - \beta)$.

14. (i) $4\sqrt{3}(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi) = -6 + 2i\sqrt{3}$;
(ii) $12(\cos 0 + i\sin 0) = 12$.

15. (i) $\dfrac{r_1}{r_2} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$, $\theta_1 - \theta_2 = \frac{1}{3}\pi - \frac{1}{4}\pi$;
$\frac{1}{2}\{(\sqrt{3} + 1) + i(\sqrt{3} - 1)\}$;
(ii) $\dfrac{r_1}{r_2} = \frac{3}{2}$, $\theta_1 - \theta_2 = \frac{1}{6}\pi$; $\frac{1}{4}(3\sqrt{3} + 3i)$.

16. (i) $r = \frac{1}{4}\sqrt{2}$, $\theta = -\frac{1}{4}\pi$;
(ii) $r = \sqrt{2}\cos\frac{1}{2}\alpha$, $\theta = \frac{1}{2}\alpha - \frac{1}{4}\pi$;
(iii) $r = \dfrac{1}{2\cos\alpha}$, $\theta = -\alpha$.

17. $-5 - i$. **18.** r^{-1}, $-\theta$.

19. (i) $(x - 1)^2 + y^2 = 4$, (ii) $x = 2$.

20. greatest value is $3\sqrt{2}$, $z = -2i$;
least value is zero, $z = i$.

21. $(2i + 1)z_1 = (2 + i)z_2$.

22. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

23. $\theta = \frac{1}{2}\pi$.

24. $c = (a - p^2b)/(1 - p^2)$,

$$r = \frac{p}{1 - p^2}|a - b| \text{ or } r^2 = \frac{p^2}{(1 - p^2)^2}(a - b)(a^* - b^*).$$

25. Centre is $\alpha/(\alpha^2 - r^2)$; exceptional case is when $\alpha^2 = r^2$, locus of z is a circle passing through the origin.

27. Centre $z = i$, radius 2.

28. Locus of P is a circle on P_1P_2 as diameter.

7.2. **2.** $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$,

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

6. $2\sin\frac{1}{2}|\theta - \varphi|$.

7. (i) $\cos\alpha + i\sin\alpha$, (ii) $\cos 5\theta - i\sin 5\theta$.

7.3. **1.** $-i$, $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, $-\frac{1}{2}\sqrt{3} + \frac{1}{2}i$;
$(x^2 + 1)\{(x - \frac{1}{2}\sqrt{3})^2 + \frac{1}{4}\}\{(x + \frac{1}{2}\sqrt{3})^2 + \frac{1}{4}\}$.

2. $z_k = a + ib + R\left\{-\sin\dfrac{2k\pi}{n} + i\cos\dfrac{2k\pi}{n}\right\}$,
$k = 1, 2, \ldots, (n - 1)$.

3. $\pm 2^{-\frac{3}{4}}(\sqrt{3} + i)$, $\pm 2^{-\frac{3}{4}}i(\sqrt{3} + i)$.

4. $\pm\{(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}) + (\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}})i\}$, $\pm(1 - i)\sqrt{2}$,
$\pm\{(\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}) + (\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}})i\}$.

5. $\cos\frac{1}{18}\pi + i\sin\frac{1}{18}\pi$, $\cos\frac{13}{18}\pi + i\sin\frac{13}{18}\pi$, $\cos\frac{25}{18}\pi + i\sin\frac{25}{18}\pi$.
7. $\pm\frac{1}{2}\{\sqrt{6} + i\sqrt{2}\}$, $\pm\frac{1}{2}\{\sqrt{6} - i\sqrt{2}\}$.
8. $(1 + i\cot\frac{1}{5}n\pi)$, $n = 1, 2, 3, 4$.

7.4. **1.** $\sinh 3\cos 1 + i\cosh 3\sin 1$. **2.** $\pm i(2n\pi \pm \frac{1}{3}\pi)$, n an integer.
3. $-i\coth\frac{1}{2}\pi$. **4.** $e^{-\frac{1}{2}\pi}\, i$. **5.** $1 + \frac{1}{2}(4n + 1)\pi i$, n an integer.
6. $\log 5 - i(2n\pi + \cos^{-1}\frac{3}{5})$, n an integer.
7. $e^{-\frac{1}{4}\pi}\cos(\log 2) + i\, e^{-\frac{1}{4}\pi}\sin(\log 2)$.
8. $\frac{1}{2}\log 3 + \frac{1}{2}(2n + 1)\pi i$, n an integer.

8.1. **1.** $$\frac{\partial u}{\partial x} = \frac{u(1 - 2x^2) + 2v}{2x(u + v)^2}, \qquad \frac{\partial u}{\partial y} = \frac{u(1 - 2y^2) + 2v}{2y(u + v)^2},$$
$$\frac{\partial v}{\partial x} = \frac{v(2x^2 - 1) + 4ux^2}{2x(u + v)^2}, \qquad \frac{\partial v}{\partial y} = \frac{v(2y^2 - 1) + 4uy^2}{2y(u + v)^2}.$$
2. $$\frac{\partial z}{\partial x} = \frac{2v\exp(u^2 + v^2)}{3uv + 1}.$$
3. $$\frac{\partial v}{\partial y} = \frac{v\{\log v - y(2u^2 - x)\}}{2v^2 - 2u^2 + x - y}, \qquad \frac{\partial u}{\partial x} = \frac{u\{\log u - x(2v^2 - y)\}}{2u^2 - 2v^2 - x + y},$$
$$\frac{\partial u}{\partial y} = \frac{u\{\log v - y(2v^2 - y)\}}{2u^2 - 2v^2 - x + y}.$$
5. $$\frac{\partial u}{\partial x} = \frac{x^2 - v}{u(u - v)}.$$

8.2. **2.** $$\frac{\partial z}{\partial x} = -\frac{c^3x^2}{a^3z^2}, \qquad \frac{\partial x}{\partial y} = -\frac{a^3y^2}{b^3x^2}.$$
5. $$\frac{dy}{dx} = -1, \qquad \frac{dz}{dx} = 0, \qquad \frac{d^2y}{dx^2} = -\frac{4}{5}, \qquad \frac{d^2z}{dx^2} = \frac{4}{5}.$$

8.3. **3.** $$\frac{\partial z}{\partial x} = (nx^{n-1} + yx^{n-2})\,e^{-y/x}, \qquad \frac{\partial z}{\partial y} = -x^{n-1}\,e^{-y/x}, \quad n = -1.$$
5. $$\frac{-(-1)^{r+s+t}a^rb^sc^t}{(ax + by + cz)^{r+s+t}}.$$ **6.** $\alpha = 0$ or $\alpha = -1$.

8.4. **9.** $$\frac{\partial^2 z}{\partial u\partial v} = -\frac{2x}{y}\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\partial y} - \frac{2y}{x}\frac{\partial^2 z}{\partial y^2} - \frac{2}{y}\frac{\partial z}{\partial x} - \frac{2}{x}\frac{\partial z}{\partial y}.$$

8.6. **1.** $$(1 + h)^2(2 + k) + (1 + h)(2 + k)^2 + 1 = \\ = 7 + (8h + 5k) + \frac{1}{2!}(4h^2 + 12hk + 2k^2) + \frac{1}{3!}(6h^2k + 6hk^2).$$
2. $$8 + 8(x - 1) + 9(y - 2) + 12(x - 1)(y - 2) + 5(y - 2)^2 + \\ + 6(x - 1)(y - 2)^2 + (y - 2)^3 + (x - 1)(y - 2)^3$$

3. $1 + 2xy - y^2 + \frac{1}{2}(4x^2y^2 - 4xy^3 + y^4)$.

5. $1 + xy + \dfrac{x^2y^2}{2!} + \dfrac{x^3y^3}{3!}$.

6. $y + xy + \dfrac{1}{3!}(3x^2y - y^3)$.

7. $x - y$.

8.7. 1. $(0, -1)$, $(0, 3)$, $(1, 1)$ stationary points; $(1, 1)$ minimum point.
2. $(0, 0)$, $(2\sqrt{2}, 4)$, $(-2\sqrt{2}, 4)$ stationary points; $(0, 0)$ maximum point.
3. $(0, 0)$, $(3, 0)$, $(0, 4)$, $(1, \frac{4}{3})$ stationary points; $(1, \frac{4}{3})$ maximum point.
4. $(3, 2)$ maximum point. All points on the lines $x = 0$ and $y = 0$ are stationary points for which $f_{aa}f_{bb} - f_{ab}^2 = 0$.
5. $(\frac{1}{3}(\sqrt{6} - 1), \frac{2}{3})$, $(-\frac{1}{3}(\sqrt{6} + 1), \frac{2}{3})$.
6. $(\frac{1}{10}\pi, \frac{1}{5}\pi)$, $(\frac{1}{2}\pi, \pi)$, $(\frac{7}{10}\pi, \frac{2}{5}\pi)$ maximum points,
$(\frac{3}{10}\pi, \frac{3}{5}\pi)$, $(\frac{1}{2}\pi, 0)$, $(\frac{9}{10}\pi, \frac{4}{5}\pi)$ saddle points.
7. (a, a) minimum point, $(0, 0)$ saddle point.
8. $x = 3^{-\frac{1}{2}}$, $y = \frac{1}{3}\pi$ a minimum point.

8.8. 1. (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$, $(-a, -a, a)$.
2. $\left(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}\right)$ giving 8 points;
$(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.
4. $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$ for value a^2,
$(-\frac{1}{3}a, \frac{2}{3}a, \frac{2}{3}a)$, $(\frac{2}{3}a, -\frac{1}{3}a, \frac{2}{3}a)$, $(\frac{2}{3}a, \frac{2}{3}a, -\frac{1}{3}a)$ for value $\frac{11}{27}a^4$.

5. The three roots of the cubic in λ
$(a - \lambda)(b - \lambda)(c - \lambda) + 2fgh - (a - \lambda)f^2 - (b - \lambda)g^2 - (c - \lambda)h^2 = 0$.

9.3. 1. (i) $-8.815\boldsymbol{i} + 3.377\boldsymbol{j}$;
(ii) $6.285\boldsymbol{i} + 30.223\boldsymbol{j}$;
(iii) $7.477\boldsymbol{i} + 6.382\boldsymbol{j}$.

9.4. 1. $13\boldsymbol{i} - 13\boldsymbol{j} + 7\boldsymbol{k}$, $\sqrt{30}$, $5\sqrt{2}$, $\sqrt{299}$, $\sqrt{387}$.
2. $AB = -6\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$, $OP = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$, $|OP| = 3\sqrt{2}$.
3. $\frac{10}{3}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$.

9.5. 1. $\sqrt{6}$, $\sqrt{26}$, $(2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})/\sqrt{6}$,
$(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})/\sqrt{26}$, $\cos^{-1}(1/2\sqrt{39})$.
3. $\frac{1}{2}(10\boldsymbol{i} - 4\boldsymbol{j} - 17\boldsymbol{k})$, $\cos^{-1}(269/9\sqrt{1110})$,
$\cos^{-1}(136/9\sqrt{445})$, $(10\boldsymbol{i} - 4\boldsymbol{j} - 17\boldsymbol{k})/9\sqrt{5}$.

9.6. 1. $\frac{1}{3}(18\boldsymbol{i} - 8\boldsymbol{j} - 9\boldsymbol{k})$, $40\boldsymbol{i} + 27\boldsymbol{j} + 56\boldsymbol{k}$.

9.8. 1. $\boldsymbol{x} = (\boldsymbol{c} \times \boldsymbol{d})/(\boldsymbol{a}\cdot\boldsymbol{c}) + \lambda\boldsymbol{a}$, ($\lambda$ any value).
2. $\alpha(\alpha^2 + p^2)\boldsymbol{x} = \alpha^2\boldsymbol{a} - \alpha(\boldsymbol{p} \times \boldsymbol{b}) + (\boldsymbol{p}\cdot\boldsymbol{a})\boldsymbol{p}$,
$\alpha(\alpha^2 + p^2)\boldsymbol{y} = \alpha^2\boldsymbol{b} - \alpha(\boldsymbol{p} \times \boldsymbol{a}) + (\boldsymbol{p}\cdot\boldsymbol{b})\boldsymbol{p}$.

10.1. **2.** (i) $\sqrt{74}$, $(3, 7, -4)/\sqrt{74}$;
(ii) $3\sqrt{10}$, $(1, -5, -8)/3\sqrt{10}$;
(iii) $10\sqrt{2}$, $(3, -5, 4)/5\sqrt{2}$;
$\cos^{-1}(-2/15\sqrt{5})$.

3. $\cos^{-1}(7/\sqrt{78})$; $(3, -10, 7)/5$

$$\frac{x-1}{-1} = \frac{y+2}{0} = \frac{z-1}{2}.$$

4. $(4, 3, 2)/\sqrt{29}$.

5. $5\sqrt{2}$; $(-5, 4, -3)/5\sqrt{2}$.

10.2. **1.** $\dfrac{x}{-2} = \dfrac{y-1}{7} = \dfrac{z-2}{13}$,

$\boldsymbol{r}\cdot(2\boldsymbol{i} - 7\boldsymbol{j} - 13\boldsymbol{k}) = 1$.

2. $\boldsymbol{r}\cdot(\boldsymbol{i} - 20\boldsymbol{j} + 27\boldsymbol{k}) = 14$.

3. $\boldsymbol{r}\cdot(13\boldsymbol{i} + 20\boldsymbol{j} - 69\boldsymbol{k}) = 0$.

4. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z}{4}$.

5. $\boldsymbol{r}\cdot(p_2\boldsymbol{u}_1 - p_1\boldsymbol{u}_2) = 0$.

6. $29x - 27y - 22z = 85$.

7. $(-\frac{3}{7}, \frac{8}{7}, \frac{5}{7})$.

10.3. **1.** $\dfrac{x+1}{3} = \dfrac{y}{0} = \dfrac{z-1}{-1}$, $\sqrt{10}$, $x - 5y + 3z = 2$.

2. $(1, 2, -1)$; $59x + 191y + 63z = 378$.

3. $\dfrac{x+1}{5} = \dfrac{y+1}{8} = \dfrac{z+2}{11}$, $5x + 8y + 11z = -35$.

4. $(\frac{2}{3}, \frac{2}{3}, \frac{5}{3})$. **5.** $\frac{4}{7}\sqrt{61}$.

8. $\sqrt{35}$, $\dfrac{x-2}{3} = \dfrac{y+3}{-5} = \dfrac{z-4}{1}$.

9. $\sqrt{14}$; $(1, 2, 1)$; $(-2, 3, -1)$.

10. $(-1, -1, 1)/\sqrt{3}$; $2\sqrt{3}$; $(-11, -16, -9)/3$; $(-5, -10, -15)/3$.

11. $x + y - z = -6$; $2\sqrt{3}$.

13. $4x + y - 3z = 12$, $\dfrac{x-2}{4} = \dfrac{y-1}{-1} = \dfrac{z+1}{5}$,

$$\frac{x-2}{1} = \frac{y-1}{-16} = \frac{z+1}{-4}.$$

14. $\dfrac{x-4}{3} = \dfrac{y+3}{-5} = \dfrac{z-7}{4}$; $(1, 2, 3)$; $(-2, 7, -1)$.

15. $(2, -1, 2)$; $\dfrac{x-2}{-2} = \dfrac{y+1}{6} = \dfrac{z-2}{3}$.

16. $\dfrac{x+2}{1} = \dfrac{y-3}{-2} = \dfrac{z}{1}$; $3x + 2y + z = 0$.

19. The line defined by the two planes
$9x - 3y - 8 = 0, \quad 15x - 3z - 16 = 0.$

11.1. **1.** $3r^2 \dfrac{dr}{du} \boldsymbol{r} + r^3 \dfrac{d\boldsymbol{r}}{du} + \boldsymbol{a} \times \dfrac{d\boldsymbol{r}}{du}.$

2. $2(a\boldsymbol{r} + r\boldsymbol{b}) \cdot \left(a \dfrac{d\boldsymbol{r}}{du} + \dfrac{dr}{du} \boldsymbol{b}\right).$

3. $\dfrac{1}{r^3} \dfrac{d\boldsymbol{r}}{du} - \dfrac{3}{r^4} \dfrac{dr}{du} \boldsymbol{r}.$

4. $2\boldsymbol{r} \cdot \dfrac{d\boldsymbol{r}}{du} - \dfrac{2}{r^4} \left(\boldsymbol{r} \cdot \dfrac{d\boldsymbol{r}}{du}\right).$

5. $m\left(\dfrac{d\boldsymbol{r}}{du} \cdot \dfrac{d^2\boldsymbol{r}}{du^2}\right).$

6. $\dfrac{1}{(\boldsymbol{a} \cdot \boldsymbol{r})^2} \left(\dfrac{d\boldsymbol{r}}{du} . \boldsymbol{b}\right) - \dfrac{2(\boldsymbol{r} \cdot \boldsymbol{b})}{(\boldsymbol{a} \cdot \boldsymbol{r})^3} \left(\boldsymbol{a} \cdot \dfrac{d\boldsymbol{r}}{du}\right).$

7. (i) $\boldsymbol{r} \cdot (\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}), \ \boldsymbol{r} \cdot (\ddot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}) + \boldsymbol{r} \cdot (\dot{\boldsymbol{r}} \times \ddddot{\boldsymbol{r}}),$

(ii) $\dot{\boldsymbol{r}} \times (\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) + \boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}),$

$\ddot{\boldsymbol{r}} \times (\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) + 2\boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}) + \boldsymbol{r} \times (\ddot{\boldsymbol{r}} \times \dddot{\boldsymbol{r}}) + \boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \ddddot{\boldsymbol{r}}).$

9. $\boldsymbol{r}_u = (\boldsymbol{i} \cos \theta + \boldsymbol{j} \sin \theta) - 2u\boldsymbol{k},$
$\boldsymbol{r}_\theta = u(-\boldsymbol{i} \sin \theta + \boldsymbol{j} \cos \theta),$

$-\sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j} + \boldsymbol{k}, \ -2\boldsymbol{k}, \ -\dfrac{1}{\sqrt{2}} (\boldsymbol{i} - \boldsymbol{j}), -\dfrac{1}{\sqrt{2}} (\boldsymbol{i} + \boldsymbol{j}).$

10. (i) $\boldsymbol{r} = \frac{1}{2}u^2\boldsymbol{a},$
(ii) $\boldsymbol{r} = \lambda\boldsymbol{a} - \frac{1}{2}u^2(\boldsymbol{a} \times \boldsymbol{b})/a^2$, λ any value,
(iii) $\boldsymbol{r} = \frac{1}{6}u^3\boldsymbol{a} + \frac{1}{2}u^2\boldsymbol{b}.$

11.2. **1.** $\frac{1}{2}a \log \frac{1}{2}\left(1 - \dfrac{1}{\sqrt{2}}\right).$ **2.** $m = \pm\sqrt{2}.$

11.3. **1.** (i) $\boldsymbol{r} \cdot \{2\boldsymbol{i} - 2u\boldsymbol{j} + (3u^2 - 2u)\boldsymbol{k}\} = 2u^3 - 9u^2 + 6u,$
(ii) $\boldsymbol{r} \cdot (3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = 5.$

2. $\boldsymbol{r} \cdot \{b(2a \cos \theta - a\theta \sin \theta)\boldsymbol{i} + b(2a \sin \theta + a\theta \cos \theta)\boldsymbol{j} +$
$- (2a^2 + a^2\theta^2)\boldsymbol{k}\} = -a^2b\theta^3,$

$$\kappa = \frac{\{b^2(4a^2 + a^2\theta^2) + (2a^2 + a^2\theta^2)^2\}^{\frac{1}{2}}}{(a^2 + b^2 + a^2\theta^2)^{\frac{3}{2}}},$$

$$\tau = \frac{b(6a^2 + a^2\theta^2)}{b^2(4a^2 + a^2\theta^2) + (2a^2 + a^2\theta^2)^2}.$$

4. $\boldsymbol{r} = \frac{1}{4}u_0^4\boldsymbol{i} + \frac{1}{3}u_0^3\boldsymbol{j} + \frac{1}{2}u_0^2\boldsymbol{k} + \lambda(u_0^3\boldsymbol{i} + u_0^2\boldsymbol{j} + u_0\boldsymbol{k});$
$u_0 = -1, \ (x = \frac{1}{4}, y = -\frac{1}{3}, z = \frac{1}{2});$
$u_0 = -2, \ (x = 4, y = -\frac{8}{3}, z = 2).$

5. $\boldsymbol{r} \cdot (2u^2\boldsymbol{i} - 2u\boldsymbol{j} + \boldsymbol{k}) = 2u^3.$

7. $\boldsymbol{r} = x_0\boldsymbol{i} + \sqrt{(2ax_0 - x_0^2)}\boldsymbol{j} - a\log\left(1 - \dfrac{x_0}{2a}\right)\boldsymbol{k} +$

$$+ \lambda\left\{\boldsymbol{i} + \frac{a - x_0}{\sqrt{(2ax_0 - x_0^2)}}\boldsymbol{j} + \frac{a}{2a - x_0}\boldsymbol{k}\right\},$$

$$a\log\left\{\frac{\sqrt{(2a)} + \sqrt{x_0}}{\sqrt{(2a)} - \sqrt{x_0}}\right\}.$$

11.4. **1.** π. **2.** (i) 5, (ii) $\frac{123}{20}$. **3.** $-\frac{250}{3}$.
4. -4π. **5.** -100π. **6.** $-\frac{3}{2}$.

7. $\frac{5}{8}\pi - 1 + \dfrac{3}{2\pi} - \dfrac{8}{\pi^2}$.

8. 0. **9.** 1. **10.** $\log 6 - 2$.

12.1. **1.** $\begin{pmatrix} 4 & 0 & 2 & 5 \\ 2 & 4 & -2 & 5 \end{pmatrix}$, $\begin{pmatrix} 2-i & -4+3i & 5-4i & -1 \\ -4-5i & -3+i & -5-i & -4 \end{pmatrix}$,

$\begin{pmatrix} 1+8i & -2 & -3-6i & -4i \\ 2-8i & 4-2i & -1 & -1-2i \end{pmatrix}$,

$\begin{pmatrix} 3-5i & 10-9i & 6i & 2-4i \\ -4+7i & 9-5i & 12+3i & 22-2i \end{pmatrix}$,

$\begin{pmatrix} 3+2i & -10-3i & 22+4i & -12 \\ -32+5i & -13-i & 6+i & -2 \end{pmatrix}$,

$\begin{pmatrix} 6-3i & -12i & 22+10i & -10-4i \\ -36+12i & -4-6i & 18+4i & 20-2i \end{pmatrix}$,

$\begin{pmatrix} 9-i & 10+3i & -16-4i & 27 \\ 38-5i & 25+i & -12-i & 17 \end{pmatrix}$.

2. $\lambda = -3$, $\mu = 2$.

3. (i) None.
(ii) $\alpha = \pm\pi$, $\alpha = \pm 2\pi$, ...
(iii) $\alpha = \pm 2\pi$, $\alpha = \pm 4\pi$, ...

12.2. **1.** $\boldsymbol{AD} = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \end{pmatrix}$, $\boldsymbol{AE} = \begin{pmatrix} 8 & 0 & 1 & -5 \\ 2 & 3 & -2 & 1 \end{pmatrix}$,

$\boldsymbol{BC} = \begin{pmatrix} 17 & 20 \\ 4 & 1 \\ -13 & 8 \end{pmatrix}$, $\boldsymbol{CA} = \begin{pmatrix} 8 & -2 \\ 3 & 9 \\ 3 & 0 \end{pmatrix}$,

$\boldsymbol{CD} = \begin{pmatrix} 10 & 16 & 12 \\ -6 & -7 & -2 \\ 3 & 5 & 4 \end{pmatrix}$, $\boldsymbol{CE} = \begin{pmatrix} 32 & 8 & -2 & -14 \\ -14 & 3 & -4 & 11 \\ 10 & 3 & -1 & -4 \end{pmatrix}$,

$\boldsymbol{DB} = \begin{pmatrix} 5 & 12 & 1 \\ -3 & -3 & -3 \end{pmatrix}$, $\boldsymbol{DC} = \begin{pmatrix} 13 & 18 \\ -5 & -6 \end{pmatrix}$,

$\boldsymbol{EF} = \begin{pmatrix} 7 & 8 & 13 \\ -3 & -1 & -4 \end{pmatrix}$,

$$\boldsymbol{FB} = \begin{pmatrix} 1 & 9 & -1 \\ 3 & 6 & 0 \\ 4 & 9 & 5 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{FC} = \begin{pmatrix} 9 & 8 \\ 6 & 11 \\ 13 & 9 \\ 0 & 0 \end{pmatrix}.$$

3. $\boldsymbol{A}$ and $\boldsymbol{B}$ are square $(n \times n)$ matrices, and $\boldsymbol{AB} = \boldsymbol{BA}$.

5. $$\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & -2 & 3 & -2 \\ 4 & 3 & -1 & 0 \\ 2 & -2 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & -2 & 2 & 2 \\ 2 & 0 & 0 & -3 \\ -2 & 0 & 0 & 6 \\ -2 & 3 & -6 & 0 \end{pmatrix}.$$

12.3. **1.** $\delta_{rs} + \delta_{r(s-1)}$.

3. $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \quad \boldsymbol{A}^n = 2^{n-1}\boldsymbol{A}.$

4. $\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$

12.4. **1.** $379, \; 103 - 47i$. **2.** $519, \; 0$.

3. $(\alpha + \beta + \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma)(-\alpha + \beta + \gamma)$,
$(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta)$.

4. $(\alpha + 3)(\alpha - 1)^3, \; (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$,
$-(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma), \; (\alpha - \beta + \gamma)^2, \; \alpha^4$.

13.1. **1.** $[(\alpha - 4)(\alpha + 1)]^{-1} \begin{pmatrix} \alpha - 2 & -2 \\ -3 & \alpha - 1 \end{pmatrix}; \quad \alpha = 4, -1.$

2. $\frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}, \; \frac{1}{3}\begin{pmatrix} -2 & 1 & 7 \\ -7 & 2 & 20 \\ -10 & 2 & 29 \end{pmatrix}$, none.

3. $$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

13.2. **1.** $x_1 = 3, \; x_2 = x_3 = 2$.

2. (i) $\alpha = 5; \quad x = \frac{1}{2}z = -\frac{1}{2}y;$
$\alpha = 26; \; x/26 = -y/10 = z/94.$
(ii) $\alpha = 1; \; x/12 = y/7 = -z/2;$
$\alpha = 2; \; x/4 = y/2 = -z.$

3. $[(\alpha - 1)(\alpha + 2)]^{-1} \begin{pmatrix} \alpha - 3 & \alpha + 3 & -2 \\ 1 - 2\alpha & \alpha^2 + 2\alpha - 1 & -\alpha \\ 5 & -3\alpha - 7 & \alpha + 4 \end{pmatrix} \begin{pmatrix} 10 \\ 6 \\ 3, 2 \end{pmatrix}; \; \alpha = 1, -2.$

13.3. **1.** (i) $\lambda_1 = 1, \; \boldsymbol{x}_1 \propto (2, -1); \; \lambda_2 = 4, \; \boldsymbol{x}_2 \propto (1, 1).$
(ii) $\lambda_1 = 1 + i, \; \boldsymbol{x}_1 \propto (2 + i, 1); \; \lambda_2 = 1 - i, \; \boldsymbol{x}_2 \propto (2 - i, 1).$
(iii) $\lambda_1 = 2, \; \boldsymbol{x}_1 \propto (3, -2); \; \lambda_2 = 7 + 5i, \; \boldsymbol{x}_2 \propto (1, 1).$

2. (i) $\lambda_1 = 0,\ \boldsymbol{x}_1 \propto (1, 2, 2);\ \lambda_2 = 1,\ \boldsymbol{x}_2 \propto (1, 1, 1);$
$\lambda_3 = 2,\ \boldsymbol{x}_3 = (0, 1, 2).$

(ii) $\lambda_1 = 1,\ \boldsymbol{x}_1 \propto (1, 2, -1);\ \lambda_2 = -2,\ \boldsymbol{x}_2 \propto (1, 1, -2);$
$\lambda_3 = 3,\ \boldsymbol{x}_3 \propto (1, 3, 1).$

13.4. **1.** (i) $\lambda_1 = 5,\ \boldsymbol{u}_1 = \dfrac{1}{\sqrt{5}}(2, 1);\ \lambda_2 = -5,\ \boldsymbol{u}_2 = \dfrac{1}{\sqrt{5}}(1, -2).$

(ii) $\lambda_1 = 8,\ \boldsymbol{u}_1 = \dfrac{1}{\sqrt{5}}(2, 1);\ \lambda_2 = -7,\ \boldsymbol{u}_2 = \dfrac{1}{\sqrt{5}}(1, -2).$

2. (i) $\lambda_1 = -1,\ \boldsymbol{u}_1 = \tfrac{1}{2}(\sqrt{2}, -1, -1);\ \lambda_2 = 3,\ \boldsymbol{u}_2 = \tfrac{1}{2}(\sqrt{2}, 1, 1);$
$\lambda_3 = 5,\ \boldsymbol{u}_3 = \dfrac{1}{\sqrt{2}}(0, -1, 1).$

(ii) $\lambda_1 = 0,$
$\boldsymbol{x}_1 \propto (28a^2 - 48ab + 18b^2,\ 18a^2 - 36ab + 36b^2,\ 36a^2 - 39ab - 36b^2);$
$\lambda_2 = a + 9b,$
$\boldsymbol{x}_2 \propto (32a^2 - 102ab + 72b^2, -16a^2 + 54ab - 18b^2, -24a^2 + 87ab - 72b^2);$
$\lambda_3 = a - 9b,$
$\boldsymbol{x}_3 \propto (32a^2 - 30ab - 36b^2,\ -16a^2 + 54ab - 18b^2,\ -24a^2 + 15ab + 90b^2).$

13.5. **1.** $\lambda_1 = 2,\ \boldsymbol{u}_1 = \tfrac{1}{3}(1, -2, 2);\ \lambda_2 = -2,\ \boldsymbol{u}_2 = \tfrac{1}{3}(2, -1, -2);$
$\lambda_3 = 3,\ \boldsymbol{u}_3 = \tfrac{1}{3}(2, 2, 1);$

$$\boldsymbol{U} = \tfrac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix};\quad 2y_1'^2 - 2y_2'^2 + 3y_3'^2 = 3.$$

2. $3x_1^2 + 5x_2^2 - x_3^3 = 1;$

$\boldsymbol{u}_1 = \tfrac{1}{2}(\sqrt{2}, 1, 1);\ \boldsymbol{u}_2 = \dfrac{1}{\sqrt{2}}(0, -1, 1);\ \boldsymbol{u}_3 = \tfrac{1}{2}(\sqrt{2}, -1, -1).$

3. $2x_1^2 - x_2^2 + x_3^2 = 1,$
$3x_1^2 + x_2^2 \qquad = 1.$

5. (i) $\boldsymbol{A}' = \boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U}$ Hermitian when $\boldsymbol{U}$ is unitary; (ii), (iii), (iv) unaltered.

6. (i) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};\quad \boldsymbol{U} = \dfrac{1}{3}\begin{pmatrix} 1 & 2(1+i) \\ -2(1-i) & 1 \end{pmatrix}.$

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix};\quad \boldsymbol{U} = \tfrac{1}{2}\begin{pmatrix} \sqrt{2} & i & 1 \\ i\sqrt{2} & 1 & -i \\ 0 & -i\sqrt{2} & \sqrt{2} \end{pmatrix}.$

14.1. **1.** (i) $2(\xi, \eta)\,\delta\xi,\ \ -2(\eta, -\xi)\,\delta\eta;$
ξ in $(0, \infty)$, η in $(-\infty, \infty)$.

(ii) $\delta\xi[e^\xi\{(1-\eta)\cos\eta - \xi\sin\eta\},\ e^\xi\{(1-\eta)\sin\eta + \xi\cos\eta\}],$
$\delta\eta[-e^\xi\{(1-\eta)\sin\eta + \xi\cos\eta\},\ e^\xi\{(1-\eta)\cos\eta - \xi\sin\eta\}];$
ξ in $(0, \infty)$, η in $(0, 2\pi)$.

(iii) $\dfrac{c\,\delta u}{(\cosh u - \cos v)^2}(-\sinh u \sin v,\ 1 - \cosh u \cos v),$

$$\frac{-c\delta v}{(\cosh u - \cos v)^2}(1 - \cosh u \cos v, \sinh u \sin v);$$
u in $(-\infty, \infty)$, v in $(0, \pi)$.

2. (i) λ in $(0, \infty)$, μ in $(0, \pi)$, φ in $(0, \pi)$.
(ii) ξ in $(0, \infty)$, η in $(0, 2\pi)$, φ in $(0, \pi)$.

14.2. **1.** $\frac{37}{24}\pi a^6$. **2.** $\frac{1}{2}\pi ab$.
3. $\frac{1}{8}(e^{16} - 16e)$. **4.** $\sqrt{(-a_3)}(\sqrt{a_2} - \sqrt{a_1})$.
5. $\frac{1}{10}e(e^2 - 1) - \frac{1}{2}e \log \frac{5}{3}$. **6.** $\frac{9a^2}{16}[\frac{1}{12}\pi + \frac{1}{3}\tan^{-1}\frac{1}{3}]$.
7. $\frac{15}{2}\pi$. **8.** 0.
9. $\frac{1}{2}\alpha \operatorname{cosec} \alpha$.

14.3. **1.** (i) $\frac{12}{5}\pi a^6$; (ii) $\frac{1}{2}\pi a^6 + \pi h^4 a^2 + \frac{3}{2}\pi h a^5 + \frac{2}{5}\pi h^5 a$;
(iii) $\pi c^6 \int_0^{\frac{1}{2}\pi} d\mu[3\cosh^4\alpha \cos^4\mu + 4\sinh^4\alpha \sin^4\mu]\cdot$
$\cdot(\cosh^2\alpha - \cos^2\mu)^{\frac{1}{2}} \cosh\alpha \cos\mu$.
2. (i) $\frac{12}{35}\pi a^7$; (ii) $\frac{1}{4}\pi a^6 h + \frac{1}{5}\pi a^2 h^5$;
(iii) $\frac{4}{35}\pi c^7[2\sinh\alpha\cosh^6\alpha + \sinh^5\alpha\cosh^2\alpha]$.
3. $\frac{16}{3}a^3$; $8a^2$. **4.** $\frac{1}{4}\pi a^2 + \frac{1}{4}\pi[a^2/k(1 - k^2)^{\frac{1}{2}}]\sin^{-1}(1 - k^2)^{\frac{1}{2}}$.
5. $\pi a^2\sqrt{2}$; $35\pi a^6/6\sqrt{2}$.
6. $\frac{1}{15}a^2b^2c^2$. **8.** $-\frac{11}{36}$.

15.1. **1.** $\frac{\exp[1 + z^2/(x^2 + y^2)]}{x^2z^2(x^2 + y^2)^{\frac{1}{2}}}[z\{(x^2 + y^2)(2x^2 - y^2) - 2x^2z^2\},$
$xyz\{3(x^2 + y^2) - 2z^2\}, x(x^2 + y^2)(2z^2 - x^2 - y^2)]$;
$\sec\varphi \exp(1 + z^2/\rho^2)\left[\frac{2(\rho^2 - z^2)}{\rho z}, \frac{\rho\tan\varphi}{z}, \frac{2z^2 - \rho^2}{z^2}\right]$;
$\sec\varphi \exp(\operatorname{cosec}^2\theta)[\tan\theta\sin\theta, \sin\theta\{\sec^2\theta + 1 - 2\operatorname{cosec}^2\theta\}, \tan\theta\tan\varphi]$.
2. $(\operatorname{grad}\psi)_{\xi,\eta} = \frac{c^3\cos\varphi}{8(\cosh^2\xi - \cos^2\eta)^{\frac{1}{2}}}[\cosh 4\xi \sin 4\eta +$
$+ 2\cosh 2\xi\sin 2\eta, \sinh 4\xi\cos 4\eta + 2\sinh 2\xi\cos 2\eta]$,
$(\operatorname{grad}\psi)_\varphi = \frac{-c^3\sin\varphi}{32\cosh\xi\cos\eta}[\sinh 4\xi\sin 4\eta + 4\sinh 2\xi\sin 2\eta]$.

15.2. **2.** $(z - x, 0, y + z)$;
$[(z - \rho\cos\varphi)\cos\varphi, (\rho\cos\varphi - z)\sin\varphi, \rho\sin\varphi + z]$;
$[r\cos\theta(\cos\theta + \sin\theta\sin\varphi) + r\sin\theta\cos\varphi(\cos\theta - \sin\theta\cos\varphi),$
$-r\sin\theta(\cos\theta + \sin\theta\sin\varphi) + r\cos\theta\cos\varphi(\cos\theta - \sin\theta\cos\varphi),$
$-r\sin\varphi(\cos\theta - \sin\theta\cos\varphi)]$.

15.3. **2.** $-\frac{5}{3}a^3$.

15.4. **2.** $(\rho + 2z)\cos\varphi \equiv x\left[1 + \dfrac{2z}{(x^2 + y^2)^{\frac{1}{2}}}\right].$

16.1. **1.** (i) $y = (2x^3 - 3x)\sin 2x + (3x^2 - \frac{3}{2})\cos 2x + C,$
(ii) $y = \frac{1}{2}x^2 \sin^{-1} x + \frac{1}{4}\cos^{-1} x + \frac{1}{4}x(1 - x^2)^{\frac{1}{2}} + C,$
(iii) $y = A(1 + x^2)^{\frac{1}{2}},$
(iv) $y = \sin^{-1}\{\sinh^{-1}(x + \alpha)\},$
(v) $y = \tan\{\frac{1}{2}x(1 + x^2)^{\frac{1}{2}} + \frac{1}{2}\sinh^{-1} x + \alpha\}.$

2. (i) $y = x\cos^{-1} x - (1 - x^2)^{\frac{1}{2}} + \frac{3}{8}\pi\sqrt{2} + (1 - \frac{1}{16}\pi^2)^{\frac{1}{2}},$
(ii) $y = \exp\{(1 + x^2)^{\frac{1}{2}} - 3\},$
(iii) $y = 2\tan\{2\sinh^{-1} x + \tan^{-1}\frac{5}{4}\pi + 2\sinh^{-1}\frac{5}{2}\pi\}.$

16.2. **1.** (i) $y = e^{2(x-x_0)}\{y_0 + \frac{1}{8}(4x_0^3 + 6x_0^2 + 22x_0 + 11) +$
$- \frac{1}{8}(4x^3 + 6x^2 + 22x + 11)\}.$
(ii) $y = e^{-x}\{y_0e^{x_0} + 2(x - x_0) + [\frac{1}{2}e^u\{u\cos u + (u - 1)\sin u\}]_{x_0}^x\}.$
(iii) $y = (1 + x^2)^{-\frac{1}{2}}\{y_0(1 + x_0^2)^{\frac{1}{2}} + 4[\tanh(\sinh^{-1} u)]_{x_0}^x\}.$
(iv) $y = \exp\left(\dfrac{1}{2x^2}\right)\left\{y_0\exp\left(-\dfrac{1}{2x_0^2}\right) + \frac{1}{4}\left[\exp\left(-\dfrac{1}{u^2}\right) - \dfrac{1}{u^2}\right]_{x_0}^x\right\}.$
(v) $y = 4\operatorname{cosec} x[(2 - u^2)\cos u + 2u\sin u - \frac{1}{5}e^{-2u}(2\sin u + \cos u)]_{x_0}^x.$

16.3. **1.** $\dfrac{1}{p-3}, \dfrac{p}{p^2-4}, \dfrac{2(3 + 2p^2 + p^4)}{p^4},$
$p^3\mathscr{L}[f] - p^2f(0) - pf'(0) - f''(0),$
$(p^2 + a_1p + a_2)\mathscr{L}[f] - (p + a_1)f(0) - f'(0).$

2. $\dfrac{1}{\sqrt{3}}\sinh x\sqrt{3}, \quad \dfrac{e^{ax} - e^{bx}}{a - b}, \quad \frac{1}{4}(\cosh 2x - 1),$
$\dfrac{x^{n-1}}{(n-1)!}, \quad \frac{1}{16}(\sinh 2x - \sin 2x), \quad \dfrac{a\sin bx - b\sin ax}{ab(a^2 - b^2)},$
$\dfrac{b^2(\cos ax - 1) - a^2(\cos bx - 1)}{a^2b^2(a^2 - b^2)}.$

16.4. **1.** (i) $2(1 - e^{-2x}).$
(ii) $-\frac{113}{32}e^{-4x} + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{17}{32}.$
(iii) $\sin x + \cos 2x - \frac{3}{2}\sin 2x.$
(iv) $\frac{25}{4}e^{2x} - 40e^x + 3x^3 + \frac{27}{2}x^2 + \frac{63}{2}x + \frac{135}{4}.$
(v) $\frac{1}{2}x(\sin 2x - \cos 2x) + \frac{1}{4}\sin 2x.$

2. (i) $(y_0 + \frac{4}{9})e^{3x} - 2x^3 - 2x^2 - \frac{4}{3}x - \frac{4}{9}.$
(ii) $(y_0 + x)e^x.$
(iii) $\frac{1}{2}y_1\sinh 2x + (\frac{1}{2}y_0 + \frac{4}{3})\cosh 2x - \frac{4}{3}\cosh x.$
(iv) $(1 + y_2)e^{-x} + \frac{1}{16}e^{-4x} + (y_1 + y_2 + \frac{3}{4})x + y_0 - y_2 - \frac{15}{16}.$
(v) $y_0\cosh x + y_1\sinh x + \frac{1}{2}(y_2 - y_0)x\sinh x +$
$+ \frac{1}{2}(y_3 + y_1)(x\cosh x - \sinh x).$

16.5. **1.** (i) $\dfrac{A}{n^2}\{(2 + \frac{1}{2}nx\sin\lambda)\cos nx + (1 - \frac{1}{2}\sin\lambda + \frac{1}{2}nx\cos\lambda)\sin nx\}.$

(ii) $\dfrac{1}{m^2 - n^2}\,e^{mx} + \dfrac{1}{2n(n+m)}\,e^{-nx} + \dfrac{1}{2n(n-m)}\,e^{nx} + {}$
$$+ \frac{1}{n^2}\{nxe^{nx} - \sinh nx\}.$$

(iii) $\frac{1}{3}\Big[2(2+x)e^x - 2x(2\cos x + \sin x) + 4\sin x + {}$
$$- 4e^{-\frac{1}{2}x}\cos\frac{x\sqrt{3}}{2} - \frac{8}{\sqrt{3}}e^{-\frac{1}{2}x}\sin\frac{x\sqrt{3}}{2}\Big].$$

[*Hint*: write $\cos x = \mathrm{Re}(e^{ix})$ and use the partial fraction method.]

(iv) $\frac{1}{74}(7 - 2\sqrt{3})\,e^x\left\{e^{-\frac{3}{2}x}\sin\dfrac{x\sqrt{3}}{2} + \sin 2x\right\} - \frac{2}{3}e^x + {}$
$$- \tfrac{2}{3}e^{-\frac{1}{2}x}\cos\frac{x\sqrt{3}}{2} - \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin\frac{x\sqrt{3}}{2}.$$

(v) $\dfrac{x^2e^{ax}}{4a^2+b^2} - \dfrac{8axe^{ax}}{(4a^2+b^2)^2} - \dfrac{4(4a^3 - 3ab^2)\,e^{-ax}\sin bx}{b(4a^2+b^2)^3} + {}$
$$- \frac{2(12a^2 - b^2)(e^{-ax}\cos bx - e^{ax})}{(4a^2+b^2)^3}.$$

(vi) $\dfrac{A(me^{-nx} - ne^{-mx})}{m^2(m-n)} + \dfrac{A(e^{-nx} - e^{-mx})}{(m-n)^2} - \dfrac{Axe^{-mx}}{m-n}.$

(vii) $\frac{11}{8}A(\sin x - x\cos x) - \frac{1}{8}Ax^2\sin x.$

(viii) $-\frac{4}{3}\cos 2x + \frac{1}{12}\sin x + (3 - \frac{1}{6}x)\cos x + {}$
$$+ \tfrac{1}{12}\sin(x-4) + \tfrac{1}{18}\sin 2x - \tfrac{1}{36}\sin(x-6) + \tfrac{1}{12}\sin(x+2).$$

2. (i) $y_0\cos x + y_1\sin x + \frac{1}{2}\sin x[1 + (x-1)e^x] + {}$
$$+ \tfrac{1}{10}xe^x(2\cos x - \sin x) - \tfrac{1}{50}e^x(4\cos x + 3\sin x) + {}$$
$$+ \tfrac{2}{25}\cos x - \tfrac{3}{50}\sin x.$$

(ii) $y_0\cos x\sqrt{2} + \frac{1}{4}\sqrt{2}[(2y_0 + y_2)x + 2y_1]\sin x\sqrt{2} + {}$
$$+ \tfrac{1}{8}(y_3 + 2y_1)(\sqrt{2}\sin x\sqrt{2} - 2x\cos x\sqrt{2}) + {}$$
$$+ \tfrac{1}{12}\cos x\sqrt{2} - \left[\frac{1}{12\sqrt{2}} + \frac{71}{1681\sqrt{2}}\right]\sin x\sqrt{2} + {}$$
$$- \tfrac{9}{1681}\sin x\cosh 2x - \tfrac{40}{1681}\cos x\sinh 2x.$$

[*Hint*: write
$\sin x\sinh 2x = \mathrm{Re} - i[\exp\{(i+2)x\} - \exp\{(i-2)x\}]$;
use $(p^2 + a)^{-2} = -\partial(p^2+a)^{-1}/\partial a$.]

(iii) $y_0e^x + (y_1 - y_0)\sinh x + \frac{1}{2}(y_3 - y_1)\{\sinh x - \sin x\} + {}$
$$+ \tfrac{1}{2}(y_2 - y_0)\{\cosh x - \cos x\} + \tfrac{1}{5}e^x\{4\sin x + e^{-2x} - 5\} + {}$$
$$+ \tfrac{4}{5}\cos x + \tfrac{2}{5}\sin x.$$

(iv) $y_0e^{-x} + (y_1 + y_0)(e^{-x} - e^{-2x}) + {}$
$$+ \tfrac{1}{4}(y_2 + 3y_1 + 2y_0)(5e^{-x} + 3e^{-3x} - 8e^{-2x} - 2xe^{-x}) + {}$$
$$+ \tfrac{1}{4}(y_3 + 7y_2 + 14y_1 + 8y_0)(2xe^{-x} + 4e^{-2x} - 3e^{-x} - e^{-3x}) + {}$$
$$+ \tfrac{1}{2}\exp[-2(x-3)]\{2x(x-6) + 2xe^x - 7xe^3 - xe^{-3} + {}$$
$$- 7e^x - e^{-x} + 4(e^3 + e^{-3}) + 18\}.$$

3. (i) $y = \frac{1}{6}\{(y_0 + 1 - 2z_0)e^x + (1 - 2z_0 - 5y_0)e^{-5x}\}$,
$z = -\frac{1}{6}\{2(y_0 + 1 - 2z_0)e^x + (1 - 2z_0 - 5y_0)e^{-5x}\}$.

(ii) $y = -\frac{61}{20}e^{2x} + \frac{117}{40}e^{-2x} + \frac{1}{16}\cos 2x + \frac{323}{240}\sin x + \frac{151}{48}\sin 2x + \frac{3}{8}xe^{2x}.$

(iii) $a^2 > 4b,\ (a^2 - 4b)^{\frac{1}{2}} = \lambda$:

$$y = (y_1 \cos \tfrac{1}{2}ax + z_1 \sin \tfrac{1}{2}ax)\ 2\lambda^{-1} \sin \tfrac{1}{2}\lambda x,$$
$$z = (z_1 \cos \tfrac{1}{2}ax - y_1 \sin \tfrac{1}{2}ax)\ 2\lambda^{-1} \sin \tfrac{1}{2}\lambda x.$$

$a^2 = 4b$:

$$y = y_1 x \cos \tfrac{1}{2}ax + z_1 x \sin \tfrac{1}{2}ax,$$
$$z = z_1 x \cos \tfrac{1}{2}ax - y_1 x \sin \tfrac{1}{2}ax.$$

$a^2 < 4b,\ (4b - a^2)^{\frac{1}{2}} = \lambda$:

$$y = \lambda^{-1}(y_1 \cos \tfrac{1}{2}ax + z_1 \sin \tfrac{1}{2}ax)(e^{\frac{1}{2}\lambda x} - e^{-\frac{1}{2}\lambda x}),$$
$$z = \lambda^{-1}(z_1 \cos \tfrac{1}{2}ax - y_1 \sin \tfrac{1}{2}ax)(e^{\frac{1}{2}\lambda x} - e^{-\frac{1}{2}\lambda x}).$$

(iv) $f_r(x) = \sum_{m=0}^{r} \dfrac{(\lambda x)^m}{m!} \mu_{r-m} e^{-x} \qquad (r = 0, 1, \ldots, n).$

17.1. **2.** (ii) $\cosh nx \cos ny,\ \sinh nx \sin ny.$

(iii) $\dfrac{r^{\frac{1}{2}}}{r^2 - 2r\cos\theta + 1}(r \cos \tfrac{3}{2}\theta - \cos \tfrac{1}{2}\theta,\ r \sin \tfrac{3}{2}\theta - \sin \tfrac{1}{2}\theta).$

(v) $\tfrac{1}{2}\log[(x^2 + y^2 - a^2 - b^2)^2 + (xy - ab)^2],$

$\tan^{-1} \dfrac{xy - ab}{x^2 + y^2 - a^2 - b^2}$, where $\zeta = a + ib$.

(vi) $\dfrac{(\cos \alpha x \cosh \alpha y,\ -\sin \alpha x \sinh \alpha y)}{\tfrac{1}{2}(\cos 2\alpha x + \cosh 2\alpha y)}.$

17.3. **1.** -1.

2. $\dfrac{a}{(a-b)(a-c)},\ \dfrac{b}{(b-a)(b-c)},\ \dfrac{c}{(c-a)(c-b)}$ at $z = a, b, c$.

3. $1 \quad (z = 1),\ -1 \quad (z = 0).$

4. $\pm(a \sinh a - \cosh a)/4a^3$ at $z = \pm a$.

5. $\dfrac{(2n-2)!\,(\mp i)}{(n-1)!\,(2a)^{2n-1}}$ at $z = \pm ia$.

6. $-\tfrac{1}{16}e^{-1\pm i}(3 \pm 2i).$

17.4. **1.** 1. **2.** $\dfrac{7\pi}{6 - \pi\sqrt{3}}.$

3. 0. **4.** $e^{\pi},\ -e^{-\pi}.$

5. $\tfrac{1}{32}[8 + (4n+1)^2\pi^2].$

17.5. **12.** $\dfrac{1}{\zeta} + 2\sum_{n=1}^{\infty} \dfrac{(-1)^n \zeta}{\zeta^2 - n^2\pi^2}.$

18.1. **1.** (i) $\tfrac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left\{\dfrac{1}{n^2}\cos nx - \dfrac{\pi}{n}\sin nx\right\}.$

(ii) $\tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}\cos nx.$

2. (i) $$\frac{e^{b\pi}\sinh b\pi}{b\pi} + \frac{2e^{b\pi}\sinh b\pi}{b\pi}\sum_{n=1}^{\infty}\frac{(b\cos nx - n\sin nx)}{b^2+n^2}.$$

(ii) $$\frac{\sinh b\pi}{b\pi} + \frac{2\sinh b\pi}{b\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(b\cos nx - n\sin nx)}{b^2+n^2}.$$

3. (i) $$\frac{a(1-\cos 2a\pi)}{\pi}\left[\frac{1}{2a^2} - \sum_{n=1}^{\infty}\frac{\cos nx}{n^2-a^2}\right] - \frac{\sin 2a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2-a^2}.$$

(ii) $$\frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n-1}\sin nx}{n^2-a^2}.$$

4. (i) $$\frac{a\sin 2a\pi}{\pi}\left[\frac{1}{2a^2} - \sum_{n=1}^{\infty}\frac{\cos nx}{n^2-a^2}\right] + \frac{1-\cos 2a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2-a^2}.$$

(ii) $$\frac{2a\sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^2-a^2}\right].$$

5. (i) $$\frac{1}{2\pi(a^2+b^2)}\{b(e^{2b\pi}\cos 2a\pi - 1) + ae^{2b\pi}\sin 2a\pi\} +$$
$$+\frac{1}{\pi}(e^{2b\pi}\cos 2a\pi - 1)\cdot$$
$$\cdot\sum_{n=1}^{\infty}\frac{\{b(a^2+b^2+n^2)\cos nx + n(a^2-b^2-n^2)\sin nx\}}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}} +$$
$$+\frac{1}{\pi}e^{2b\pi}\sin 2a\pi\sum_{n=1}^{\infty}\frac{\{a(a^2+b^2-n^2)\cos nx - 2abn\sin nx\}}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}}.$$

(ii) $$\frac{1}{\pi(a^2+b^2)}\{b\cos a\pi\sinh b\pi + a\sin a\pi\cosh b\pi\} +$$
$$+\frac{2}{\pi}\cos a\pi\sinh b\pi\cdot$$
$$\cdot\sum_{n=1}^{\infty}\frac{(-1)^n\{b(a^2+b^2+n^2)\cos nx + n(a^2-b^2-n^2)\sin nx\}}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}} +$$
$$+\frac{2}{\pi}\sin a\pi\cosh b\pi\cdot$$
$$\cdot\sum_{n=1}^{\infty}\frac{(-1)^n\{a(a^2+b^2-n^2)\cos nx - 2abn\sin nx\}}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}}.$$

6. (i) $$\frac{2}{a^2}\sin 2a\pi - \frac{2\pi}{a}\cos 2a\pi - \frac{1}{\pi a^3}(1-\cos 2a\pi) +$$
$$+\sum_{n=1}^{\infty}\left\{\frac{4a\pi\cos 2a\pi}{n^2-a^2} + \frac{4(a^2+n^2)\sin 2a\pi}{(n^2-a^2)^2} +\right.$$
$$\left.+\frac{2a(a^2+3n^2)(1-\cos 2a\pi)}{(n^2-a^2)^3}\right\}\cos nx + \sum_{n=1}^{\infty}\left\{-\frac{4n\pi\sin 2a\pi}{n^2-a^2} +\right.$$
$$\left.+\frac{8an\cos 2a\pi}{(n^2-a^2)^2} + \frac{2n(3a^2+n^2)\sin 2a\pi}{(n^2-a^2)^3}\right\}\sin nx.$$

(ii) $$\sum_{n=1}^{\infty}(-1)^n\left\{-\frac{2n\pi\sin a\pi}{n^2-a^2} + \frac{8an\cos a\pi}{(n^2-a^2)^2} +\right.$$
$$\left.+\frac{4n(3a^2+n^2)\sin a\pi}{(n^2-a^2)^3}\right\}\sin nx.$$

7. (i), (ii) $$\frac{\pi^2}{48} + \sum_{n=1}^{\infty}\left\{\frac{\pi}{4n}\sin\tfrac{1}{2}n\pi + \frac{1}{n^2}\cos\tfrac{1}{2}n\pi - \frac{2}{\pi n^3}\sin\tfrac{1}{2}n\pi\right\}\cos nx +$$
$$+ \sum_{n=1}^{\infty}\left\{-\frac{\pi}{4n}\cos\tfrac{1}{2}n\pi + \frac{1}{n^2}\sin\tfrac{1}{2}n\pi + \frac{2}{\pi n^3}\cos\tfrac{1}{2}n\pi\right\}\sin nx.$$

18.2. **1.** (i) $$\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}\pi^2}{n} - \frac{2}{n^3}[1-(-1)^n]\right\}\sin nx.$$

(ii) $$\tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

2. (i) $$\frac{2}{\pi}\sin a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\sin nx}{n^2-a^2}.$$

(ii) $$\frac{2a}{\pi}\sum_{n=0}^{\infty}\frac{\{(-1)^n\cos a\pi - 1\}}{n^2-a^2}\cos nx.$$

3. (i) $$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[\{(-1)^n e^{b\pi}\cos a\pi - 1\}n(a^2-b^2-n^2) - 2(-1)^n abn e^{b\pi}\sin a\pi]}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}}\cdot$$
$$\cdot\sin nx.$$

(ii) $$\frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\cdot$$
$$\cdot\frac{[\{(-1)^n e^{b\pi}\cos a\pi - 1\}b(a^2+b^2+n^2) + 2(-1)^n a(a^2+b^2-n^2)e^{b\pi}\sin a\pi]}{\{b^2+(n+a)^2\}\{b^2+(n-a)^2\}}.$$

4. (i) $$\sum_{n=1}^{\infty}\left[-\frac{\pi}{2n}\cos\tfrac{1}{2}n\pi + \frac{2}{n^2}\sin\tfrac{1}{2}n\pi + \frac{4}{\pi n^3}(\cos\tfrac{1}{2}n\pi - 1)\right]\sin nx.$$

(ii) $$\frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left[\frac{\pi}{2n}\sin\tfrac{1}{2}n\pi + \frac{2}{n^2}\cos\tfrac{1}{2}n\pi - \frac{4}{\pi n^3}\sin\tfrac{1}{2}n\pi\right]\cos nx.$$

18.3. **1.** $a_n' = 0$, $b_n' = (-1)^{n-1}\,4n^{-1}$, giving for the derived function
$$2x = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx,$$
which is the same as Ch. 18, Example 1 (ii).

2. $a_n' = ba_n$, $b_n' = bb_n$, which is correct since $\mathrm{d}(e^{bx})/\mathrm{d}x = be^{bx}$.

3. Fourier series for $\cosh bx$ in $(-\pi, \pi)$ has
$$a_n = \frac{2(-1)^n b\sinh b\pi}{\pi(b^2+n^2)},\quad b_n = 0.$$
Thus
$$a_n' = 0,\quad b_n' = \frac{2nb(-1)^{n-1}\sinh b\pi}{\pi(b^2+n^2)},$$
giving
$$b\sinh bx = \frac{2b\sinh b\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\sin nx}{b^2+n^2},$$
which is correct.

4. $\frac{1}{2}\cos\frac{1}{2}x = \dfrac{1}{\pi} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}\cos nx}{4n^2-1}$.

5. $a\cos ax \equiv \dfrac{\sin a\pi}{\pi} + \dfrac{2a^2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}\cos nx}{n^2-a^2}$,

which is correct from 18.1, Example 4 (ii).

18.4. **2.** (i) $F(k) = \pi a(k) = \pi,\ \ b(k) = 0.$

(ii) $F(k) = \pi a(k)$

$$= \frac{\pi}{a^5}\left[\tfrac{1}{3}e^{-ka} + \frac{1}{14\sqrt{6}}\,e^{-ka/\sqrt{2}}\,\{\sqrt{3}\cos(ka\sqrt{\tfrac{3}{2}}) + 5\sin(ka\sqrt{\tfrac{3}{2}})\}\right],$$

$b(k) = 0.$

(iii) $F(k) = \pi a(k) = a\pi^{\frac{1}{2}}\exp(-k^2a^2),\ \ b(k) = 0.$

(iv) $F(k) = \frac{1}{2}\pi\{a(k) + ib(k)\} = \dfrac{a+ik}{a^2+k^2}$.

19.4. **4.** $m = n + 1$: $\quad 2,\ -\dfrac{2(n+2)(n+1)^2}{(2n+3)(2n-1)}$.

$m = n - 1$: $\quad 0,\ \dfrac{2n^2(n-1)(n+2)}{4n^2-1}$.

20.1. **1.** $\bar{x} = 2.29;\quad \sigma = 1.47.$

2. $\bar{x} = 66.99;\quad \sigma = 2.52.$

20.2. **1.** $2^{-n}\dbinom{n}{m}$.

2. (i) $\dfrac{39\cdot 26\cdot 13}{51\cdot 50\cdot 49} \approx 0.105.$

(ii) $\dfrac{12\cdot 11\cdot 10}{51\cdot 50\cdot 49} \approx 0.0106.$

(iii) $\dfrac{13\cdot 12\cdot 13\cdot 12\cdot (4!)}{52\cdot 51\cdot 50\cdot 49\cdot (2!)^2} \approx 0.0225.$

3. $\langle x\rangle = \frac{1}{6}(n+1)(2n+1).$
$\sigma^2 = \frac{1}{180}[16n^4 + 30n^3 - 5n^2 - 210n - 119].$

4. $\dfrac{35\cdot 17\cdot 364\cdot 363\cdot\ldots\cdot 332}{(365)^{34}} \approx 0.265.$

6. $\langle x\rangle = -\dfrac{3a}{4\pi};\ \ \sigma = \dfrac{a}{4\pi}(8\pi^2 - 5)^{\frac{1}{2}}.$

7. (i) $\langle x\rangle = \frac{1}{4}a\pi;\ \ \sigma \approx 0.788a.$
(ii) $\langle x\rangle = 4a\pi^{-1};\ \ \sigma \approx 0.619a.$

8. (i) $\langle n \rangle = 6$; $\sigma = \sqrt{30}$.
(ii) $\langle n \rangle = 36$; $\sigma = 2\sqrt{15}$.

20.3. **1.** Binomial distribution $p_r = \dfrac{5^{12-r}}{6^{12}} \dbinom{12}{r}$:

r	0	1	2	3	4	5	6	7	8
$10^3 p_r$	112	269	296	197	89	28	7	1	0

$\langle r \rangle = 2$; $\bar{r} = 1.96$.
σ (predicted) $= 1.29$; σ (observed) $= 1.26$.

2. $\langle n \rangle = 2.3$, $\bar{n} = 2.29$;
σ (predicted) $= 1{,}52$, σ (observed) $= 1.47$.

20.4. **1.** Significant: 73″, (72″ nearly).
Highly significant: 58.5″, 75″.

2. p (45 to 50) $= 0.381$,
p (49 to 55) $= 0.289$,
p (41 to 53) $= 0.770$,
p (36 to 46) $= 0.406$.

20.5. **1.** Observation not significant, assuming equal probability of 'heads' and tails'.

2. $\epsilon = 0.028$; $|p_1' - p_2'| = 0.016 < \epsilon$.
Proportions likely to be equal.

20.6. **1.** (N, T) *correlation*; take $N = x$, $T = y$.
Lines of regression:
(N on T) $y - 44.95 = 2.55(x - 31.33)$,
(T on N) $y - 44.95 = 0.715(x - 31.33)$.
Correlation coefficient $r = 0.53$.
(N, P) *correlation*; take $N = x$, $P = y$.
Lines of regression:
(N on P) $y - 10.05 = -1.69(x - 31.15)$,
(P on N) $y - 10.05 = -0.56(x - 31.15)$.
Correlation coefficient $r = 0.58$.

2. x_r = result in test A,
y_s = result in test B.
Lines of regression:
(x on y) $y - 3.035 = 1.506(x - 3.360)$,
(y on x) $y - 3.035 = 0.6203(x - 3.360)$.
$r = 0.6228$.

3. $y = 3.615 + 0.433x + 0.216x^2$.

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INDEX

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